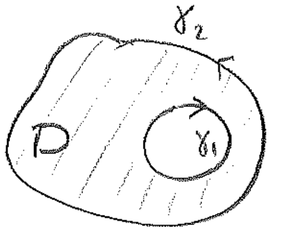
Kap. 9.2 Greens formel.



$\gamma = \partial D$ = randen till D.
$\gamma = \gamma_1 \cup \gamma_2$ är <u>positivt orienterad</u> om området ligger till vänster om γ.

Sats (Greens formel)

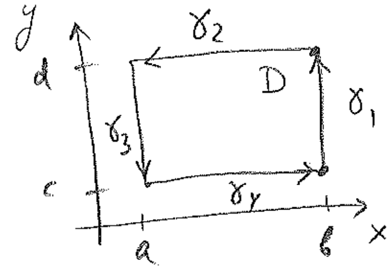
Antag
- $P, Q : \Omega \to \mathbb{R}$ är C^1 i en öppen mängd $\Omega \subset \mathbb{R}^2$.
- $D \subset \Omega$ är kompakt
- ∂D är styckvis C^1.

Då gäller
$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Bevis (i fallet då $D = [a,b] \times [c,d]$ är en rektangel.

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_c^d \left(\int_a^b \frac{\partial Q(x,y)}{\partial x} dx\right) dy - \int_a^b \left(\int_c^d \frac{\partial P(x,y)}{\partial y} dy\right) dx =$$

$$= \int_c^d \big(Q(b,y) - Q(a,y)\big)\,dy - \int_a^b \big(P(x,d) - P(x,c)\big)\,dx =$$

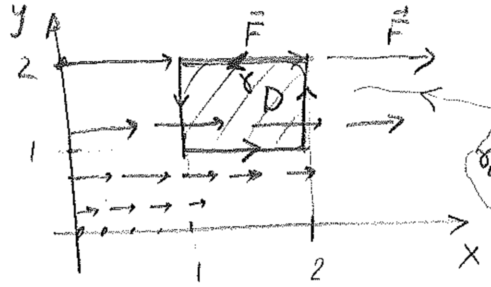


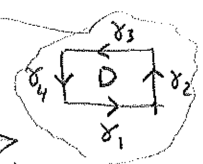
$$= \int_{\gamma_1} Q\,dy + \int_{\gamma_3} Q\,dy + \int_{\gamma_2} P\,dx + \int_{\gamma_4} P\,dx =$$

$$\left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4}\right)(P\,dx + Q\,dy) = \int_{\partial D} P\,dx + Q\,dy \qquad \boxtimes$$

Ex Låt $D = [1,2] \times [1,2]$; $\vec{F} = (P, Q) = (\frac{y}{2}, 0)$

$$\oint_{\partial D} P\,dx + Q\,dy = \int_1^2 (\tfrac{1}{2}, 0)\cdot(1,0)\,dx + \underbrace{\int_1^2 (\tfrac{y}{2}, 0)\cdot(0,1)\,dy}_{\gamma_2} - \underbrace{\int_1^2 (2,0)\cdot(1,0)\,dx}_{\gamma_3} + $$
$$- \underbrace{\int_1^2 (\tfrac{y}{2}, 0)\cdot(0,1)\,dy}_{\gamma_4} =$$





$\gamma_1 : \vec{r}_1 = (x, 0)$
$\gamma_2 : \vec{r}_2 = (0, y)$

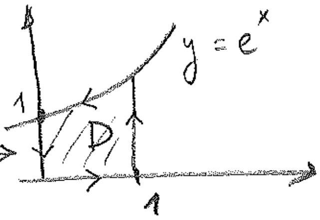
$\Leftrightarrow \frac{1}{2} - 1 = \underline{-\frac{1}{2}.}$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_1^2\int_1^2 \left(0 - \tfrac{1}{2}\right) dx\,dy = \underline{-\tfrac{1}{2}}. \qquad \text{Ok!}$$

<u>Ex 2</u> (se ex. 3 i Kap. 9.1)

$\vec{F} = (e^x y^2, \cos x)$, D

Cirkulation längs ∂D?



$$\int_{\partial D} \vec{F}\cdot d\vec{r} = \int_{\partial D} \underbrace{e^x y^2}_{P}\, dx + \underbrace{\cos x}_{Q}\, dy =$$

$\underset{\text{sats}}{\underline{\text{Greens}}}$ $$\iint_D \left(\frac{\partial}{\partial x}\cos x - \frac{\partial}{\partial y} e^x y^2\right) dx\,dy =$$

$$= \int_0^1\int_0^{e^x} \left(-\sin x - 2y e^x\right) dy\,dx = -\int_0^1 \Big[y\cdot\sin x + y^2 e^x\Big]_{y=0}^{e^x} dx =$$

$$= -\int_0^1 \left(e^x \sin x + e^{3x}\right) dx = \frac{e}{2}(\cos 1 - \sin 1) - \frac{e^3}{3} - \frac{1}{6}.$$