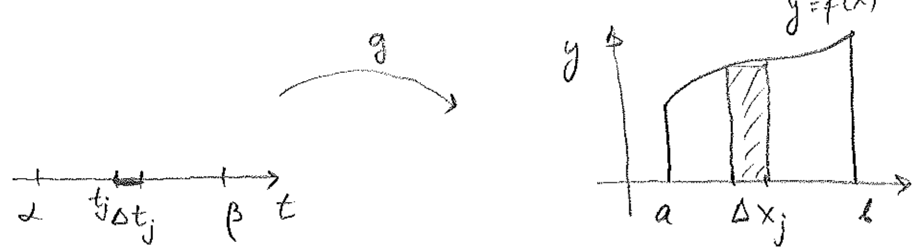
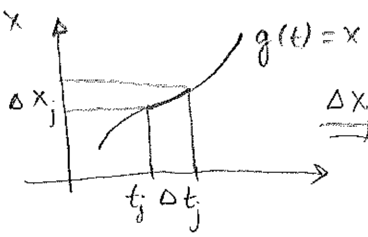
Kap 6.4 Variabelbyte i dubbelintegraler.

Ex. (1-dim.) Om g är monoton, så gäller

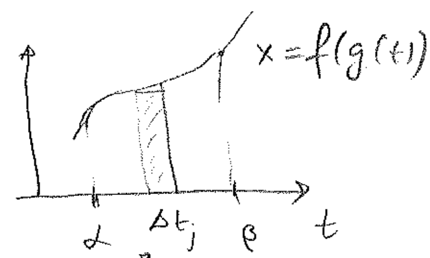
$$A = \int_a^b f(x)\,dx = \int_\alpha^\beta f(g(t)) \cdot \underbrace{|g'(t)|\,dt}_{dx}$$



$$\int_a^b f(x)\,dx \approx \sum^N f(x_j)\,\Delta x_j$$
Riemannsumma.



$$\underline{\Delta x_j \approx |g'(t_j)| \cdot \Delta t_j}$$



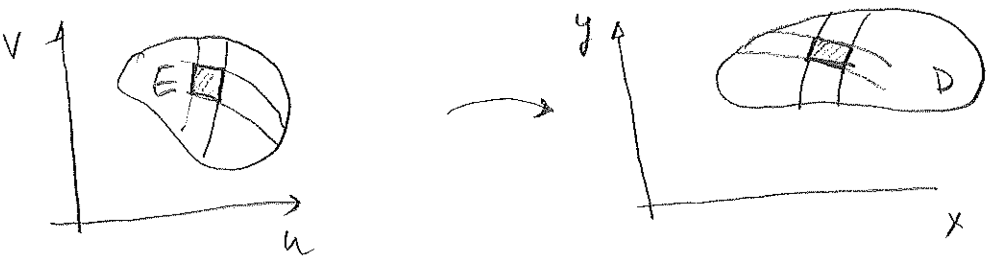
$$\sum^N f(\underbrace{g(t_j)}_{x_j}) \cdot \Delta x_j \approx \sum^N f(g(t_j)) \cdot |g'(t_j)|\,\Delta t_j$$

Sats Låt $\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$ vara en bijektiv C^1-avbildning $E \mapsto D$ där E, D är öppna kvadrerbara områden i $\mathbb{R}^2$.

Antag $J(u,v) = \dfrac{d(x,y)}{d(u,v)} \neq 0$ i E

funktionaldeterminanten (eller Jakobianen)

$$J(u,v) = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

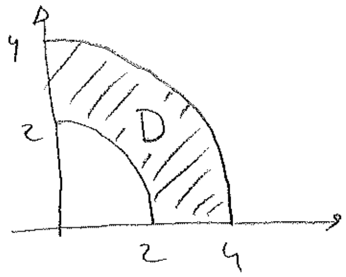


$|J(u,v)|$ = areaförstorings faktor.

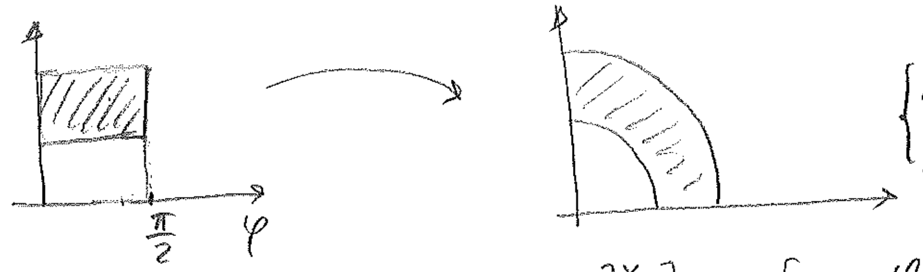
Då är $\displaystyle\iint_D f(x,y)\,dx\,dy = \iint_E f(g(u,v), h(u,v)) \cdot |J(u,v)|\,du\,dv.$

om funktionerna under integraltecken är integrerbara.

Ex 1 Beräkna $\iint_D (x+y)\,dx\,dy$ där



Lösn. Använd polära koordinater:



$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

$$\frac{d(x,y)}{d(r,\varphi)} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{bmatrix} = \det\begin{bmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{bmatrix} = r.$$

Alltså, $\iint_D (x+y)\,dx\,dy = \int_0^{\pi/2}\left(\int_2^4 (r\cos\varphi + r\sin\varphi)\, r\,dr\right)d\varphi =$

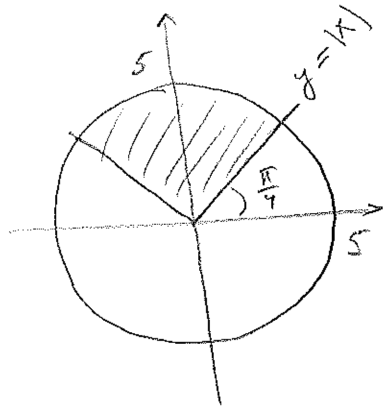
$= \int_0^{\pi/2} (\cos\varphi + \sin\varphi)\left[\frac{r^3}{3}\right]_2^4 d\varphi = \frac{56}{3}\int_0^{\frac{\pi}{2}} (\cos\varphi + \sin\varphi)\,d\varphi$

$= \frac{56}{3}\left[\sin\varphi - \cos\varphi\right]_0^{\pi/2} = \frac{112}{3}.$

Ex (Uppg. 6.22)

$D = \{(x,y) \mid x^2+y^2 \le 25,\ \ y \ge |x|\}.$

$\iint_D x^2 e^{x^2+y^2}\,dx\,dy = ?$



Polära koordinater: $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$

$$\begin{cases} x^2 e^{x^2+y^2} = (r\cos\varphi)^2 e^{r^2} \\ dx\,dy = r\,dr\,d\varphi \end{cases}$$

$\iint_D x^2 e^{x^2+y^2}\,dx\,dy = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^5 r^2\cos^2\varphi\, e^{r^2}\, r\,dr\,d\varphi =$

$= \underbrace{\left(\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos^2\varphi\,d\varphi\right)}_{I_1}\underbrace{\left(\int_0^5 r^3 e^{r^2}\,dr\right)}_{I_2} =$

$$I_1 = \int_{\pi/4}^{3\pi/4} \frac{1+\cos 2\varphi}{2}\,d\varphi = \left[\frac{1}{2}\varphi + \frac{\sin 2\varphi}{4}\right]_{\pi/4}^{3\pi/4} = \frac{\pi}{4} + \frac{\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}}{4} =$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

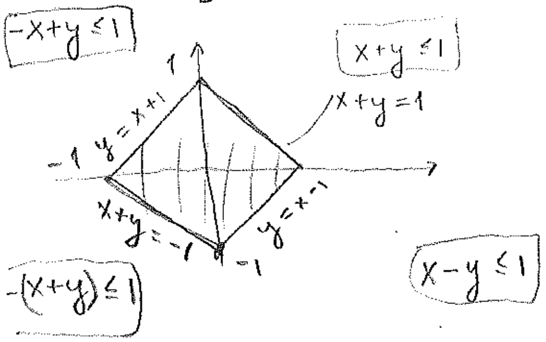
$$I_2 = \int_0^5 \underbrace{\frac{r^2}{2}}_{u} \cdot \underbrace{\left(e^{r^2}\right)'}_{v'} \qquad v = e^{r^2}$$

$$= \left[\frac{r^2}{2} e^{r^2}\right]_0^5 - \int_0^5 \underbrace{r}_{u'}\,\underbrace{e^{r^2}}_{v}\,dr = \left[\frac{r^2}{2} e^{r^2} - \frac{e^{r^2}}{2}\right]_0^{5} =$$

$$= 12\, e^{25} + \frac{1}{2}.$$

Den ursprungliga integr. $= I_1 \cdot I_2 = \left(\frac{\pi}{4} - \frac{1}{2}\right)\left(12\, e^{25} + \frac{1}{2}\right)$.

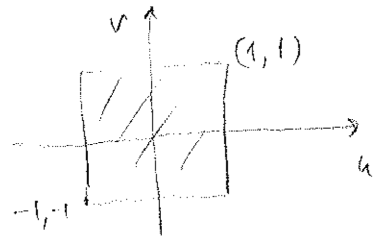
__Ex__ $\iint_D e^{-x-y}\,dx\,dy$ där $D = \{(x,y) \mid |x|+|y| \le 1\}$.



Inför nya koordinater:

$$\begin{cases} u = x+y \\ v = -x+y \end{cases} \Downarrow \quad \begin{cases} x = \frac{u-v}{2} \\ y = \frac{u+v}{2} \end{cases}$$

(x,y)	(u,v)
$(0,1)$	$(1,1)$
$(-1,0)$	$(-1,1)$
$(1,0)$	$(1,-1)$
$(0,-1)$	$(-1,-1)$



$$J = \det \frac{d(x,y)}{d(u,v)} = \det \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}$$

$$\iint_D e^{-x-y}\,dx\,dy = \int_{-1}^{1}\left(\int_{-1}^{1} e^{-u}\cdot\frac{1}{2}\,du\right)dv = \frac{1}{2}\int_{-1}^{1} dv \cdot \int_{-1}^{1} e^{-u}\,du = \frac{1}{2}\cdot 2\cdot\left[-e^{-u}\right]_{-1}^{1} =$$

$$= \underline{e - \frac{1}{e}}.$$