Review

Sets, Relations and Transition Systems

(Naïve) Set Theory

- Sets contain elements, which can be sets:
  - $A = \{a, \{d\}, \{a\}\}$ (Set A contains a, and the sets $\{d\}$ and $\{a\}$)
- Sometimes we do not want to list all the members of a set, then we can write:
  - $\{ x \mid x$ has some property$\}$, e.g. $\{ x \mid x$ is an even number$\}$, the set of all even numbers
- Order not important, number of occurrences not important:
  - $\{a, b\} = \{a, a, b\} = \{b, a\}$
  - $\{a, \{a\}\} \neq \{a\}$

(Naïve) Set Theory

- Exception: *multi-sets*, number of occurrences *are* important:
  - $\{a, b\} \neq \{a, a, b\}$

- Formally a multi-set is a function that maps elements of its domain to a range of positive integers
  - Multi-set {a, a, b} is the set {a → 2, b → 1}

- If an element x belongs to a set S, we write x ∈ S

- The empty set, ∅. I.e. for all x, the following is true: not (x∈ ∅)

Operations on Sets, Orders

- Cartesian product × :
  - $X \times Y$, the set $\{ab \mid a \in X \text{ and } b \in Y\}$
  - Example, $X=\{a, b, c\}$ $\mathrm{Y}=\{c, d, e\}$

$X \times Y = \{ac, ad, ae, bc, bd, be, cc, cd, ce\}$

- Union ∪:
  - $X \cup Y$ is the set $\{ x \mid x \in X \text{ or } x \in Y \}$
  - Example: $X=\{a, b\}$ $Y=\{a, d, e\}$
  - $X \cup Y = \{ a, b, d, e \}$

Operations on sets, orders

- Sets are ordered by the subset, $\subseteq$, relationship:

- $X \subseteq Y$ iff $a \in X$ then $a \in Y$, for all $a \in X$
  - $\{a, b\} \subseteq \{a, b, c\}$
  - $\{a, b, c\} \subseteq \{a, b, c\}$
  - $\{a, d\} \subseteq \{a, b, c\}$

Binary Relations

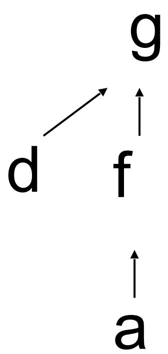
- A relation X is a subset of S×S
- Example:
  - S = { a, b, d, e }
  - X = { aa, be, ee }, sometimes { (a,a), (b,e), (e,e) }

Partial Order (poset)

- A pair, a set and a relation (S, ≤) which, for any x and y in S is:
  - Reflexive:
    - $x \leq x$
  - Anti-symmetric:
    - if $x \leq y$ and $y \leq x$ then $x = y$
  - Transitive:
    - if $x \leq y$ and $y \leq z$ then $x \leq z$

Partial Order example

- S = { a, d, f, g }
- ≤ = {aa, dd, ff, gg, af, fg, ag, dg}



Total Order

- A set and an relation (S, ≤) which, for any x and y in S is:
  - Reflexive:
    - $x \leq x$
  - Anti-symmetric:
    - if $x \leq y$ and $y \leq x$ then $x = y$
  - Transitive:
    - if $x \leq y$ and $y \leq z$ then $x \leq z$
  - Dichotomy:
    - either $x \leq y$ or $y \leq x$

Total order example

- S = { a, d, f, g }
- ≤ = {aa, dd, ff, gg, af, fg, ag, dg, dg, df, da}

$$\begin{tikzcd} g \\ f \arrow[u] \\ a \arrow[u] \\ d \arrow[u] \end{tikzcd}$$

State Transition System (informal)

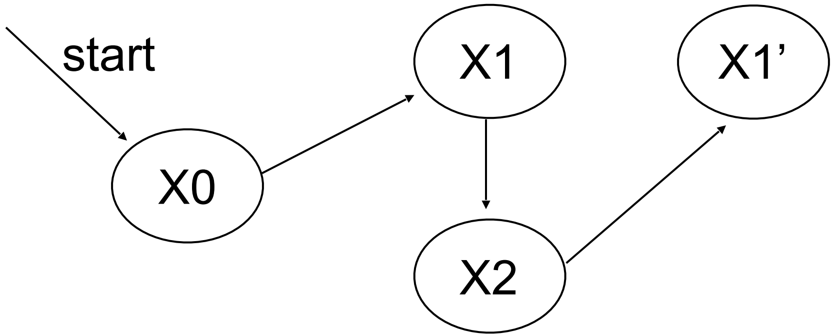
- A state transition system consists of
  - A set of states
  - Rule for which state to go to from each state (transition function/binary relation)
  - The set of starting states (*initial states*)

State transition system - example

- Example algorithm:

```
X:=0;
while (X<2) do
    X = X + 1;
end
X:=1
```

Using graphs:



- Formally:
  - States {X0, X1, X2, X1'}
  - Transitions function {X0→X1, X1→X2, X2→X1'}
  - Initial states {X0}

Distributed Algorithms

- Sequential (centralized) algorithms are modeled as a function from input to output
- Distributed systems and algorithms are systems that runs forever and interacts with its environments
- How do we make a useful abstract model of this?
- These are modeled as transition systems

Transition Systems

- Transition system is a triple TS = $(S, \rightarrow, I)$
  - S a set of states
  - $\rightarrow$ is binary transition relation on S
  - $I \subseteq S$, set of initial states
- Transition relation $\rightarrow$ is a subset of $S \times S$
  - $(s_1, s_2) \in \rightarrow$ is written $s_1 \rightarrow s_2$
- Execution of TS
  - Is a maximal sequence $E = (s_0, s_1, s_2, \ldots)$ where $s_0 \in I$, for all $i \geq 0$, $s_i \rightarrow s_{i+1}$

Transition Systems

- Terminal state
  - a state s_f for which there is no x such that $s_f \rightarrow x$
- Maximal sequence
  - $E = (s_0, s_1, s_2, \ldots)$ is maximal if it is infinite or ends in a terminal state
- State y is reachable from x, if there is a sequence $E = (x = s_0, s_1, s_2, \ldots, s_k = y)$ with $s_i \rightarrow s_{i+1}$ for all $0 \leq i \leq k$
- State y is reachable if it is reachable from an initial state$s_i \rightarrow s_{i+1}$

Labeled Transition Systems

- Labeled Transition system is a 4-tuple TS = (S, A, →, I)
  - S a set of states
  - A is a set of actions
  - $a\rightarrow$ is tertiary transition relation takes a state, an action, moves to new state
  - $I \subseteq S$, set of initial states
- Transition relation $a\rightarrow$ is a subset of S × A × S
  - $(s_1, a, s_2) \in \rightarrow$ is written $(s_1, a) \rightarrow s_2$
- Execution of TS
  - Is a maximal sequence $E = (s_0, a_0, s_1, a_1, s_2, \ldots)$
    where $s_0 \in I$, for all $i \geq 0$, $(s_i, a_i) \rightarrow s_{i+1}$

Transition Systems

- Terminal state
  - a state s_f for which there is no x such that $(s_f , a) \rightarrow x$
- Maximal sequence
  - $E = (s_0, a_0, s_1, a_1, s_2, \ldots)$ is maximal if it is infinite or ends in a terminal state
- State y is reachable from x, if there is a sequence $E = (x = s_0, a_0, s_1, a_1, s_2, \ldots, s_k = y)$ with $(s_i, a_i) \rightarrow s_{i+1}$ for all $0 \leq i \leq k$
- State y is reachable if it is reachable from an initial state$s_i \rightarrow s_{i+1}$