

Lappskrivning 2: Lösningsförslag

Måndag 25 april 2016 8:15-9:45

Differential- och integralkalkyl II, del 2, SF1603, Flervariabelanalys

Inga hjälpmedel är tillåtna.

Max: 12 poäng

Version A

1. (4 poäng) Beräkna volymen av kroppen $K \subset \mathbb{R}^3$ som begränsas av de tre planen

$$y = 0, \quad z = 0, \quad z = 3 - x + y,$$

samt av den paraboliska cylindern $y = 1 - x^2$.

Lösning: Låt $D \subset \mathbb{R}^2$ vara projektionen av K på xy -planet. Då D begränsas av $y = 0$ och $y = 1 - x^2$ har vi

$$D = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1, 0 \leq y \leq 1 - x^2\}.$$

Kroppen K består av volymen över D som begränsas uppåt av planet $z = 3 - x + y$ och nedåt av planet $z = 0$ (notera att $3 - x + y \geq 0$ för $(x, y) \in D$). Vi finner

$$\begin{aligned} \text{Volym}(K) &= \iint_D (3 - x + y - 0) dx dy \\ &= \int_{-1}^1 \int_0^{1-x^2} (3 - x + y) dy dx \\ &= \int_{-1}^1 \left[(3 - x)y + \frac{y^2}{2} \right]_{y=0}^{1-x^2} dx \\ &= \int_{-1}^1 \left((3 - x)(1 - x^2) + \frac{(1 - x^2)^2}{2} \right) dx \\ &= \int_{-1}^1 \left(\frac{7}{2} - x - 4x^2 + x^3 + \frac{x^4}{2} \right) dx \\ &= 2 \int_0^1 \left(\frac{7}{2} - 4x^2 + \frac{x^4}{2} \right) dx \\ &= 2 \left[\frac{7x}{2} - 4\frac{x^3}{3} + \frac{x^5}{10} \right]_{x=0}^1 \\ &= 2 \left(\frac{7}{2} - \frac{4}{3} + \frac{1}{10} \right) \\ &= \frac{68}{15}. \end{aligned}$$

Svar: $\text{Volym}(K) = \frac{68}{15}$.

2. (4 poäng) Beräkna $\iint_Y f(x, y, z) dS$ där $f(x, y, z) = y^2 \sqrt{x}$ och ytan Y har parameterframställningen

$$\mathbf{r}(u, v) = (v^2, u, u + v), \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 1.$$

Lösning: Integralen ges av

$$\iint_Y f(x, y, z) dS = \int_0^1 \int_0^1 f(\mathbf{r}(u, v)) |\mathbf{r}'_u \times \mathbf{r}'_v| dv du.$$

Eftersom

$$\mathbf{r}'_u \times \mathbf{r}'_v = (0, 1, 1) \times (2v, 0, 1) = (1, 2v, -2v),$$

får vi

$$|\mathbf{r}'_u \times \mathbf{r}'_v| = \sqrt{1 + (2v)^2 + (-2v)^2} = \sqrt{1 + 8v^2}.$$

Det följer att

$$\begin{aligned} \iint_Y f(x, y, z) dS &= \int_0^1 \int_0^1 u^2 v \sqrt{1 + 8v^2} dv du \\ &= \left(\int_0^1 u^2 du \right) \left(\int_0^1 v \sqrt{1 + 8v^2} dv \right) \\ &= \frac{1}{3} \left[\frac{(1 + 8v^2)^{3/2}}{24} \right]_0^1 \\ &= \frac{1}{3} \cdot \frac{9^{3/2}}{24} = \frac{1}{3} \cdot \frac{3^3}{24} = \frac{13}{36}. \end{aligned}$$

Svar: $\iint_Y f(x, y, z) dS = \frac{13}{36}.$

3. (4 poäng) Bestäm ett uttryck för ekvationen

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 4 \frac{\partial^4 u}{\partial z^4} = 0$$

i cylindriska koordinater.

Lösning: De cylindriska koordinaterna (r, φ, z) uppfyller

$$x = r \cos \varphi, \quad y = r \sin \varphi, \quad z = z.$$

Alltså är

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \cos \varphi & -r \sin \varphi \\ \sin \varphi & r \cos \varphi \end{pmatrix},$$

vilket ger

$$\begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix}^{-1} = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\frac{1}{r} \sin \varphi & \frac{1}{r} \cos \varphi \end{pmatrix}.$$

Det följer att

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial x} \frac{\partial}{\partial \varphi} = \cos \varphi \frac{\partial}{\partial r} - \frac{1}{r} \sin \varphi \frac{\partial}{\partial \varphi}$$

och

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y} \frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial y} \frac{\partial}{\partial \varphi} = \sin \varphi \frac{\partial}{\partial r} + \frac{1}{r} \cos \varphi \frac{\partial}{\partial \varphi}.$$

Detta ger

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \left(\cos \varphi \frac{\partial}{\partial r} - \frac{1}{r} \sin \varphi \frac{\partial}{\partial \varphi} \right) \left(\cos \varphi \frac{\partial}{\partial r} - \frac{1}{r} \sin \varphi \frac{\partial}{\partial \varphi} \right) u \\ &\quad + \left(\sin \varphi \frac{\partial}{\partial r} + \frac{1}{r} \cos \varphi \frac{\partial}{\partial \varphi} \right) \left(\sin \varphi \frac{\partial}{\partial r} + \frac{1}{r} \cos \varphi \frac{\partial}{\partial \varphi} \right) u \\ &= \left(\cos^2 \varphi \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \cos \varphi \sin \varphi \frac{\partial}{\partial \varphi} - \frac{1}{r} \cos \varphi \sin \varphi \frac{\partial^2}{\partial r \partial \varphi} \right. \\ &\quad \left. + \frac{1}{r} \sin^2 \varphi \frac{\partial}{\partial r} - \frac{1}{r} \sin \varphi \cos \varphi \frac{\partial^2}{\partial r \partial \varphi} + \frac{1}{r^2} \sin \varphi \cos \varphi \frac{\partial}{\partial \varphi} + \frac{1}{r^2} \sin^2 \varphi \frac{\partial^2}{\partial \varphi^2} \right) u \\ &\quad + \left(\sin^2 \varphi \frac{\partial^2}{\partial r^2} - \frac{1}{r^2} \sin \varphi \cos \varphi \frac{\partial}{\partial \varphi} + \frac{1}{r} \sin \varphi \cos \varphi \frac{\partial^2}{\partial r \partial \varphi} \right. \\ &\quad \left. + \frac{1}{r} \cos^2 \varphi \frac{\partial}{\partial r} + \frac{1}{r} \cos \varphi \sin \varphi \frac{\partial^2}{\partial r \partial \varphi} - \frac{1}{r^2} \cos \varphi \sin \varphi \frac{\partial}{\partial \varphi} + \frac{1}{r^2} \cos^2 \varphi \frac{\partial^2}{\partial \varphi^2} \right) u. \end{aligned}$$

Förenkling ger

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right) u.$$

Svar: Ekvationen ges i cylindriska koordinater av

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} + 4 \frac{\partial^4 u}{\partial z^4} = 0.$$