Differential Geometry
Lecture Notes

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Motivation

The central objects of this lecture are manifolds which are some kind of curved objects. Roughly speaking, one can describe manifolds as objects which locally look like Euclidean space but in generally not globally. One usually visualizes submanifolds as surfaces in the three-dimensional Euclidean space, for example the sphere

$$\mathbb{S}^2 = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 0\right\}$$

or the torus

$$\mathbb{T}^2 = \left\{(x, y, z) \in \mathbb{R}^3 : \sqrt{x^2 + y^2 - 2} + z^2 - 1 = 0\right\}.$$

In contrast, objects such as the union of the coordinate axes

$$\left\{(x, y, z) \in \mathbb{R}^3 : xyz = 0\right\}$$

or the union of the z-axix and the orthogonal plane

$$\left\{(x, y, z) \in \mathbb{R}^3 : xz = yz = 0\right\}$$

are not considered as manifolds. They have points whose neighborhoods certainly do not look like an Euclidean space. There is a well-developed theory for surfaces of $\mathbb{R}^3$ which can be generalized to a theory of n-dimensional submanifolds of $\mathbb{R}^m$. In this theory, submanifolds can be for example defined as zero sets of suitable functions. This theory can be understood with a solid background in multi-dimensional analysis.

For our purposes we need to define and understand manifolds as independent objects which do not nessecarily lie in a surrounding space $\mathbb{R}^m$. A central motivation of this comes from general relativity and astrophysics, because the universe is a curved object which however is not located in a surrounding space. In an abstract and intrinsic definition of manifolds, we aim to describe the manifold by charts, which are bijective maps from parts of the manifold to parts of Euclidean space. A collection of charts whose domains cover the whole manifold forms an atlas of the manifold provided that the transition between the charts behave well. Thinking of the earth, we would try to describe its surface just by studying the charts in an ordinary atlas. In order to make these definitions precise, we need addition to multi-dimensional analysis also some solid background in topology.

As it turns out, such an atlas is all we need to describe the geometry and topology of the manifold as well as the physics that happens on it. With the additional structure of a semi-Riemannian metrics, it allows us to define and describe length of curves, distances between points, volumes of subsets, curvature, motion of particles, distribution of heat and electromagnetic waves and many other things.

The theory of manifolds has many applications in other fields. Within mathematics, manifold theory is very relevant in topology. Moreover, solution sets (for example of differential equations) need sometimes to be regarded as manifolds. Outside of mathematics, the main application is certainly theoretical physics. General relativity, particle physics, and especially more modern theories like string theory require a lot of knowledge in differential geometry. But also outside mathematics and physics, there are surprisingly many applications, for example in biology (cell membrane structures), geology (structure description), engineering (digital signal processing), computer science (digital analysis of shapes), medicine (medical image analysis) and architecture (statics, design).

Chapter 1

Foundations of Manifold theory

1.1 Topology

Definition 1.1.1: Topological space

Let X be a set and $\mathscr{O}$ a collection of subsets of X. Then the pair $(X, \mathscr{O})$ is called a **topological space**, if $\mathscr{O}$ satisfies the following:

(i) $\varnothing, X \in \mathscr{O}$;

(ii) Given an index set I, and open sets $U_i \in \mathscr{O}$ (for $i \in I$), we have $\bigcup_{i\in I} U_i \in \mathscr{O}$.

(iii) Given $U_1, U_2 \in \mathscr{O}$, we have $U_1 \cap U_2 \in \mathscr{O}$.

The collection $\mathscr{O}$ is a topology and its elements are called **open sets**. Given $x \in X$ and $U \in \mathscr{O}$ with $x \in U$, we say U is an **open neighborhood** (or briefly a **neighborhood**) of x.

By induction, (iii) holds for any finite intersection of open sets.

Example 1.1.2. Consider $X = \mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $r > 0$, we define

$$B_r(x) = \{y \in \mathbb{R}^n : \|x - y\| < r\}.$$

We call $B_r(x)$ an **open ball centered at** x **with radius** r. We can equip $\mathbb{R}^n$ with different topologies:

(i) The **Euclidean topology** $\mathscr{O}_e$ generated by the basis

$$\mathcal{B} = \{B_r(x) : x \in \mathbb{R}^n, r > 0\}$$

of all open balls in $\mathbb{R}^n$. (See the remark following Defn. 1.1.3)

(ii) The **coarse topology** $\mathscr{O}_c := \{\varnothing, \mathbb{R}^n\}$

(iii) The **trivial topology** $\mathscr{O}_t := \mathcal{P}(\mathbb{R}^n) = \{U : U \subset \mathbb{R}^n\}$.

Definition 1.1.3: Basis

Let $(X, \mathscr{O})$ be a topological space. A collection of sets $\mathcal{B} \subset \mathscr{O}$ is called a **basis** of the topology if every $U \in \mathscr{O}$ can be written as a union of elements in $\mathcal{B}$. In other words, given $U \in \mathscr{O}$, we can find open sets $B_i \in \mathcal{B}$ such that $U = \bigcup_{i\in I} B_i$.

Remark. Note that every topology has a basis: take $\mathcal{B} = \mathscr{O}$. We have the alternative characterization of a basis: A **basis** is a collection $\mathcal{B}$ of subsets of X satisfying

(i) $\bigcup_{B\in\mathcal{B}} B = X$ (or equivalently, given any $x \in X$, there exists $B \in \mathcal{B}$ with $x \in B$),

(ii) Given any $B_1, B_2 \in \mathcal{B}$, and any $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that

$$x \in B_3 \subset B_1 \cap B_2.$$

In some sense, basis sets are a "generalization of open balls". Given a basis $\mathcal{B}$ of X, we may form a topology $\mathcal{O}$ on X as follows: define a subset $U \subset X$ to be open if and only if for all $x \in U$, we can find $B \in \mathcal{B}$ such that $x \in B \subset U$.

This topology $\mathcal{O}$ is called the **topology generated by** $\mathcal{B}$. It can be verified that this is indeed a topology, and that $\mathcal{B}$ is a basis for $\mathcal{O}$ in the sense of Defn. 1.1.3.

We are interested in specific kinds of topological spaces.

Definition 1.1.4: Hausdorff & Second Countable

A topological space $(X, \mathcal{O})$ is called

(i) **Hausdorff** if for any two distinct points $x, y \in X$, we may find disjoint open sets U and V such that $x \in U$ and $y \in V$.

(ii) **Second Countable** if $\mathcal{O}$ has a basis $\mathcal{B}$ that is second countable.

Remark. In some sense, the Hausdorff-condition ensures there are "not too few" open sets, while the second countable condition ensures there are "not too many" open sets. The addition of these conditions also make topological spaces more similar to $\mathbb{R}^n$ under the euclidean topology. To be more precise, it can be shown the following statements hold in the corresponding spaces. (See the definitions in the remainder of this section.)

Hausdorff: (i) Compact sets are closed.

(ii) Limits of convergent sequences are unique.

(iii) One point sets are closed.

Second Countable: (i) Every open set may be written as a countable union of open sets.

(ii) Every open cover of X has a countable subcover.

(iii) X has a countable dense subset. That is, X has a countable subset whose closure is equal to X.

Example 1.1.5. Let us consider the different topologies on $\mathbb{R}^n$ given in Example 1.1.2.

(i) The Euclidean topology is Hausdorff and second countable. Given $x \neq y$, we can pick

$$r < \frac{1}{2} \|x - y\| .$$

The triangle inequality shows $B_r(x)$ and $B_r(y)$ are disjoint neighborhoods of x and y. Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, a countable basis is given by

$$\mathcal{B}_e = \{B_r(x) : x \in \mathbb{Q}^n, r \in \mathbb{Q} \cap (0, \infty)\} .$$

(ii) The coarse topology is second countable but not Hausdorff. (Too few open sets.)

(iii) The trivial topology is Hausdorff but not second countable. (Too many open sets.)

Definition 1.1.6: Constructions of New Topological Spaces

Let $(X, \mathscr{O}_X)$ be a topological space.

(i) If $Y \subset X$, then the **subspace topology** on Y is defined by

$$\mathscr{O}_Y := \{U \cap Y : U \in \mathscr{O}_X\} .$$

(ii) If $f : X \to Y$ is a map, we may define the **quotient topology** on Y by

$$\mathscr{O}_Y := \left\{V \subset Y : f^{-1}(V) \in \mathscr{O}_X\right\} .$$

This construction is particularly used when $f : X \to X/\sim$ is the projection map of an equivlence relation $\sim$.

(iii) If $(Y, \mathscr{O}_Y)$ is a topological space, we may define the **product topology** on $X \times Y$ to be the topology *generated by* the basis

$$\mathscr{B}_{X\times Y} := \{U \times V : U \in \mathscr{O}_X, V \in \mathscr{O}_Y\} .$$

In other words, the open sets of the product topology consist of all sets of the form $\bigcup_{i\in I}(U_i \times V_i)$, where I is an index set, $U_i \in \mathscr{O}_X$, and $V_i \in \mathscr{O}_Y$.

Notation 1.1.7

To avoid cumbersome notation, we will denote topological space just by X instead of $(X, \mathscr{O})$ whenever the topology is clear from the context.

Definition 1.1.8: Continuity & Homeomorphisms

Let X, Y be topological spaces and $f : X \to Y$. Then

(i) f is **continuous** if for every open $V \subset Y$, its preimage $f^{-1}(V) \subset X$ is open.

(ii) f is a **homeomorphism** if f is bijective and continuous, and the inverse map f^{-1} is also continuous.

(iii) X and Y are **homeomorphic** if there exists a homeomorphism between them.

Note that the quotient topology is constructed precisely so that $f : X \to Y$ is continuous.

Definition 1.1.9: A few more topological definitions

Let X be a topological space. Then

(i) X is **compact** if every open cover of X admits a finite subcover. That is, for any collection $\{U_i\}$ of open sets with $\bigcup_i U_i = M$, there exist $i_1, \dots, i_n$ so that $M = U_{i_1} \cup \dots \cup U_{i_n}$.

(ii) X is **connected** if X cannot be written as a union of disjoint open sets.

(iii) A **path** c in X is a continuous map $c : [0, 1] \to X$, where $[0, 1]$ is viewed as a subspace of $\mathbb{R}$ under the Euclidean topology. We say that c is a path from x to y if $c(0) = x$ and $c(1) = y$.

(iv) X is **path-connected** if for any points $x, y \in X$, there exists a path c from x to y.

Exercise 1.1.10. Let X, Y be topological spaes and $Z \subset X$ an arbitrary subset. Show that Z (equipped with the subspace topology) is alos Hausdorff and second countable.

Exercise 1.1.11. Let X, Y be topological spaces and $f : X \to Y$ be a continuous map. Show that if X is compact/connected/path-connected, then $f(X)$ is also compact/connected/path-connected.

1.2 Smooth Manifolds

From now on $\mathbb{R}^n$ is always equipped with the Euclidean topology and subsets of $\mathbb{R}^n$ are equipped with the corresponding subspace topology, unless stated otherwise. For open subsets $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$, we call a map $\varphi : U \to V$ a diffeomorphism, if it is infinitely often differentiable (we say C^∞, or smooth), bijective, and the inverse is also infinitely often differentiable.

Definition 1.2.1: Manifolds & Charts

A topological space M is a **manifold** of dimension n if the following hold:

(i) M is second countable.

(ii) M is Hausdorff.

(iii) For every $p \in M$, there exists an open neighborhood U of p such that $U \cong \mathbb{R}^n$.

Let M be a manifold and $p \in M$. Let $\varphi : U \cong \mathbb{R}^n$ be a homeomorphism. The pair (U, φ) is called a **chart** of M.

Two charts (U_1, φ_1), (U_2, φ_2) are C^∞**-compatible** if $U_1 \cap U_2 = \varnothing$ or the **transition map**

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is a diffeomorphism.

Any transition map is a priori just a homeomorphism. Thus, C^∞-compatibility demands much more.

Notation 1.2.2

(i) If we want to emphasize the dimension of a manifold, we sometimes w rite M^n.

(ii) If (U, φ) is a chart, we get component functions $x^i : U \to \mathbb{R}$ such that

$$x = \varphi(p) = \left(x^1(p), \dots, x^n(p)\right).$$

If we want to emphasize the component functions, we sometimes write $(U, \varphi = (x^1, \dots, x^n))$.

Definition 1.2.3: Atlas

Let M be a topological manifold. A set $\mathscr{A} = \{(U_i, \varphi_i) : i \in I\}$ of charts of M is called a C^∞**-atlas**(or **smooth atlas** of M if

(i) the charts are pairwise C^∞-compatible,

(ii) $M = \bigcup_i U_i$

Two C^∞-atlases $\mathscr{A}_1$ and $\mathscr{A}_2$ of a topological manifold M are **equivalent**, written $\mathscr{A}_1 \sim \mathscr{A}_2$ if any two charts in $\mathscr{A}_1$ and $\mathscr{A}_2$ are C^∞-compatible. In other words,

$$\mathscr{A}_1 \sim \mathscr{A}_2 \iff \mathscr{A}_1 \cup \mathscr{A}_2 \text{ is a } C^\infty \text{ atlas.}$$

We will verify this forms an equivalence relation.

Proof. **Reflexive:** Follows from Defn. 1.2.3.

Symmetric: Follows from the definition of C^∞-compatibility.

Transitive: Suppose $\mathscr{A}_1 \sim \mathscr{A}_2$ and $\mathscr{A}_2 \sim \mathscr{A}_3$. Let $(U_1, \varphi_1) \in \mathscr{A}_1$ and $(U_3, \varphi_3) \in \mathscr{A}_3$. Suppose $U_1 \cap U_3 \neq \varnothing$. Recall $\varphi_3 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_3) \to \varphi_3(U_1 \cap U_3)$ is a homeomorphism. We will show $\varphi_3 \circ \varphi_1^{-1}$ is a diffeomorphism. It will thus suffice to show that for all $x \in \varphi_1(U_1 \cap U_3)$, there is an open neighborhood V of x such that $\varphi_3 \circ \varphi_1^{-1}|_V : V \to \varphi_3 \circ \varphi_1^{-1}(V)$ is a diffeomorphism.

Let $x \in \varphi_1(U_1 \cap U_3)$ and set $p = \varphi_1^{-1}(x) \in U_1 \cap U_3$. Since $\mathscr{A}_2$ is an atlas, there exists a chart $(U_2, \varphi_2) \in \mathscr{A}_2$ such that $p \in U_2$. Set

$$U = U_1 \cap U_2 \cap U_3$$

and define $V = \varphi_1(U)$. Since $\mathscr{A}_2 \sim \mathscr{A}_3$, we see that $(\varphi_3 \circ \varphi_2^{-1})|_{\varphi_2 \circ \varphi_1^{-1}(V)}$ is a diffeomorphism. Since $\mathscr{A}_1 \sim \mathscr{A}_2$, we see that $\varphi_2 \circ \varphi_1^{-1}|_V$ is a diffeomorphsim. Hence,

$$\varphi_3 \circ \varphi_1^{-1}|_V = (\varphi_3 \circ \varphi_2^{-1})|_{\varphi_2 \circ \varphi_1^{-1}(V)} \circ \varphi_2 \circ \varphi_1^{-1}|_V$$

is a diffeomorphism. This shows $\mathscr{A}_1 \sim \mathscr{A}_3$. □

Usually, it is straightforward to see that transition maps are diffeomorphisms. However, checking the Hausdorff and second countable conditions and that the charts are homeomorphisms is quite tedious in general. We will discuss some criterions which will give us these conditions for free.

Remark 1.2.4. If M is a priori just a set, we can turn it into a smooth manifold as follows: Suppose that there exist an index set I and for each $i \in I$ a subset $U_i \subset M$, an open subset $V_i \subset \mathbb{R}^n$ and a bijective map $\varphi_i : U_i \to V_i$ such that

(i) $\bigcup_{i \in I} U_i = M$

(ii) $\varphi_i(U_i \cap U_j) \subset \mathbb{R}^n$ is open for all pairs $i, j \in I$

(iii) All transition maps $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are continuous

Now we define a subset $U \subset M$ to be open if and only if $\varphi_i(U_i \cap U)$ is open for all $i \in I$. The obtained topology $\mathscr{O}$ is unique topology in M for which all $U_i \subset M$ are open and the maps $\varphi_i : U_i \to V_i$ are homeomorphisms, i.e. the unique topology which satisfies (ii) in Definition 1.2.1 of a topological manifold. If additionally

(iv) I is countable, then $\mathscr{O}$ is second countable

(v) If for all $p, q \in M$ $p \neq q$, there exists U_i such that $p, q \in U_i$ or disjoint sets U_i, U_j such that $p \in U_i$ or $q \in U_j$, then the topology is Hausdorff.

Thus, if (i)-(v) hold, $(M, \mathscr{O})$ is a topological manifold. Finally if

(vi) All transition maps $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are C^∞, $\mathscr{A} := \{(U_i, \varphi_i) : i \in I\}$ is a C^∞-atlas on M, so that $(M, [\mathscr{A}])$ is a C^∞-manifold.

In view of this remark, we can also forget about any potential a-priori topology on M since the maps φ_i do not leave much choice for the topology on M.

Example 1.2.5. We start with some examples one usually has in mind when thinking about manifolds.

(i) The atlas $\mathscr{A} = \{(\mathbb{R}^n, \mathrm{id})\}$, induces a smooth structure on $\mathbb{R}^n$, called the **canonical smooth structure** on $\mathbb{R}^n$. All the points (i)-(vi) in Remark 1.2.4 are obvious. Unless stated otherwise, $\mathbb{R}^n$ will from now on always be equipped with this smooth structure.

(ii) We define the following maps on $\mathbb{S}^1 := \{x \in \mathbb{R}^2 : \|x\| = 1\}$:

$$\begin{aligned}
\varphi_1 : U_1 &:= \{x \in \mathbb{S}^1 : x^1 > 0\} \to (-1,1), \qquad && \varphi_1(x) = x^2, \\
\varphi_2 : U_2 &:= \{x \in \mathbb{S}^1 : x^2 > 0\} \to (-1,1), && \varphi_2(x) = x^1 \\
\varphi_3 : U_3 &:= \{x \in \mathbb{S}^1 : x^1 < 0\} \to (-1,1), && \varphi_3(x) = x^2 \\
\varphi_4 : U_4 &:= \{x \in \mathbb{S}^1 : x^2 < 0\} \to (-1,1), && \varphi_4(x) = x^1
\end{aligned}$$

It is obvious that $M = \bigcup_{i=1}^4 U_i$. The sets $\varphi_i\left(U_i \cap U_j\right)$ are all of the form $(-1,0)$ or $(0,1)$, hence open and the transition maps are all of the form $t \mapsto \pm\sqrt{1-t^2}$, thus smooth. Finally (iv)-(v) in Remark 1.2.4 are also clear.
In this example, it is also pretty straightforward to see that the topology induced by the maps φ_i (according to 1.2.4) coincide with the subspace topolgy coming from $\mathbb{R}^2$.

The advantage of the abstract notion of manifold that we are using is that it allows more sophisticated constructions, as the following examples show:

(iii) Define $\mathbb{R}/\mathbb{Z} := \{[x] : x \in \mathbb{R}\}$, where $x \sim y$ if and only if $x - y \in \mathbb{Z}$. We define two (hence countably many) maps

$$\begin{aligned} {}_1 : U_1 &:= \mathbb{R}/\mathbb{Z} \setminus \{[0]\} \to (0,1), && [x] \mapsto \text{ unique representative in } (0,1) \\ \varphi_2 : U_1 &:= \mathbb{R}/\mathbb{Z} \setminus \{[1/2]\} \to (-1/2, 1/2), && [x] \mapsto \text{ unique representative in } (-1/2,1/2). \end{aligned}$$

Clearly, $M = U_1 \cup U_2$. It is straightfoward to check that

$$\begin{aligned} \varphi_1(U_1 \cap U_2) &= (0, 1/2) \cup (1/2, 1), \\ \varphi_2(U_1 \cap U_2) &= (-1/2, 0) \cup (0, 1/2) \end{aligned}$$

and thus are open. Furthermore,

$$\varphi_2 \circ \varphi_1^{-1} : x \mapsto \begin{cases} x, & \text{if } x < 1/2 \\ x - 1 & \text{if } x > 1/2, \end{cases}$$

$$\varphi_1 \circ \varphi_2^{-1} : x \mapsto \begin{cases} x + 1, & \text{if } x < 0 \\ x & \text{if } x > 0, \end{cases}$$

which are both smooth. Thus we have verified all points in Remark 1.2.4 except (v), which also holds for all pairs of points $p, q \in \mathbb{R}/\mathbb{Z}$, except the pair $p = [0], q = [1/2]$. In this case, we check it manually: Let $\varepsilon \in (0, 1/4)$ and $V_1 := \{[x] : |x| < \varepsilon\}$, $V_2 := \{[x] : |x - 1/2| < \varepsilon\}$. By the triangle inequality, these sets are disjoint. One quickly verifies that $\varphi_i(U_i \cap V_j)$ is open for $1 \le i, j \le 2$, hence V_1 and V_2 are the desired disjoint open neighborhoods of p, q,respectively.

(iv) Consider the set $\mathbb{RP}^n := (\mathbb{R}^n \setminus \{0\})/\sim$, where

$$x \sim y \iff \exists \lambda \in \mathbb{R} \setminus \{0\} \text{ such that } x = \lambda y.$$

This set can be often thought as being the set of lines through the origin. We use projective coordinates here, that is $[x^1 : \ldots : x^{n+1}] := [x]$, if $x = (x^1, \cdots x^n)$. In particular $[x^1 : \cdots : x^{n+1}] = [\lambda x^1 : \cdots : \lambda x^{n+1}]$ for any $\lambda \in \mathbb{R} \setminus \{0\}$. Now for $i \in \{1, \ldots, n+1\}$, let

$$U_i := \{[x^1 : \cdots : x^{n+1}] : x^i \neq 0\}$$

and define $\varphi_i : U_i \to \mathbb{R}^n$ by

$$\varphi_i([x^1 : \cdots : x^{n+1}]) := \left(\frac{x^1}{x^i}, \ldots \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \ldots \frac{x^{n+1}}{x^i}\right)$$

Clearly $M = \bigcup_{i=1}^{n+1} U_i$ since each point in $\mathbb{RP}^n$ has at least one nonvanishing (projective) coordinate. Moreover, the φ_i are bijective maps and

$$\varphi_i(U_i \cap U_j) = \begin{cases} \mathbb{R}^n \setminus \{x^{j-1} = 0\} & \text{if } i < j, \\ \mathbb{R}^n \setminus \{x^j = 0\} & \text{if } j < i, \end{cases}$$

hence open in all cases. For $i < j$, the transition map is

$$\begin{aligned} \varphi_j \circ \varphi_i^{-1} : \mathbb{R}^n \setminus \{x^{j-1} = 0\} &\to \mathbb{R}^n \setminus \{x^i = 0\} \\ (x^1, \ldots, x^n) &\mapsto \left(\frac{x^1}{x^{j-1}}, \ldots \frac{x^{i-1}}{x^j}, \frac{1}{x^{j-1}}, \frac{x^i}{x^{j-1}}, \ldots, \frac{x^{j-2}}{x^{j-1}}, \frac{x^j}{x^{j-1}}, \ldots \frac{x^n}{x^{j-1}}\right) \end{aligned}$$

while for $j < i$, it is

$$\begin{aligned} \varphi_j \circ \varphi_i^{-1} : \mathbb{R}^n \setminus \{x^j = 0\} &\to \mathbb{R}^n \setminus \{x^{i-1} = 0\} \\ (x^1, \ldots, x^n) &\mapsto \left(\frac{x^1}{x^j}, \ldots, \frac{x^{j-1}}{x^j}, \frac{x^{j+1}}{x^j}, \ldots \frac{x^{i-1}}{x^j} \frac{1}{x^j}, \frac{x^i}{x^j}, \ldots \frac{x^n}{x^j}\right), \end{aligned}$$

so the all the transition maps are smooth. Moreover, for any pair of points p, q, we find U_i containing both points, unless p, q have no common nonvanishing coordinates. In this case, we have to add a manual argument in the spirit of example (iii).

Finally, we discuss two constructions where we obtain a new manifold from given ones:

(v) If $(M, [\mathscr{A}])$ is a smooth manifold, any open set $U \subset M$ is again a C^∞-manifold in a canonical way: If $\mathscr{A} = \{(U_i, \varphi_i) : i \in I\}$ is a smooth atlas of M, $\mathscr{A}|_U = \{(U_i \cap U, \varphi|_{U_i \cap U}) : i \in I\}$ is a smooth atlas on U. From now on we equip an open subset U of a smooth manifold $(M, [\mathscr{A}])$ always with $[\mathscr{A}|_U]$, unless stated otherwise.

(iv) If $(M, [\mathscr{A}_M]), (N, [\mathscr{A}_N])$ are smooth manifolds, the product $M \times N$ can be equipped with a canonical C^∞-structure as follows: If $\mathscr{A}_M = \{(U_i, \varphi_i) : i \in I\}$ and $\mathscr{A}_N = \{(V_j, \psi_j) : j \in J\}$ are smooth atlases of M, N, respectively, then

$$\mathscr{A}_{M\times N} = \{(U_i \times V_j, \varphi_i \times \psi_j) : (i, j) \in I \times J\}$$

is a smooth atlas of $M \times N$. We call $(M \times N, [\mathscr{A}_{M\times N}])$ a **product manifold**.

Notation 1.2.6

(i) From now on, we will usually write M instead of $(M, [\mathscr{A}])$ if the smooth structure is clear from the context. If M is a smooth manifold, we will also say from now on that (U, φ) is a **chart** of M if it is C^∞-compatible with any chart in a given atlas $\mathscr{A}$ of M. The open set U is often called **coordinate neighborhood**.

(ii) A 1-dimensional manifold is called **line**, a 2-dimensional manifold **surface**.

Finally, in order to complement the above examples of manifolds, we also want to mention some spaces which are locally homeomorphic to $\mathbb{R}^n$ but not Hausdorff or not second countable.

Example 1.2.7. Let M be a set, $U_i, V_i, \varphi_i, i \in I$ as in Remark 1.2.4 and suppose (i)-(iii) in Remark 1.2.4 hold. Then with the induced topology, M is not always a topological manifold as the topology will in general not be Hausdorff or countable:

(i) Let $M = \mathbb{R} \setminus \{0\} \cup \{p_1\} \cup p_2$. For $i = 1, 2$, set $U_i = \mathbb{R} \setminus \{0\} \cup \{p_i\}$ and define the map $\varphi_i : U_i \to \mathbb{R}$ as $\varphi_i(x) = x$ for $x \in \mathbb{R} \setminus \{0\}$ and $\varphi_i(p_i) = 0$. Then, we obviously have $M = U_1 \cup U_2$, $\varphi_i(U_i \cap U_j) = \mathbb{R} \setminus \{0\}$, hence open, and the transition maps are given by the identity, hence smooth and in particular, continuous. Because we only have two charts, the topology is also second countable. However, it is not Hausdorff, because p_1 and p_2 can not be separated by open neighborhoods: For any open neighborhood V_i of p_i $i = 1, 2$, $\varphi_1(V_i)$ contains an interval of the form $(-\varepsilon, 0) \cup (0, \varepsilon)$ which means that $V_1 \cap V_2 \neq \emptyset$.

(ii) Consider $M = \mathbb{R}^2$ with the sets $U_x = \{x\}\times\mathbb{R}$, $x \in \mathbb{R}$ and the bijective maps $\varphi_x : U_x \to V_x = \mathbb{R}$, $\varphi_x((x, y)) = y$. Clearly $M = \bigcup_{x\in\mathbb{R}} U_x$ and with the induced topology on M, each U_x is an open set homeomorphic to $\mathbb{R}$. The U_x are pairwise disjoint, so there are no transition maps. Thus, (i)-(iii) in Remark 1.2.4 hold and it is also not hard to check that M is Hausdorff. However, it is not countable because M contains uncountably many pairwise disjoint open subsets.

Definition 1.2.8: Smooth Manifold

Let M be a manifold and $\mathscr{A}$ an atlas. Then the pair $(M, [\mathscr{A}])$, where $[\mathscr{A}]$ is the equivalence class of $\mathscr{A}$, is called a C^∞**-manifold** (or **smooth manifold**). We call $[\mathscr{A}]$ a C^∞**-structure** (or **smooth structure**) on M.

Remark 1.2.9. If M is a topological manifold, a C^∞-structure is uniquely determined by picking **one** C^∞-atlas $\mathscr{A}$ on M. (This is because $\mathscr{A}$ determines $[\mathscr{A}]$.)

Lemma 1.2.10

Any C^∞-structure $[\mathscr{A}]$ on a topological manifold M contains a countable atlas $\widetilde{\mathscr{A}} \in [\mathscr{A}]$.

Proof. This follows from M being second countable and $\mathscr{A}$ being an open cover. $\square$

1.3 Smooth functions

Definition 1.3.1: Smooth map

Let M, N be C^∞ manifolds. A continuous map $f : M \to N$ is called C^∞ (or **smooth**) if for every $p \in M$, there exist charts (U, φ) of M and (V, ψ) of N with $p \in U$, $f(p) \in V$ such that

$$\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \to \psi(V)$$

is smooth (in the classical sense).

We adopt the notation

$$C^\infty(M, N) := \left\{ M \xrightarrow{f} N : f \text{ is } C^\infty \right\}$$
$$C^\infty(M) := C^\infty(M, \mathbb{R}).$$

That is, $C^\infty(M, N)$ refers to the set of all smooth maps from M to N. The elements of $C^\infty(M)$ are called C^∞-functions (or **smooth functions**) on M.

Definition 1.3.2: Diffeomorphism

Let M,N be smooth manifolds and $f \in C^\infty(M, N)$. Then f is a **diffeomorphism** if f is bijective and $f^{-1} \in C^\infty(M, N)$. In this case, we say M and N are **diffeomorphic**.

Example 1.3.3. (i) Any map $f : \mathbb{R}^n \to \mathbb{R}^m$ which is smooth in the classical sense is also smooth in the manifold sense. We can take the identity charts in this case to see that the composition

$$\mathrm{id} \circ f \circ \mathrm{id}^{-1} = f$$

is smooth.

(ii) If M is a smooth manifold and (U, φ) is a chart on M, then $\varphi : U \to \varphi(U)$ is a diffeomorphism.

(iii) The manifolds $\mathbb{S}^1$ and $\mathbb{R}/\mathbb{Z}$ are diffeomorphic (exercise).

Lemma 1.3.4

Let M, N be smooth manifolds and $f : M \to N$. The following are equivalent.

(i) f is smooth.

(ii) For any chart (U, φ) of M and (V, ψ) of N, the composition

$$\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \to \psi(V). \tag{1.3.1}$$

is smooth.

Proof. **(ii) ⇒ (i):** Clear.

(i) ⇒ (ii): Let (U, φ), (V, ψ) be as in (ii). Let $p \in U \cap f^{-1}(V)$. It will suffice to show that $\psi \circ f \circ \varphi^{-1}$ is C^∞ in a neighborhood of $\varphi(p)$. Since f is smooth, there exist charts $(\widetilde{U}, \widetilde{\varphi})$ of M and $(\widetilde{V}, \widetilde{\varphi})$ in N with $p \in \widetilde{U}$ and $f(p) \in \widetilde{V}$ such that

$$\widetilde{\psi} \circ f \circ \widetilde{\varphi}^{-1} : \widetilde{\varphi}\left(\widetilde{U} \cap f^{-1}\left(\widetilde{V}\right)\right) \to \widetilde{\psi}\left(\widetilde{V}\right)$$

is smooth. Since $\widetilde{\varphi}, \widetilde{\psi}$ are homeomorphisms and smoothness in $\mathbb{R}^n$ is a local property, we may make $\widetilde{U}$ and $\widetilde{V}$ smaller so that $\widetilde{U} \subset U$ and $\widetilde{V} \subset V$. (For example, replace $\widetilde{U}$ with $\widetilde{U} \cap U$ and V with $\widetilde{V} \cap V$.) Since the transition maps are smooth, we see that on $\varphi\left(\widetilde{U} \cap f^{-1}\left(\widetilde{V}\right)\right)$, we have

$$\psi \circ f \circ \varphi^{-1} = \underbrace{\left(\psi \circ \widetilde{\psi}^{-1}\right)}_{C^\infty} \circ \underbrace{\left(\widetilde{\psi} \circ f \circ \widetilde{\varphi}^{-1}\right)}_{C^\infty} \circ \underbrace{\left(\widetilde{\varphi} \circ \varphi^{-1}\right)}_{C^\infty}.$$

This shows (1.3.1) is smooth. □

Lemma 1.3.5

The collection of smooth manifolds with smooth maps forms a category.

Proof. We must show the identity map is smooth and that the composition of two smooth maps is smooth. Given $p \in M$, pick a chart (U, φ) such that $p \in U$. Then set $(V, \psi) = (U, \varphi)$ since $p = \mathrm{id}(p)$. Observe

$$\psi \circ \mathrm{id} \circ \varphi^{-1} = \varphi \circ \varphi^{-1} = \mathrm{id}$$

so that id is a smooth map.

Next, consider smooth maps $M \xrightarrow{f} N \xrightarrow{g} P$. We must show $g \circ f$ is smooth. (U, φ) be a chart of M and (W, χ) be a chart of P. Pick $p \in U \cap (g \circ f)^{-1}(W)$. By Lemma 1.3.4, it will suffice to show

$$\chi \circ g \circ f \circ \varphi^{-1} : \varphi(U \cap (g \circ f)^{-1}(W)) \to \chi(W)$$

is smooth near $\varphi(p)$. Let (V, ψ) be a chart of N with $f(p) \in V$. Then near $\varphi(p)$, on $\varphi(U \cap f^{-1}(V \cap g^{-1}(W)))$, we have

$$\chi \circ g \circ f \circ \varphi^{-1} = \underbrace{\chi \circ g \circ \psi^{-1}}_{C^\infty} \circ \underbrace{\psi \circ f \circ \varphi^{-1}}_{C^\infty}.$$

□

Remark 1.3.6. Let M be a smooth manifold.

(i) Let $\mathrm{Diff}(M)$ denote the set of all diffeomorphisms $f : M \to M$. Under the natural composition of maps, $\langle \mathrm{Diff}(M), \circ \rangle$ is a group, the **diffeomorphism group** of M.

(ii) With the natural addition and multiplication of functions, $\langle C^\infty(M), +, \cdot \rangle$ is a commutative ring with a unit (the constant function 1).

Remark 1.3.7. Let M be a topological manifold and $\mathscr{A}_1$ and $\mathscr{A}_2$ two smooth atlases on M. If $[\mathscr{A}_1] \neq [\mathscr{A}_2]$, it may still happen that the two smooth manifolds $(M, [\mathscr{A}_1])$ and $(M, [\mathscr{A}_2])$ are diffeomorphic. In this case we call the two smooth structures $[\mathscr{A}_1]$ and $[\mathscr{A}_2]$ equivalent.

One can show that on $\mathbb{R}^n$, with $n \neq 4$, there exists up to equivalence only one smooth structure. However, $\mathbb{R}^4$ admits uncountably many inequivalent smooth structures!

Exercise 1.3.8. Prove that two atlases $\mathscr{A}_1$ and $\mathscr{A}_2$ on a topological manifold M are equivalent if and only if the identity map

$$\mathrm{id} : (M, [\mathscr{A}_1]) \to (M, [\mathscr{A}_2]), \quad x \mapsto x,$$

is a diffeomorphism with respect to the smooth structures $[\mathscr{A}_1]$ and $[\mathscr{A}_2]$.

Exercise 1.3.9. Find a smooth atlas for the unit sphere in $\mathbb{R}^3$, i.e. for

$$\mathbb{S}^2 = \left\{ x \in \mathbb{R}^3 : \|x\| = 1 \right\}.$$

Show that the charts of your atlas are C^∞-compatible and cover all of $\mathbb{S}^2$.

Exercise 1.3.10. Consider the atlas $\mathscr{A}_1 = \{(\mathbb{R}, \mathrm{id})\}$ on $\mathbb{R}$ and recall that $[\mathscr{A}_1]$ is the canonical smooth structure on $\mathbb{R}$.

(a) Find a homeomorphism φ on $\mathbb{R}$ which is not a diffeomorphism (with respect to $[\mathscr{A}_1]$).

(b) Is there a smooth structure $[\mathscr{A}_2]$ on $\mathbb{R}$ such that $\varphi : (\mathbb{R}, [\mathscr{A}_1]) \to (\mathbb{R}, [\mathscr{A}_2])$ is a diffeomorphism?

Exercise 1.3.11. Show that the manifolds $\mathbb{S}^1$ and $\mathbb{R}/\mathbb{Z}$ constructed in Example 1.2.5 are diffeomorphic, i.e. find a bijection between these two manifolds and show that it is a diffeomorphism.

Exercise 1.3.12. Let M be a smooth manifold and G a group which acts on M via diffeomorphisms, that is we have a group homomorphism $\varphi : G \to \mathrm{Diff}(M)$. Prove that if G acts strictly discontinuously on M, the quotient space M/G carries a unique smooth structure such that the projection map $\pi : M \to M/G$ is smooth.

1.4 The Tangent Space

Definition 1.4.1: Smooth Curves

A **smooth curve** on a smooth manifold M is an element $c \in C^\infty(I_\varepsilon, M)$, where $I_\varepsilon = (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. For $p \in M$, let

$$C_p := \{c \in C^\infty(I_\varepsilon, M) : c(0) = p\}$$

be the set of all curves through p.

We say that two curves $c_1, c_2 \in C_p$ are **equivalent** if there exists a chart (U, φ) with $p \in U$ such that

$$\frac{d}{dt}(\varphi \circ c_1)\Big|_{t=0} = \frac{d}{dt}(\varphi \circ c_2)\Big|_{t=0}. \tag{1.4.1}$$

Lemma 1.4.2

(i) If (1.4.1) holds for one chart, it holds for any chart.

(ii) $\sim$ is an equivalence relation on C_p.

Proof. (i) Let (V, ψ) be another chart with $p \in V$. The chain rule shows that for $i = 1, 2$,

$$\frac{d}{dt}(\psi \circ c_i)\Big|_{t=0} = \frac{d}{dt}(\psi \circ \varphi^{-1} \circ \varphi \circ c_i)\Big|_{t=0} = D(\psi \circ \varphi^{-1})\big|_{\varphi(x)} \frac{d}{dt}(\varphi \circ c_i)\Big|_{t=0}. \tag{1.4.2}$$

Then (1.4.1) implies

$$\begin{aligned} \frac{d}{dt}(\psi \circ c_1)\Big|_{t=0} &= D(\psi \circ \varphi^{-1})\big|_{\varphi(x)} \frac{d}{dt}(\varphi \circ c_1)\Big|_{t=0} \\ &= D(\psi \circ \varphi^{-1})\big|_{\varphi(x)} \frac{d}{dt}(\varphi \circ c_2)\Big|_{t=0} \\ &= \frac{d}{dt}(\psi \circ c_2)\Big|_{t=0}. \end{aligned}$$

(ii) Symmetry and reflexivity are clear. We show transitivity. Suppose $c_1 \sim c_2 \sim c_3$ and let (U, φ) be any chart of M with $p \in U$. Then by (i),

$$\frac{d}{dt}(\varphi \circ c_1)\Big|_{t=0} = \frac{d}{dt}(\varphi \circ c_2)\Big|_{t=0} = \frac{d}{dt}(\varphi \circ c_3)\Big|_{t=0}.$$

□

Definition 1.4.3: Tangent Vectors & Tangent Space

Let M be a smooth manifold and $p \in M$. A **tangent vector** v to M at p is an equivalence class of C^∞ curves $c \in C_p$. That is, $v = [c]_p$ for some $c \in C_p$. The **tangent space** to M at p, denoted T_pM, is defined to be the set of all tangent vectors to M at p. That is,

$$T_pM = C_p/\sim .$$

Proposition 1.4.4

Let M^n be a C^∞ manifold, $p \in M$, and (U, φ) be a chart of M with $p \in M$. Then the map

$$\begin{aligned} T_p\varphi : T_pM &\to \mathbb{R}^n \\ [c]_p &\mapsto \frac{d}{dt}(\varphi \circ c)\Big|_{t=0} \end{aligned}$$

is well-defined and bijective. By defining

$$\begin{aligned} v +_\varphi w &:= (T_p\varphi)^{-1}\big(T_p\varphi(v) + T_p\varphi(w)\big) \\ \alpha \cdot_\varphi w &:= (T_p\varphi)^{-1}\big(\alpha T_p\varphi(w)\big), \end{aligned}$$

T_pM becomes a real vector space. This structure is independent of the choice of chart. This implies T_pM can be equipped with a **canonical** vector space structure and $\dim T_p(M) = \dim(M) = n$.

Proof. (1.4.1) and Lemma 1.4.2 show $T_p\varphi$ is well-defined and injective. We will show $T_p\varphi$ is surjective. Let $w \in \mathbb{R}^n$; define $c \in C_p$ by

$$c(t) := \varphi^{-1}\big(\varphi(p) + tw\big).$$

Note $c(t) \in \varphi(U)$ when t is small. Observe that

$$T_p\varphi[c]_p = \frac{d}{dt}(\varphi \circ c)\Big|_{t=0} = \frac{d}{dt}(\varphi(p) + tw)\Big|_{t=0} = w.$$

The definitions of addition and scalar multiplication force $T_p\varphi$ to become a linear map. That is,

$$\begin{aligned} T_p\varphi(v +_\varphi w) &= T_p\varphi\Big((T_p\varphi)^{-1}\big(T_p\varphi(v) + T_p\varphi(w)\big)\Big) = T_p\varphi(v) + T_p\varphi(w) \\ T_p\varphi\left(\alpha \cdot_\varphi w\right) &= T_p\varphi\Big((T_p\varphi)^{-1}\big(\alpha T_p\varphi(w)\big)\Big) = \alpha T_p\varphi(w). \end{aligned}$$

Thus, the vector space structure of $\mathbb{R}^n$ is imported into T_pM in a way such that $\mathbb{R}^n \cong T_pM$. Let (V, ψ) be another chart of M with $p \in V$. Then by (1.4.2),

$$\begin{aligned} T_p\psi[c]_p &= \frac{d}{dt}(\psi \circ c)\big|_{t=0} \\ &= D(\psi \circ \varphi^{-1})\big|_{\varphi(p)} \frac{d}{dt}(\varphi \circ c)\big|_{t=0} \\ &= D(\psi \circ \varphi^{-1})\big|_{\varphi(p)} \circ T_p\varphi[c]_p \end{aligned}$$

so that

$$T_p\psi[c]_p = D(\psi \circ \varphi^{-1})\big|_{\varphi(p)} \circ T_p\varphi[c]_p. \tag{1.4.3}$$

Note (1.4.3) is equivalent to

$$T_p\psi \circ (T_p\varphi)^{-1} = D(\psi \circ \varphi^{-1})\big|_{\varphi(p)},$$

which implies

$$\begin{aligned} (T_p\varphi)^{-1} &= (T_p\psi)^{-1} \circ T_p\psi \circ (T_p\varphi)^{-1} \\ &= (T_p\psi)^{-1} \circ D(\psi \circ \varphi^{-1})\big|_{\varphi(p)}. \end{aligned}$$

We have thus shown

$$(T_p\varphi)^{-1} = (T_p\psi)^{-1} \circ D(\psi \circ \varphi^{-1})\big|_{\varphi(p)}. \tag{1.4.4}$$

Now, let $v, w \in T_pM$. Then by linearity of $D(\psi \circ \varphi^{-1})|_{\varphi(p)}$,

$$\begin{aligned} v +_\psi w &= (T_p\psi)^{-1}\big(T_p\psi(v) + T_p\psi(w)\big) \\ &= (T_p\psi)^{-1}\big(D(\psi \circ \varphi^{-1})\big|_{\varphi(p)} \circ T_p\varphi(v) + D(\psi \circ \varphi^{-1})\big|_{\varphi(p)} \circ T_p\varphi(w)\big), && \text{by (1.4.3)} \\ &= (T_p\psi)^{-1} \circ D(\psi \circ \varphi^{-1})\big|_{\varphi(p)}\big(T_p\varphi(v) + T_p\varphi(w)\big) \\ &= (T_p\varphi)^{-1}\big(T_p\varphi(v) + T_p\varphi(w)\big)\Big), && \text{by (1.4.4)} \\ &= v +_\varphi w. \end{aligned}$$

The argument for scalar multiplication is analogous. □

Remark. We could have started out by showing that the map

$$\begin{aligned} F : C_p &\to \mathbb{R}^n \\ c &\mapsto \frac{d}{dt}(\varphi \circ c)\Big|_{t=0} \end{aligned}$$

is surjective and that the equivalence relation defined on C_p is precisely the equivalence relation induced by F. (That is, the relation $c_1 \sim c_2 \iff F(c_1) = F(c_2)$.) Then it is a standard result that F descends to a bijective map (see Figure 1.1). We may then define $T_pM = {}^{C_p}/_{\sim}$ and $T_p\varphi = \widetilde{F}$.

$$\begin{tikzcd} C_p \arrow[r, "F"] \arrow[d] & \mathbb{R}^n \\ {}^{C_p}/_{\sim} \arrow[ur, dashed, "\widetilde{F}"'] & \end{tikzcd}$$

Figure 1.1

Remark 1.4.5. If M is a smooth manifold and $U \subset M$ is open, then $T_pU = T_pM$ for every $p \in U$. This is because we could restrict ourselves to curves contained in U when constructing C_p.

Definition 1.4.6: Coordinate Vectors

Let M^n be a smooth manifold and (U, φ) be a chart of M. For all $p \in U$, the **coordinate vectors** at p with respect to (U, φ) are given by

$$\frac{\partial}{\partial x^i}\Big|_p := \left(T_p\varphi\right)^{-1}(e_i) \in T_pM \qquad (1 \le i \le n).$$

The collection $\frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^n}\big|_p$ is the **coordinate basis** of T_pM with respect to (U, φ).

Note the coordinate basis does indeed inform a basis since $T_p\varphi$ is a vector space isomorphism.

Definition 1.4.7: Tangent Map

Let $f \in C^\infty(M, N)$ and $p \in M$. The map

$$\begin{aligned} T_pf : T_pM &\to T_{f(p)}N \\ [c]_p &\mapsto [f \circ c]_{f(p)} \end{aligned}$$

is called the **tangent map** of f at p.

We will show that the tangent map is well-defined and linear.

Proof. First, note that $f \circ c \in C_{f(p)}$. Let (U, φ) be a chart of M and (V, ψ) be a chart of N such that $p \in U$, $f(p) \in V$.

Then by the chain rule and Prop. 1.4.4,

$$\begin{aligned}
T_p f[c]_p &= [f \circ c]_{f(p)} \\
&= \left(T_{f(p)}\psi\right)^{-1} \frac{d}{dt} (\psi \circ f \circ c)\,\big|_{t=0} \\
&= \left(T_{f(p)}\psi\right)^{-1} \frac{d}{dt} \left(\psi \circ f \circ \varphi^{-1} \circ \varphi \circ c\right)\big|_{t=0} \\
&= \left(T_{f(p)}\psi\right)^{-1} D\left(\psi \circ f \circ \varphi^{-1}\right)\big|_{\varphi(p)} \frac{d}{dt} (\varphi \circ c)\,\big|_{t=0} \\
&= \left(T_{f(p)}\psi\right)^{-1} D\left(\psi \circ f \circ \varphi^{-1}\right)\big|_{\varphi(p)} T_p\varphi[c]_p.
\end{aligned}$$

This implies

$$T_p f = \underbrace{\left(T_{f(p)}\psi\right)^{-1}}_{\text{linear}} \circ \underbrace{D\left(\psi \circ f \circ \varphi^{-1}\right)|_{\varphi(p)}}_{\text{linear}} \circ \underbrace{T_p\varphi}_{\text{linear}}\ . \tag{1.4.5}$$

□

Remark 1.4.8. (i) For smooth manifolds M,N, the definition of $T_p f$ also makes sense for functions that are only defined on a neighborhood U of p.

(ii) For open subsets $U \subset \mathbb{R}^n$ and $p \in U$, there is a **canonical isomorphism**

$$\begin{aligned}
T_p\,\mathrm{id} : T_p U &\cong \mathbb{R}^n \\
[c]_p &\mapsto \frac{d}{dt} c\Big|_{t=0},
\end{aligned}$$

where we identify a vector $[c]_p \in T_p M$ with the vector $\frac{d}{dt}(\mathrm{id} \circ c)\big|_{t=0} \in \mathbb{R}^n$ that uniquely determines its equivalence class.

(iii) Via this canonical isomorphism, the definition of $T_p\varphi$ for a chart in Prop. 1.4.4 coincides with the definition of

$$\begin{aligned}
T_p\varphi : T_p M = T_p U &\to T_p\varphi(U) \\
[c]_p &\mapsto [\varphi \circ c]_{\varphi(p)}
\end{aligned}$$

in Defn. 1.4.7 More concretely, the definitions coincide via

$$T_p U \xrightarrow{T_p\varphi} T_{\varphi(p)}\varphi(U) \xrightarrow{T_p\,\mathrm{id}_{\mathbb{R}^n}} \mathbb{R}^n,$$

where elements get mapped by

$$[c]_p \xmapsto{T_p\varphi} [\varphi \circ c]_{\varphi(p)} \xmapsto{T_p\,\mathrm{id}} \frac{d}{dt}(\varphi \circ c)\Big|_{t=0}.$$

Proposition 1.4.9: Functoriality of T_p

T_p is a functor from the category of pointed smooth manifolds to the category of real vector spaces. More explicitly, this means

(i) $T_p \operatorname{id}_M = \operatorname{id}_{T_p M}$;

(ii) If $M \xrightarrow{f} N \xrightarrow{g} P$ are smooth, then

$$T_p(g \circ f) = T_{f(p)} g \circ T_p f \qquad (p \in M).$$

Proof. (i) follows immediately from Defn. 1.4.7. To show (ii), observe that

$$T_p(g \circ f)[c]_p = [g \circ f \circ c]_{g \circ f(p)} = T_{f(p)} g [f \circ c]_{f(p)} = T_{f(p)} g \circ T_p f [c]_p. \qquad \square$$

Proposition 1.4.10

Suppose

(i) M^n, N^m are C^∞ manifolds,

(ii) $f : M \to N$ is smooth,

(iii) $(U, \varphi = (x^1, \ldots, x^n))$ is a chart of M around p,

(iv) $(V, \psi = (y^1, \ldots, y^n))$ is a chart of N around $f(p)$,

Then the matrix representation of $T_p f$ with respect to coordinate bases $\left\{ \frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^n}\big|_p \right\}$ and $\left\{ \frac{\partial}{\partial y^1}\big|_{f(p)}, \ldots, \frac{\partial}{\partial y^m}\big|_{f(p)} \right\}$ is given by the Jacobian matrix of $\psi \circ f \circ \varphi^{-1}$ at $\varphi(p)$. In other words,

$$T_p f \left(\frac{\partial}{\partial x^i}\bigg|_p \right) = \sum_{j=1}^m \frac{\partial \left(y^j \circ f \circ \varphi^{-1} \right)}{\partial x^i} (\varphi(p)) \frac{\partial}{\partial y^j}\bigg|_{f(p)}. \tag{1.4.6}$$

Proof. Recall that $\left\{ \frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^n}\big|_p \right\}$ is a basis of $T_p M$ and $\left\{ \frac{\partial}{\partial y^1}\big|_{f(p)}, \ldots, \frac{\partial}{\partial y^m}\big|_{f(p)} \right\}$ is a basis of $T_{f(p)} N$. (See Defn. 1.4.3.) The assertion follows from computing

$$\begin{aligned}
T_p f \left(\frac{\partial}{\partial x^i}\bigg|_p \right) &= \left(T_{f(p)} \psi \right)^{-1} \circ D \left(\psi \circ f \circ \varphi^{-1} \right)\Big|_{\varphi(p)} \circ T_p \varphi \left(\frac{\partial}{\partial x^i}\bigg|_p \right), && \text{by (1.4.5)} \\
&= \left(T_{f(p)} \psi \right)^{-1} \circ D \left(\psi \circ f \circ \varphi^{-1} \right)\Big|_{\varphi(p)} (e_i), && \text{by Defn. 1.4.6} \\
&= \left(T_{f(p)} \psi \right)^{-1} \left(\sum_{j=1}^m \frac{\partial y^j \circ f \circ \varphi^{-1}}{\partial x^i} (\varphi(p)) \cdot e_j \right) \\
&= \sum_{j=1}^m \underbrace{\frac{\partial y^j \circ f \circ \varphi^{-1}}{\partial x^i} (\varphi(p))}_{\in \mathbb{R}} \left(T_{f(p)} \psi \right)^{-1} (e_j), && \text{by linearity} \\
&= \sum_{j=1}^m \frac{\partial y^j \circ f \circ \varphi^{-1}}{\partial x^i} (\varphi(p)) \frac{\partial}{\partial y^j}\bigg|_{f(p)}, && \text{by Defn. 1.4.6.}
\end{aligned}$$

$\square$

Remark 1.4.11. Prop. 1.4.10 has many interesting special cases.

(i) If $M \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^m$ are open, then $T_p f$ coincides with the standard differential $Df|_p$ via the canonical isomorphism from Remark 1.4.8(iii). In that sense, the tangent map can be seen as a generalization of the

Jacobian. More explicitly, set $\psi = \varphi = \mathrm{id}$ and see that

$$T_p f\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \sum_{j=1}^m \frac{\partial y^j \circ f \circ \mathrm{id}}{\partial x^i}(p) \cdot \frac{\partial}{\partial x^j}\Big|_p = \sum_{j=1}^m \frac{\partial f_j}{\partial x^i}(p) \cdot \frac{\partial}{\partial x^j}\Big|_p \cong \sum_{j=1}^m \frac{\partial f_j}{\partial x^i}(p) \cdot e_j.$$

(ii) If $(U, \varphi = (x^1, \ldots, x^n))$ and $(V, \psi = (y^1, \ldots, y^n))$ are two different charts of M with $p \in U \cap V$, then (1.4.6) with $f = \mathrm{id}$ yields

$$\frac{\partial}{\partial x^i}\Big|_p = \sum_{j=1}^m \frac{\partial\left(y^j \circ \varphi^{-1}\right)}{\partial x^i}(\varphi(p))\frac{\partial}{\partial y^j}\Big|_p. \tag{1.4.7}$$

Note that we have also used the functoriality of T_p to obtain $T_p\,\mathrm{id}\left(\frac{\partial}{\partial x^i}\big|_p\right) = \frac{\partial}{\partial x^i}\big|_p$. Then (1.4.7) is the transformation rule between coordinate bases. Compare with (1.4.3).

(iii) If $M = I \subset \mathbb{R}$ is an open interval, t the standard coordinate on I, and $c \in C^\infty(I, N)$ a smooth curve, we denote

$$\frac{\partial c}{\partial t} := T_t c\left(\frac{\partial}{\partial t}\Big|_t\right) \in T_{c(t)}M.$$

In this case, (1.4.6) yields

$$\frac{\partial c}{\partial t} = \sum_{j=1}^n \frac{\partial y^j \circ c}{\partial t}(t)\frac{\partial}{\partial y^j}\Big|_{c(t)}. \tag{1.4.8}$$

(iv) If $N = \mathbb{R}$, then the theorem shows that (under the identification $T_{f(p)}\mathbb{R} \cong \mathbb{R}$)

$$T_p f\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial\left(f \circ \varphi^{-1}\right)}{\partial x^i}(\varphi(p))\frac{\partial}{\partial y}\Big|_{f(p)} \cong \frac{\partial\left(f \circ \varphi^{-1}\right)}{\partial x^i}(\varphi(p)).$$

Thus, if $v = \sum_{i=1}^n v^i \cdot \frac{\partial}{\partial x^i}\big|_p \in T_pM$, then

$$T_p f(v) = T_p f\left(\sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\Big|_p\right) \cong \sum_{i=1}^n v^i \frac{\partial\left(f \circ \varphi^{-1}\right)}{\partial x^i}(\varphi(p)). \tag{1.4.9}$$

We also write

$$\partial_v f := v(f) := T_p f(v).$$

This is the **directional derivative** of f in the direction of v.

(v) Sometimes, we write $\partial_{x^i}\big|_p$ for $\frac{\partial}{\partial x^i}\big|_p$ or even $\partial_i\big|_p$ if the coordinates being chosen are understood. For $f \in C^\infty(M)$, we often write (by abuse of notation)

$$\frac{\partial f}{\partial x^i}(p) := \frac{\partial\left(f \circ \varphi^{-1}\right)}{\partial x^i}(\varphi(p)).$$

Sometimes we also write $\partial_i f(p) := \frac{\partial f}{\partial x^i}(p)$. These abbreviations allow us to write the transformation rule (1.4.7) as

$$\frac{\partial}{\partial x^i}\Big|_p = \sum_{j=1}^n \frac{\partial y^j}{\partial x^i}(p)\frac{\partial}{\partial y^j}\Big|_p.$$

Exercise 1.4.12. Let M, N be smooth manifolds and consider the product manifold $M \times N$. Use the tangent maps of the inclusion maps

$$i_q : M \ni p \to (p, q), \qquad i_p : N \ni i_p : q \to (p, q)$$

and the projection maps

$$\pi_M : M \times N \ni (p, q) \to p, \qquad \pi_N : M \times N \ni (p, q) \to q$$

to construct a (canonical) isomorphism

$$T_{(p,q)}(M \times N) \cong T_pM \oplus T_qN.$$

Exercise 1.4.13. We equip the manifold $M = \mathbb{R}^2 \setminus \{x \in \mathbb{R}^2 : x_2 = 0 \text{ and } x_1 \leq 0\} \subset \mathbb{R}^2$ with the charts (M, φ) and (M, ψ) where $\varphi(x) = (x_2, x_1)$ and

$$\psi^{-1} : (0, \infty) \times (-\pi, \pi) \to M\,, \quad (r, \theta) \mapsto (r\cos\theta, r\sin\theta)\,.$$

Show that φ and ψ are compatible and express the coordinate vectors $\partial_{x_1}|_p$ and $\partial_{x_2}|_p$ associated with φ as well as the coordinate vectors $\partial_r|_p$ and $\partial_\theta|_p$ associated with ψ at a point $p \in M$ in terms of the canonical basis vectors e_1 and e_2 in $T_pM \cong \mathbb{R}^2$.

1.5 Submanifolds

Definition 1.5.1: Immersions and Submersions

Let M^m, N^n be C^∞ manifolds and $f \in C^\infty(M, N)$. Then f is called

(i) **immersion** if $m \leq n$ and T_pf is injective for all $p \in M$,

(ii) **submersion** if $m \geq n$ and T_pf is surjective for all $p \in M$,

(iii) **embedding** if f is an injective immersion which is a homeomorphism onto its image.

Example 1.5.2. Let $M = \mathbb{R}^m, N = \mathbb{R}^n$.

(i) $m \leq n$ and $f : x = (x^1, \dots, x^m) \mapsto (x^1, \dots, x^m, 0, \dots, 0)$ is an injective immersion since

$$T_pf \cong Df\big|_p \cong \begin{bmatrix} 1 & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & 1 \\ 0 & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & 0 \end{bmatrix}.$$

Note f is also a homeomorphism onto its image $g : (y^1, \dots, y^m, y^{m+1}, \dots, y^n) \mapsto (y^1, \dots, y^m)$ restricts to a continuous inverse of f.

(ii) If $m \geq n$, $f : x = (x^1, \dots, x^n, x^{n+1}, \dots, x^m) \mapsto (x^1, \dots, x^n)$ is a (surjective) submersion since

$$T_pf \cong Df\big|_p \cong \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

(iii) The map

$$\begin{aligned} f : (0, 2\pi) &\to \mathbb{R}^2 \\ t &\mapsto (\sin 2t, \sin t) \end{aligned}$$

is not an injective immersion. Note $f'(t) = (2\cos t2t, \cos t2) \neq (0, 0)$ for all $t \in (0, 2\pi)$, but f is not a homeomorphism onto its image.

Lemma 1.5.3: Inverse function theorem for manifolds

Let M, N be smooth manifolds of the same dimension and $f \in C^\infty(M, N)$.

(i) If f is a diffeomorphism, then T_pf is a linear isomorphism and $(T_pf)^{-1} = T_{f(p)}f^{-1}$.

(ii) If T_pf is a linear isomorphism for some $p \in M$, there exist open neighborhoods $U \subset M$, $V \subset N$ of p and $f(p)$, respectively, such that $f(U) = V$ and $f|_U : U \to V$ is a diffeomorphism.

Definition 1.5.4: Submanifolds

Let M^m be a C^∞ manifold. A subset $N \subset M$ is an n**-dimensional submanifold** if for every $p \in N$, there is a chart (U, φ) of M around p such that

$$\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^n \times \{0\}^{m-n}).$$

Such a chart (U, φ) is called a **submanifold chart** of N. The number $m - n$ is the **codimension** of N in M.

Example 1.5.5. (i) $N \subset M$ is a submanifold of codimension 0 $\iff$ N is an open subset of M.

(ii) $N \subset M$ is a submanifold of dimension 0 $\iff$ N is a discrete subset of M.

(iii) **Affine subspaces:** Let $M = \mathbb{R}^m$ and $N = N' + p$, where $N' \subset \mathbb{R}^m$ is an n-dimensional subspace and $p \in \mathbb{R}^m$ is fixed. Pick $A \in \mathrm{GL}(m, \mathbb{R})$ such that $A(N') = \mathbb{R}^n \times \{0\}$. Then $\varphi : U = \mathbb{R}^m \to \mathbb{R}^m$ given by

$$\varphi(q) = A(q - p)$$

is a submanifold chart. Such an A exists because N' is an n-dimensional subspace.

(iv) **Graphs:** Let M_1, M_2 be C^∞ manifolds and $f \in C^\infty(M_1, M_2)$. Let $M = M_1 \times M_2$ and

$$N = \Gamma_f := \{(p, q) \in M_1 \times M_2 = M : f(p) = q\}$$

the graph of f. For $i = 1, 2$, choose charts (U_i, φ_i) around $p_i \in M_i$ such that $f(U_1) \subset U_2$. For $x \in \varphi_1(U_1)$, $y \in \varphi_2(U_2)$, set

$$\psi(x, y) = \left(x, y - \left(\varphi_2 \circ f \circ \varphi_1^{-1}\right)(x)\right).$$

Then $\varphi = \psi \circ (\varphi_1 \times \varphi_2)$ is a submanifold chart. Note that for all $(x, y) \in (U_1 \times U_2) \cap \Gamma_f$, we have

$$\begin{aligned}
\varphi(x, y) &= \psi(\varphi_1(x), \varphi_2(y)) \\
&= \left(\varphi_1(x), \varphi_2(y) - (\varphi_2 \circ f \circ \varphi_1^{-1})(\varphi_1(x)\right) \\
&= (\varphi_1(x), \varphi_2(y) - \varphi_2(f(x))) \\
&= (\varphi_1(x), \varphi_2(y) - \varphi_2(y)) \\
&= (\varphi_1(x), 0).
\end{aligned}$$

This shows

$$\varphi\left((U_1 \times U_2) \cap \Gamma_f\right) = \varphi(U_1 \times U_2) \cap (\mathbb{R}^n \times \{0\}).$$

Lemma 1.5.6: Generalized inverse function theorem

Let $U \subset \mathbb{R}^m$, let $x \in U$, and let $f \in C^\infty(U, \mathbb{R}^n)$.

(i) If $m \le n$ and $Df|_x$ is injective, there exist neighborhoods $V \subset U$, $W \subset \mathbb{R}^n$ of x and $f(x)$ respectively, with $f(V) \subset W$ and a diffeomorphism $\varphi : W \to \varphi(W) \subset \mathbb{R}^n$ such that

$$\varphi \circ f|_V = i : (x^1, \ldots, x^m) \mapsto \left(x^1, \ldots, x^m, 0, \ldots, 0\right).$$

(ii) If $m \ge n$ and $Df|_x$ is surjective, there is a neighborhood $V \subset \mathbb{R}^m$ of x and a diffeomorphism $\varphi : \mathbb{R}^m \supset W \to V$ such that

$$f \circ \varphi|_W = \mathrm{pr} : \left(x^1, \ldots, x^n, x^{n+1}, \ldots, x^m\right) \mapsto \left(x^1, \ldots, x^n\right).$$

Proof. Exercise. $\square$

Theorem 1.5.7

Let M^m be a C^∞ manifold and $N \subset M$ a subset. The following are equivalent

(i) N is an n-dimensional submanifold.

(ii) For every $p \in N$, there is an open neighborhood $V \subset M$ of p and a submersion $f : V \to \mathbb{R}^{m-n}$ such that $N \cap V = f^{-1}(0)$. Such a function f is a **locally defining function** for N.

(iii) For all $p \in N$, there exists an open neighborhood $V \subset M$ of p, an open subset $U \subset \mathbb{R}^n$, and an embedding $\psi : U \to V$ such that $\psi(U) = N \cap V$. Such a map ψ is called a **local parametrization**.

Proof. Throughout the proof, let

$$
\begin{aligned}
i : \mathbb{R}^n &\hookrightarrow \mathbb{R}^m \\
(x^1, \ldots, x^n) &\mapsto (x^1, \ldots, x^n, 0, \ldots, 0) \\
\mathrm{pr} : \mathbb{R}^m &\twoheadrightarrow \mathbb{R}^{m-n} \\
(x^1, \ldots, x^n, x^{n+1}, \ldots, x^m) &\mapsto (x^{n+1}, \ldots, x^m)
\end{aligned}
$$

(i) ⇒ (ii) Let $p \in N$ and $\varphi : U \to \mathbb{R}^m$ be a submanifold chart with $p \in U$. Then $f = \mathrm{pr} \circ \varphi$ is a submersion. This is because

$$T_p f = T_{\varphi(p)} \mathrm{pr} \circ T_p \varphi,$$

where $T_{\varphi(p)} \mathrm{pr} \cong D \,\mathrm{pr}|_{\varphi(p)}$ so that $T_{\varphi(p)} \mathrm{pr}$ is surjective. Since φ is a chart, $T_p\varphi$ is surjective and hence so is $T_p f$. It remains to show $U \cap N = f^{-1}(0)$. By choice of the chart U, we have $\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^n \times \{0\}^{m-n})$. Since φ is bijective and its domain is U, this means

$$U \cap N = \varphi^{-1}(\varphi(U \cap N)) = \varphi^{-1}(\varphi(U) \cap (\mathbb{R}^n \times \{0\}^{m-n})).$$

Then by definition of pr, we get $\mathrm{pr}^{-1}(0) = \mathbb{R}^n \times \{0\}^{m-n}$. Thus,

$$
\begin{aligned}
f^{-1}(0) &= \varphi^{-1}\left(\mathrm{pr}^{-1}(0)\right) \\
&= \varphi^{-1}(\mathbb{R}^n \times \{0\}^{m-n}) \\
&= \varphi^{-1}(\varphi(U) \cap (\mathbb{R}^n \times \{0\}^{m-n})) \\
&= U \cap N.
\end{aligned}
$$

(ii) ⇒ (i):

$$
\begin{tikzcd}
 & \{p\} \arrow[d, hook] & & \\
M \arrow[dd] & U \arrow[l, hook'] \arrow[r, "\varphi", "\sim"'] \arrow[d, hook] & \varphi(U) \arrow[r, "\psi", "\sim"'] \arrow[d, hook] & \psi(\varphi(U)) \arrow[dl, hook'] \arrow[ddl, bend left, "f\circ\varphi^{-1}\circ\psi^{-1}"] \\
 & V \arrow[ul, hook] \arrow[d] \arrow[r, "f"] & \mathbb{R}^m \arrow[d, two heads, "\mathrm{pr}"] & \\
 & & \mathbb{R}^{m-n} \arrow[d, "\sim"'] & \\
T_pM \arrow[r, "="'] & T_pV \arrow[r, two heads, "T_pf"] & T_{f(p)}\mathbb{R}^{m-n} &
\end{tikzcd}
$$

Let (U, φ) be a chart of M and suppose $U \subset V$ with V as in (ii). Since f is a submersion, we have $T_p f$ is injective; since $T_p\varphi$ an isomorphism, we have that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^{m-n}$ is a submersion. Since $\varphi(U) \subset \mathbb{R}^m$ is open and $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^{m-n}$ is such that $T_{\varphi(p)}(f \circ \varphi^{-1})$ is surjective, we may apply the inverse function theorem (Lemma 1.5.6) to obtain open subsets $\widetilde{U} \subset U$, $\widetilde{W} \subset \mathbb{R}^m$ such that

$$\psi : \varphi\left(\widetilde{U}\right) \to \widetilde{W}$$

is a diffeomorphism satisfying

$$f \circ \varphi^{-1} \circ \psi^{-1}(x) = \mathrm{pr}(x) \qquad (x \in \widetilde{W}).$$

By identifying $U = \widetilde{U}$ and $\varphi = \varphi|_{\widetilde{U}}$, we may suppose $\psi : \varphi(U) \to \psi(\varphi(U))$ and $f \circ \varphi^{-1} \circ \psi^{-1} = \mathrm{pr}$ on $\psi(\varphi(U)) \subset \widetilde{W}$. Since $\mathrm{pr}^{-1}(0) = \mathbb{R}^n \times \{0\}^{m-n}$ and f is a locally defining function, we have

$$\begin{aligned} p \in U \cap N &\iff f(p) = 0 \\ &\iff f \circ \varphi^{-1} \circ \psi^{-1} \circ \psi \circ \varphi(p) = 0 \\ &\iff \psi \circ \varphi(p) \in \left(f \circ \varphi^{-1} \circ \psi^{-1}\right)^{-1}(0) \\ &\iff \psi \circ \varphi(p) \in \mathrm{pr}^{-1}(0) = (\mathbb{R}^n \times \{0\}^{m-n}) \\ &\iff \psi \circ \varphi(p) \in (\psi \circ \varphi)(U) \cap (\mathbb{R}^n \times \{0\}^{m-n}). \end{aligned}$$

Thus, $(\psi \circ \varphi)(U \cap N) = (\psi \circ \varphi)(U) \cap (\mathbb{R}^n \times \{0\}^{m-n})$. This implies $\psi \circ \varphi$ is the desired submanifold chart.

(i) ⇒ (iii):

$$\begin{tikzcd}
M & U \arrow[l, hook'] \arrow[r, "\varphi", "\sim"'] & \varphi(U) \arrow[r, hook] & \mathbb{R}^m & \\
N \arrow[u] & U \cap N \arrow[l, hook'] \arrow[u] \arrow[r, hook, two heads, "\varphi"] & \varphi(U) \cap (\mathbb{R}^n \times \{0\}) \arrow[u, hook] \arrow[r, hook] & \mathbb{R}^n \times \{0\} \arrow[u, hook] & \mathbb{R}^n \arrow[l, "\sim"] \arrow[ul, hook, "i"']
\end{tikzcd}$$

Let $p \in N$. If (U, φ) is a submanifold chart, let $\psi = \varphi^{-1} \circ i : \varphi(U) \cap (\mathbb{R}^n \times \{0\}^{m-n}) \to U$ be the embedding. Set $V = \varphi(U) \cap (\mathbb{R}^n \times \{0\}^{m-n})$. ($\psi$ is indeed an embedding since φ is a homeomorphism and i is an embedding.) ψ is an immersion because φ is a diffeomorphism and i is an immersion.

It remains to show $\psi(V) = U \cap N$. This can be done in two ways. Since the diagram above commutes, we see (by following the arrows on the bottom) that $\psi(V) = \mathrm{im}(\varphi^{-1} \circ i) = U \cap N$.

Alternatively, since φ is a submanifold chart, we have

$$\begin{aligned} \psi(V) &= \varphi^{-1} \circ i\, (\varphi\,(U) \cap (\mathbb{R}^n \times \{0\}^{m-n})) \\ &= \varphi^{-1}\,(\varphi\,(U) \cap (\mathbb{R}^n \times \{0\}^{m-n})) \\ &= \varphi^{-1}\,(\varphi\,(U \cap N)) \\ &= U \cap N. \end{aligned}$$

(iii) ⇒ (i): This is the setup:

$$\begin{tikzcd}
\mathbb{R}^n \arrow[rrr, hook, "i"] & & & \mathbb{R}^m \\
& M & & \\
U \arrow[uu, hook] \arrow[r, hook, "\psi"] \arrow[d] & V \arrow[u, hook] \arrow[r, "\varphi", "\sim"'] \arrow[d] & \varphi(V) \arrow[r, "\chi", "\sim"'] \arrow[uur, hook] \arrow[d] & \chi(\varphi(V)) \arrow[uu, hook] \arrow[d] \\
T_xU \arrow[r, hook, "T_x\psi"] & T_pM \arrow[r, "T_p\varphi", "\sim"'] & T_{\varphi(p)}\mathbb{R}^m \arrow[r, "T_{\varphi(p)}\chi", "\sim"'] & T_{\chi(\varphi(p))}\mathbb{R}^m
\end{tikzcd}$$

Plugging in for images of maps, we obtain:

$$\begin{tikzcd}
\mathbb{R}^n \arrow[rrr, hook, two heads, "i"] & & & \mathbb{R}^n \times \{0\} \\
& M & & \\
U \arrow[uu, hook] \arrow[r, "\psi", "\sim"'] & V \cap N \arrow[u, hook] \arrow[r, hook, two heads, "\varphi"] & \varphi(V \cap N) \arrow[r, hook, two heads, "\chi"] \arrow[uur, hook] & \chi(\varphi(V \cap N)) \arrow[uu, hook]
\end{tikzcd}$$

Let $p \in M$ and $\psi : U \to V$ be as in (iii). By making U and V smaller if necessary, we may pick a chart (V, φ) of p. The map $\varphi \circ \psi : U \to \mathbb{R}^m$ is an embedding since ψ is an embedding and φ is a chart. By Lemma 1.5.6, (making U and V smaller if necessary) there is a diffeomorphism $\chi : \varphi(V) \to \chi(\varphi(V))$ such that $\chi \circ \varphi \circ \psi = i$ on U.

We now verify $\chi \circ \varphi : V \to \chi \circ \varphi(V)$ is a submanifold chart. Let $q \in V$. We show $q \in V \cap N \iff \chi \circ \varphi(q) \in \mathbb{R}^n \times \{0\}$. Since ψ is a local parameterization, we have $\psi(U) = V \cap N$. Thus, $q \in V \cap N \iff$ there exists $x \in U$ with $\psi(x) = q$. For this x, we have (also by definition of i),

$$\chi \circ \varphi(q) = \chi \circ \varphi \circ \psi(x) = i(x) \in \mathbb{R}^n \times \{0\}^{m-n} .$$

□

Remark 1.5.8. A submanifold $N \subset M$ of a C^∞-manifold is itself a C^∞-manifold. If $(U, \varphi = (x^1, \ldots, x^n, \ldots, x^m))$ is a submanifold chart, then

$$\psi = (x^1, \ldots, x^n) : U \cap N \to \mathbb{R}^n$$

is a chart of N. The set of such charts forms a C^∞-atlas of N. Note here that by the proof of Theorem 1.5.7(i) $\implies$ (iii), the map $\psi^{-1} : \psi(U \cap N) \to U \cap N$ is a local parametrization of N since $\psi^{-1} = \varphi^{-1} \circ i$.

Theorem 1.5.9

Let M, P, Q be smooth manifolds with $N \subset M$ a submanifold of M and $\iota : N \to M$ the inclusion map. Then

(i) $\iota \in C^\infty(N, M)$ and $T_p\iota : T_pN \to T_pM$ is injective.

(ii) If $f \in C^\infty(M, P)$, then $f|_N \in C^\infty(N, P)$.

(iii) If $g \in C^\infty(Q, M)$ and $g(Q) \subset N$, then $g \in C^\infty(Q, N)$.

Proof. (i) Let $p \in N$ and $(U, \varphi = (x^1, \ldots, x^m))$ be a submanifold chart of M with $p \in U$. Let $(V = U \cap N, \psi = (x^1, \ldots, x^n))$ be the corresponding chart for N. Note that

$$\begin{tikzcd} N \arrow[r, hook, "\iota"] \arrow[d, "\psi"] & M \arrow[d, "\varphi"] \\ \mathbb{R}^n \arrow[r, hook, "i"] & \mathbb{R}^m \end{tikzcd}$$

so that $\iota = \varphi \circ i \circ \psi^{-1}$; this shows ι is smooth. The injectivity of $T_p\iota$ follows from the chain rule, the injectivity of $Di|_{\psi(p)} \cong T_{\psi(p)}i$, and

$$\begin{tikzcd} T_pN \arrow[r, hook, "T_p\iota"] \arrow[d, "T_p\psi", "\sim"'] & T_pM \arrow[d, "T_p\varphi", "\sim"'] \\ T_{\psi(p)}\mathbb{R}^n \arrow[r, hook, "T_{\psi(p)}i"] & T_{i\circ\psi(p)}\mathbb{R}^m \end{tikzcd}$$

(ii) Since ι and f are smooth, so is $f|_N = f \circ \iota$.

(iii) Let $p = g(q)$ and $(U, \varphi), (V, \psi)$ be as in the proof of (i). Since $g(Q) \subset N$, we have $g(g^{-1}(U)) \subset U \cap N = V$.

Since φ is a submanifold chart, the following diagram commutes:

$$\begin{tikzcd}
N \arrow[r, hook, "\iota"'] & M & Q \arrow[l, "g"'] \\
V = U \cap N \arrow[u, hook] \arrow[r, hook, "\iota"'] \arrow[dd, "\psi"] \arrow[ddr, "\varphi"] & U \arrow[u, hook] \arrow[d, "\varphi"] & g^{-1}(U) \arrow[u, hook] \arrow[l, "g"'] \\
& \mathbb{R}^m & \\
\mathbb{R}^n \arrow[r, "\sim"] & \mathbb{R}^n \times \{0\} \arrow[u, hook]
\end{tikzcd}$$

In particular, the compositions $g^i = x^i \circ g$ are smooths on $g^{-1}(U)$ for $1 \le i \le n$. The assumption $g(Q) \subset N$ implies $(g^1, \ldots, g^m) = (g^1, \ldots, g^n, 0, \ldots, 0)$ since φ is a submanifold chart. Consequently, $\psi \circ g = (g^1, \ldots, g^n)$ is a smooth map and $g \in C^\infty(Q, N)$. □

Remark 1.5.10. (i) One identifies T_pN with $T_p\iota(T_pN) \subset T_pM$ and thinks of it as a vector subspace of T_pM. In particular, if $N \subset \mathbb{R}^m$ is a submanifold, then $T_pN \subset \mathbb{R}^m$.

(ii) If ψ is a local parameterization of $N \subset M$ and f is a locally defining function for N, then

$$T_pN = \operatorname{im} T_x\psi = \ker T_pf,$$

where $\psi(x) = p$.

Example 1.5.11. (i) Let $\mathbb{S}^n = \left\{x \in \mathbb{R}^{n+1} : \|x\|^2 = 1\right\}$. Define

$$\begin{aligned} f : \mathbb{R}^{n+1} &\to \mathbb{R} \\ x &\mapsto \|x\|^2 - 1. \end{aligned}$$

Note $\mathbb{S}^n = f^{-1}(0)$. Observe that the tangent map

$$\begin{aligned} T_pf \cong Df|_p : \mathbb{R}^{n+1} &\to \mathbb{R} \\ (v^1, \ldots, v^{n+1}) &\mapsto 2\sum_{i=1}^{n+1} v^i p^i \end{aligned}$$

is surjective for $p \neq 0$. In particular, this shows f is a submersion on the neighborhood $\mathbb{R}^{n+1} \setminus \{0\}$ of $\mathbb{S}^n$. By Theorem 1.5.7, this shows $\mathbb{S}^n$ is a submanifold of $\mathbb{R}^{n+1}$, of dimension n. The tangent map

$$T_pf \cong Df|_p.$$

By the remark, we have that

$$T_p\mathbb{S}^n = \ker T_pf = \left\{v \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} v^i p^i = 0\right\} = p^\perp.$$

(ii) If M_1, M_2 are smooth manifolds and $N_1 \subset M_1, N_2 \subset M_2$ are submanifolds, then $N_1 \times N_2 \subset M_1 \times M_2$ is a submanifold as well.

(iii) If M is a smooth manifold with $N \subset M$ a submanifold and $P \subset N$ a submanifold, then $P \subset M$ is a submanifold.

Theorem 1.5.12

If M_1, M_2 are manifolds and $f \in C^\infty(M_1, M_2)$ is an embedding, then the image $f(M_1) \subset M_2$ is a submanifold of M_2. In this case, we call M_1 an **embedded submanifold** of M_2.

Proof. For every chart (U, φ) of M_1, the map $\psi = f \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^{m_1} \to M_2$ is an embedding Note that ψ is a local parametrization because

$$\psi(U) = f \circ \varphi^{-1} \circ \varphi(U) = f(U).$$

More explicitly, let $f(p) \in M_2$ and let (V, χ) be a chart around $f(p)$. Since f is continuous, we may pick a chart (U, φ) around p so that $U = f^{-1}(V)$. Pick a chart U around p such that $U \subset f^{-1}(V)$. Then $\psi = f \circ \varphi^{-1} : \varphi(U) \to M_2$ is an embedding and

$$\psi(U) = f \circ \varphi^{-1} \circ \varphi(U) = f(U) = f(f^{-1}(V)) = V \cap f(M_1).$$

This shows ψ is a local parametrization. □

Theorem 1.5.13: Whitney embedding

If M^n is a smooth manifold, then there exists an embedding $\psi : M \to \mathbb{R}^{2n}$.

The above theorem means we can view every n-dimensional manifold as a submanifold of $\mathbb{R}^{2n}$.

Exercise 1.5.14. Prove Lemma 1.5.3

Exercise 1.5.15. Prove Lemma 1.5.6.

Exercise 1.5.16. Prove Remark 1.5.10 (ii).

Exercise 1.5.17. Prove that the Examples 1.5.11 (ii) and (iii) are indeed submanifolds.

Exercise 1.5.18. Show that the orthogonal matrices $O(n) := \{Q \in \mathrm{GL}(n) \mid Q^T Q = \mathrm{Id}\}$ form a $\frac{n(n-1)}{2}$-dimensional submanifold of the manifold of $n \times n$-matrices $\mathrm{mat}(n) \cong \mathbb{R}^{n^2}$. Show also that

$$T_Q O(n) = \{B \in \mathrm{mat}(n) \mid (Q^{-1}B)^T = -Q^{-1}B\},$$

and hence, in particular,

$$T_{\mathrm{Id}} O(n) \;=\; \{B \mid B^T = -B\} \;=:\; \mathrm{skew}(n).$$

Chapter 2

Fields on Manifolds

Throughout the whole chapter, let M be a smooth manifold with dimension n, unless stated otherwise.

2.1 Vector fields

Definition 2.1.1: Vector Field

A **vector field** X on M is a map defined on M such that $X(p) \in T_pM$ for all $p \in M$.

A vector field X is called **smooth** if for all $p \in M$, there exists a chart $(U, \varphi = (x^1, \ldots, x^m))$ around p such that the coefficient charts $X^i : U \to \mathbb{R}$ defined by

$$X(q) = \sum_{i=1}^{n} X^i(q) \frac{\partial}{\partial x^i}\Big|_q \qquad (q \in U) \tag{2.1.1}$$

are C^∞.

We denote by $\mathfrak{X}(M)$ the set of all C^∞ vector fields on M.

Lemma 2.1.2

Let X be a vector field on M. Then the following are equivalent

(i) $X \in \mathfrak{X}(M)$,

(ii) For any chart $(V, \psi = (y^1, .., y^n))$, the functions $\widetilde{X}^i : V \to \mathbb{R}$ defined by

$$X(q) = \sum_{i=1}^{n} \widetilde{X}^i(q) \frac{\partial}{\partial y^i}\Big|_q \qquad (q \in V)$$

are smooth.

Proof. **(i) ⇒ (ii):** By Defn. 2.1.1, there is a chart $(U, \varphi = (x^1, \ldots, x^n))$ around p such that (2.1.1) is smooth. We will show the $\widetilde{X}^i$ are smooth on $U \cap V$.

By Remark 1.4.11, we have for all $q \in U \cap V$,

$$\frac{\partial}{\partial x^i}\Big|_q = \sum_{j=1}^{n} \frac{\partial (y^j \circ \varphi^{-1})}{\partial x^i}(\varphi(q)) \frac{\partial}{\partial y^j}\Big|_q.$$

Thus,

$$\begin{aligned}\sum_{j=1}^{n} \widetilde{X}^j(q)\frac{\partial}{\partial y^j}\Big|_q &= X(q) \\ &= \sum_{i=1}^{n} X^i(q)\frac{\partial}{\partial x^i}\Big|_q \\ &= \sum_{i=1}^{n} X^i(q)\sum_{j=1}^{n}\frac{\partial(y^j\circ\varphi^{-1})}{\partial x^i}(\varphi(q))\frac{\partial}{\partial y^j}\Big|_q.\end{aligned}$$

By fixing j, expanding out the above sum, and collecting coefficients, we have

$$\widetilde{X}^j(q) = \sum_{i=1}^{n} X^i(q)\frac{\partial(y^j\circ\varphi^{-1})}{\partial x^i}(\varphi(q)) \qquad (1 \le j \le n).$$

In particular, this means that for each j,

$$\widetilde{X}^j = \sum_{i=1}^{n} \underbrace{X^i}_{C^\infty}\ \underbrace{\frac{\partial y^j}{\partial x^i}}_{C^\infty}$$

so that $\widetilde{X}^j \in C^\infty(M)$.

(ii) ⇒ (i): Clear. □

Remark 2.1.3. (i) We have $X \in \mathfrak{X}(M) \iff X|_U \in \mathfrak{X}(U)$ for all open $U \subset M$.

(ii) If (U,φ) is a chart on M, the coordinate vector fields $\frac{\partial}{\partial x^i}$, defined on U by

$$\frac{\partial}{\partial x^i}(p) = \frac{\partial}{\partial x^i}\Big|_p \in T_pM$$

are smooth vector fields on U.

(iii) Let $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. For all $p \in M$, we have $f(p) \in \mathbb{R}$ and $X(p), Y(p) \in T_pM$. Since T_pM is a real vector space, we may define $X+Y$ and $f\cdot X$ by

$$(X+Y)(p) = X(p)+Y(p) \qquad \text{and} \qquad (f\cdot X)(p) = f(p)\cdot X(p).$$

These definitions of addition and multiplication turn $\mathfrak{X}(M)$ into a $C^\infty(M)$-module.

Definition 2.1.4: $X(f)$

For $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, we define $X(f) : M \to T_p$ (also denoted by $\partial_X f$) to be the map

$$\begin{aligned} X(f) : M &\to \bigsqcup_{p\in M} T_{f(p)}\mathbb{R} \\ p &\mapsto T_pf(X(p)) \in T_{f(p)}\mathbb{R} \cong \mathbb{R}\end{aligned}$$

We also use the notation $\partial_{X(p)}f = T_pf(X(p))$.

Note. *$X(f)$ is smooth: if (U,φ) is a chart of M and $X = \sum_{i=1}^{n} X^i\frac{\partial}{\partial x^i}$, we have*

$$X(f)|_U = \sum_{i=1}^{n} X^i\frac{\partial f}{\partial x^i} \in C^\infty(M), \tag{2.1.2}$$

where we have used the shorthand notation

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial(f\circ\varphi^{-1})}{\partial x^i}(\varphi(p)),$$

so that the above becomes

$$X(f)|_U(p) = \sum_{i=1}^n X^i(p)\frac{\partial(f\circ\varphi^{-1})}{\partial x^i}(\varphi(p)).$$

This is because each $X^i : M \to \mathbb{R}$ *is smooth and because* $f : M \to \mathbb{R}$ *is smooth. Alternatively, we can see the map* $X(f)$ *as "sort of" being the composition*

$$\begin{tikzcd} M \arrow[r, "X(f)"] \arrow[d, "X"'] & \mathbb{R} \\ T_pM \arrow[ur, "T_pf"'] & \end{tikzcd}$$

Example 2.1.5. If $M \subset \mathbb{R}^m$ is a submanifold, then any $X \in \mathfrak{X}(M)$ can be viewed as a smooth map $X : M \to \mathbb{R}^m$. In particular, we may deduce a very explicit expression for coordinate vector fields. Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$, and $\psi : U \to V$ be a local parametrization. The proof of Theorem 1.5.7 shows $\varphi = \psi^{-1}|_{M\cap V} : M \cap V \to U$ is a chart of M. Then its coordinate vectors $p \in M \cap V$ are given by

$$\frac{\partial}{\partial x^i}\Big|_p = (T_p\varphi)^{-1}(e_i) = T_{\varphi(p)}\varphi^{-1}(e_i) = T_{\varphi(p)}\psi(e_i).$$

Viewing ψ as a map $U \to V$, the tangent map agrees with the Jacobian, giving us (by Prop 1.4.10),

$$\begin{aligned}\frac{\partial}{\partial x^i}\Big|_p &= T_{\varphi(p)}\psi(e_i) = \sum_{j=1}^m \frac{\partial(\mathrm{id}^j \circ\psi\circ \mathrm{id}^{-1})}{\partial x^i}\big(\mathrm{id}(\varphi(p))\big)\frac{\partial}{\partial\, \mathrm{id}^j}\Big|_{(\psi(\varphi(p)))} \\ &= \sum_{j=1}^m \frac{\partial\psi^j}{\partial x^i}\big(\varphi(p)\big)e_j \in \mathbb{R}^m.\end{aligned}$$

Since $\varphi\circ\psi(p) = p$, the above may be rewritten as

$$\frac{\partial}{\partial x^i}\Big|_{\psi(p)} = \sum_{j=1}^m \frac{\partial\psi^j}{\partial x^i}(p)e_j \in \mathbb{R}^m.$$

For example, if $M = \mathbb{S}^2 \subset \mathbb{R}^3$, a local parametrization is given by

$$\begin{aligned}\psi : (0, 2\pi)\times\left(\frac{-\pi}{2}, \frac{\pi}{2}\right) &\to M \\ (\phi,\theta) &\mapsto \begin{pmatrix}\cos(\phi)\cos(\theta)\\ \sin(\phi)\cos(\theta)\\ \sin(\theta)\end{pmatrix}.\end{aligned}$$

Its inverse is a chart with coordinate functions ϕ, θ, and we have

$$\frac{\partial}{\partial\phi}\Big|_{\psi(\phi,\theta)} = \begin{pmatrix}-\sin(\phi)\cos(\theta)\\ \cos(\phi)\cos(\theta)\\ 0\end{pmatrix} \qquad\text{and}\qquad \frac{\partial}{\partial\theta}\Big|_{\psi(\phi,\theta)} = \begin{pmatrix}-\cos(\phi)\sin(\theta)\\ -\sin(\phi)\sin(\theta)\\ \cos(\theta)\end{pmatrix}.$$

Definition 2.1.6: Lie-bracket

Given $X, Y \in \mathfrak{X}(M)$, we define a vector field $[X, Y] \in \mathfrak{X}(M)$ locally as follows: If $(U, \varphi = (x^1, \ldots, x^n))$ is a chart of M with respect to which we have

$$X|_U = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i} \qquad \text{and} \qquad Y|_U = \sum_{j=1}^n \frac{\partial}{\partial x^j}$$

for smooth functions $X^i, Y^j : U \to \mathbb{R}$, we set

$$[X, Y]|_U = \sum_{j=1}^n \left[\sum_{i=1}^n \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \right] \frac{\partial}{\partial x^j}.$$

The vector field $[X, Y]$ is called **the Lie-bracket** or **commutator** of X and Y.

Lemma 2.1.7

For $X, Y \in \mathfrak{X}(M)$, the map $[X, Y]$ is well-defined, i.e., its definition does not depend on the chosen chart.

Proof. Exercise. □

Theorem 2.1.8

Let $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. Then

$$[X, Y](f) = X(Y(f)) - Y(X(f)). \tag{2.1.3}$$

In other words, if we view X, Y as $\mathbb{R}$-linear maps on $C^\infty(M)$, we have $[X, Y] = X \circ Y - Y \circ X$.

Proof. Observe that the function

$$\begin{aligned} \mathfrak{X}(M) \times C^\infty(M) &\to C^\infty(M) \\ (X, f) &\mapsto X(f) \end{aligned}$$

is local in the sense that $X(f)|_U = (X|_U(f|_U))$ for any open set U containing p. (This is a consequence of $T_pM = T_pU$.) Thus, it will suffice to prove (2.1.3) for coordinate neighborhoods; so let $(U, \varphi = (x^1, \ldots, x^n))$ be a chart of M and write

$$X|_U = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i} \qquad \text{and} \qquad Y|_U = \sum_{j=1}^n Y^j \frac{\partial}{\partial x^j},$$

where $X^i, Y^j : U \to \mathbb{R}$ are smooth since $X, Y \in \mathfrak{X}(M)$. Observe that

$$\begin{aligned}
X(Y(f))|_U &= X|_U(Y|_U(f|_U)) \\
&= X|_U\left(\sum_{j=1}^n Y^j \frac{\partial f}{\partial x^j}\right), && \text{by (2.1.2)} \\
&= \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}\left(\sum_{j=1}^n Y^j \frac{\partial f}{\partial x^j}\right), && \text{by (2.1.2)} \\
&= \sum_{i=1}^n \sum_{j=1}^n X^i \frac{\partial}{\partial x^i}\left(Y^j \frac{\partial f}{\partial x^j}\right) \\
&= \sum_{i,j=1}^n X^i \left(\frac{\partial Y^j}{\partial x^i}\frac{\partial f}{\partial x^j} + Y^j \frac{\partial^2 f}{\partial x^i \partial x^j}\right), && \text{by the product rule} \\
&= \sum_{i,j=1}^n X^i \frac{\partial Y^j}{\partial x^i}\frac{\partial f}{\partial x^j} + \sum_{i,j=1}^n X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j}.
\end{aligned}$$

Similarly, we get

$$-Y(X(f))|_U = -\sum_{i,j=1}^n Y^i \frac{\partial X^j}{\partial x^i}\frac{\partial f}{\partial x^j} - \sum_{i,j=1}^n Y^i X^j \frac{\partial^2 f}{\partial x^i \partial x^j}.$$

Observe that

$$\begin{aligned}
\sum_{i,j=1}^n X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j} &= \sum_{i,j=1}^n X^j Y^i \frac{\partial^2 f}{\partial x^j \partial x^i}, && \text{by swapping indices} \\
&= \sum_{i,j=1}^n X^j Y^i \frac{\partial^2 f}{\partial x^i \partial x^j}, && \text{by Schwartz theorem.}
\end{aligned}$$

Thus,

$$\sum_{i,j=1}^n X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_{i,j=1}^n Y^i X^j \frac{\partial^2 f}{\partial x^i \partial x^j} = 0.$$

In particular, this shows

$$\begin{aligned}
X(Y(f))|_U - Y(X(f))|_U &= \sum_{i,j=1}^n X^i \frac{\partial Y^j}{\partial x^i}\frac{\partial f}{\partial x^j} + \sum_{i,j=1}^n X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_{i,j=1}^n Y^i \frac{\partial X^j}{\partial x^i}\frac{\partial f}{\partial x^j} - \sum_{i,j=1}^n Y^i X^j \frac{\partial^2 f}{\partial x^i \partial x^j} \\
&= \sum_{i,j=1}^n X^i \frac{\partial Y^j}{\partial x^i}\frac{\partial f}{\partial x^j} - \sum_{i,j=1}^n Y^i \frac{\partial X^j}{\partial x^i}\frac{\partial f}{\partial x^j} + 0 \\
&= \sum_{i,j=1}^n \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right)\frac{\partial f}{\partial x^j} \\
&= [X, Y](f)|_U.
\end{aligned}$$

□

Lemma 2.1.9

The Lie bracket admits the following properties:

(i) It is an ($\mathbb{R}$-) bilinear and antisymmetric map $[.,.] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$.

(ii) For all $X, Y, Z \in \mathfrak{X}(M)$, we have the **Jacobi identity**

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \tag{2.1.4}$$

(iii) For all vector fields $X, Y, Z \in \mathfrak{X}(M)$ and functions $f, g \in C^\infty(M)$, we have the identity

$$[f \cdot X, g \cdot Y] = fg[X,Y] - gY(f)X + fX(g)Y. \tag{2.1.5}$$

(iv) For coordiante fields of a chart, we have the identity $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$ for $1 \leq i, j \leq n$.

Proof. Exercise. □

We now give a different definition of vector fields.

Definition 2.1.10: Tangent Bundle

The disjoint union

$$TM := \bigsqcup_{p \in M} T_pM$$

is the **tangent bundle** of M. We have a natural map

$$\begin{aligned} \pi : TM &\to M \\ \xi &\mapsto p \qquad \text{if } \xi \in T_pM \end{aligned}$$

called the **footprint map**.

Note. *The footprint map works as follows: given $\xi \in TM$, we have $\xi \in T_pM$ for exactly one $p \in M$. Then $\pi(\xi) = p$ so that π recovers the index of the disjoint union to which ξ belongs.*

We can give TM a smooth structure as follows:

(i) Given a chart (U, φ), we define a chart $(TU, T\varphi)$ as follows:

$$TU := \pi^{-1}(U) = \bigsqcup_{p \in U} T_pM$$

and

$$\begin{aligned} T\varphi : TU &\to \varphi(U) \times \mathbb{R}^n \subset \mathbb{R}^{2n} \\ \xi &\mapsto \left(\varphi \circ \pi(\xi), T_{\pi(\xi)}\varphi(\xi)\right) \end{aligned}$$

Basically, $T\varphi$ takes in a point $\xi \in TM$ and decomposes it into the pair (p, v), where $p = \pi(\xi)$ and $v = \xi \in T_pM$. We then map p into $\mathbb{R}^n$ via φ and v into $\mathbb{R}^n$ via $T_p\varphi$. As both φ and $T_p\varphi$ are bijective, and the map $\xi \mapsto (p, v)$ is bijective, we see that $T\varphi$ is bijective.

More explicitly, we may compute

$$\begin{aligned} (T\varphi)^{-1} : \varphi(U) \times \mathbb{R}^n &\to TU \\ (x, v) &\mapsto \left(T_{\varphi^{-1}(x)}\varphi\right)^{-1}(v) \end{aligned}$$

Note $(T\varphi)^{-1}$ takes in a pair $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$; then, we map

$$\begin{aligned} x &\mapsto \varphi^{-1}(x) = p \in U \\ v &\mapsto \left(T_p\varphi\right)^{-1}(v) = T_x\varphi^{-1}(v) \in T_pU \subset TU. \end{aligned}$$

Basically, x acts as the index for which tangent space the vector v should be mapped to.

(ii) We now check that transition maps are smooth. Let (V, ψ) be another chart and set $W = U \cap V$. Note that our discussion on $(T\varphi)^{-1}$ shows that $\pi \circ \left(T_{\varphi^{-1}(x)}\varphi\right)^{-1}(v) = \varphi^{-1}(x)$. Thus,

$$\begin{aligned} T\psi \circ (T\varphi)^{-1} : \varphi(W) \times \mathbb{R}^n &\to \psi(U \cap V) \times \mathbb{R}^n \\ (x, v) &\mapsto \left(\psi \circ \pi \circ \left(T_{\varphi^{-1}(x)}\varphi\right)^{-1}(v), T_{\pi\circ\left(T_{\varphi^{-1}(x)}\varphi\right)^{-1}(v)}\psi \circ \left(T_{\varphi^{-1}(x)}\varphi\right)^{-1}(v)\right) \\ &= \left(\psi \circ \varphi^{-1}(x), T_{\varphi^{-1}(x)}\psi \circ \left(T_{\varphi^{-1}(x)}\varphi\right)^{-1}(v)\right) \\ &= \left(\psi \circ \varphi^{-1}(x), T_{\varphi^{-1}(x)}\psi \circ T_x\left(\varphi^{-1}\right)(v)\right) \\ &= \left(\psi \circ \varphi^{-1}(x), T_x\left(\psi \circ \varphi^{-1}\right)(v)\right) \end{aligned}$$

Since $\psi \circ \varphi^{-1}$ is smooth, and since $T_x(\psi \circ \varphi^{-1}) \cong D(\psi \circ \varphi^{-1})\big|_x$ is smooth, we see that $T\psi \circ (T\varphi)^{-1}$ is smooth.

Thus, the transition maps are smooth. We can thus equip TM with a topology as in Remark 1.2.4; it turns out TM is Hausdorff and second countable because M is. Thus, TM is a smooth manifold. Given a smooth atlas $\mathscr{A} = \{(U_i, \varphi_i)\}$ of M, we have that $T\mathscr{A} = \{(TU_i, T\varphi_i)\}$ is a smooth atlas of TM.

Lemma 2.1.11

(i) $\pi \in C^\infty(TM, M)$.

(ii) A map $X : M \to TM$ is a smooth vector field $\iff$ $X \in C^\infty(M, TM)$ and $\pi \circ X = \mathrm{id}_M$.

Proof. Let $(U, \varphi = (x^1, \ldots, x^n))$ be a chart on M.

(i) Observe that

$$\begin{aligned} \varphi \circ \pi \circ (T\varphi)^{-1} : \varphi(U) \times \mathbb{R}^n &\to \varphi(U) \\ (x, v) &\mapsto \varphi \circ \pi \circ \left(T_{\varphi^{-1}(x)}\varphi\right)^{-1}(v) \\ &= \varphi \circ \varphi^{-1}(x) = x \end{aligned}$$

Thus, π is smooth by definition.

(ii) The condition $\pi \circ X = \mathrm{id}_M$ says $X(p) \in T_pM$ for all $p \in M$. Next, let $X|_U = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$. Note that for all $p \in M$, (since $\pi \circ X = \mathrm{id}_M$) the definition of the coordinate vectors $\frac{\partial}{\partial x^i}|_p$ and linearity shows

$$T_p\varphi \circ X(p) = T_p\varphi\left(\sum_{i=1}^n X^i(p)\frac{\partial}{\partial x^i}\Big|_p\right) = \sum_{i=1}^n X^i(p) T_p\varphi\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \sum_{i=1}^n X^i(p) e_i.$$

Since $\pi \circ X(p) = p$, we have

$$\begin{aligned} T\varphi \circ X \circ \varphi^{-1}(x) &= \left(\varphi(\pi(X(\varphi^{-1}(x)))), T_{\pi(X(\varphi^{-1}(x)))}\varphi(X(\varphi^{-1}(x)))\right) \\ &= \left(\varphi(\varphi^{-1}(x)), T_{(\varphi^{-1}(x)}\varphi(X(\varphi^{-1}(x)))\right) \\ &= \left(x, T_{(\varphi^{-1}(x)}\varphi \circ X \circ \varphi^{-1}(x)\right) \\ &= \left(x, X^1 \circ \varphi^{-1}(x), \ldots, X^n \circ \varphi^{-1}(x)\right). \end{aligned}$$

Note that $X \in C^\infty(M, TM)$ if and only if $T\varphi \circ X \circ \varphi^{-1}$ is smooth for an arbitrary chart (U, φ). The above shows this holds if and only if every X^i is smooth. Thus, the result follows by Defn. 2.1.1. □

2.2 1-forms

We will now define objects that are dual to vector fields. For this purpose, we recall some facts from linear algebra.

Definition 2.2.1: Dual Space

Let V be an n-dimensional real vector space. Then the space $V^* := \mathcal{L}(V, \mathbb{R})$ with natural addition and scalar multiplication of maps is also an n-dimensional real vector space, called the **dual space** of V. Given a basis $\{e_1, \ldots, e_n\}$ is a basis of V, then we define the collection of maps $e^i \in V^*$ by

$$e^i(e_j) := \delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

and linearly extend these maps to V. The collection $\left\{e^1, \ldots, e^n\right\}$ is a basis of V^*, called the **dual basis**.

The dual basis is in fact a basis. To see this, let $v^* \in V^*$ and define $v_i = v^*(e_i)$. Let $v = \sum_{i=1}^n \lambda^i e_i \in V$. Then

$$v^*(v) = v^*\left(\sum_{i=1}^n \lambda^i e_i\right) = \sum_{i=1}^n \lambda^i v^*(e_i) = \sum_{i,j=1}^n \lambda^i v_j e^j(e_i) = \sum_{j=1}^n v_j e^j \left(\sum_{i=1}^n \lambda^i e_i\right)$$

This shows $v^* = \sum_{i=1}^n v_i e^i$.

Remark 2.2.2. (i) We have isomorphic vector spaces $V \cong V^* \cong \mathbb{R}^n$ as they are all n-dimensional vector spaces. However, the spaces are not canonically isomorphic. This is why we have to treat them separately.

(ii) The map

$$\begin{aligned} i : V &\to (V^*)^* \\ v &\mapsto \left(v^* \mapsto v^*(v)\right) \end{aligned}$$

is a canonical map which is linear and injective. As $\dim(V) = n < \infty$, we have $\dim((V^*)^*) = n$ and thus i is an isomorphism. This shows $V \cong (V^*)^*$.

Definition 2.2.3: Cotangent Space

Let $p \in M$. The space $T_p^*M := (T_pM)^*$ is the **cotangent space** of M at p. An element $\xi \in T_p^*M$ is a **cotangent vector** at p.

Example 2.2.4. If $f \in C^\infty(M)$, then $T_pf : T_pM \to T_{f(p)}\mathbb{R} \cong \mathbb{R}$ is a linear map, and hence an element in T_p^*M. If we regard T_pf as a covector, we use from now on the notation $df|_p \in T_p^*M$. The notation T_pf will still be used for maps between manifolds.

$$\begin{tikzcd} T_pM \arrow[r, "df|_p"] \arrow[d, "T_pf"'] & \mathbb{R} \\ T_{f(p)}\mathbb{R} \arrow[ur, "\sim"'] & \end{tikzcd}$$

This example gives rise to an important definition.

Definition 2.2.5: Coordinate Covectors

Let $(U, \varphi = (x^1, \ldots, x^n))$ be a chart of M. Consider the smooth functions $x^1, \ldots, x^n \in C^\infty(U)$. For $p \in U$, we call $dx^1|_p, \ldots, dx^n|_p \in T_p^*M$ the **coordinate covectors** of (U, φ) at p.

Lemma 2.2.6

Let $(U, \varphi = (x^1, \ldots, x^n))$ be a chart of M and $p \in M$. The set $\{dx^1|_p, \ldots, dx^n|_p\}$ is the dual basis of $T_pM = \langle \frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^n}\big|_p \rangle$. In particular, it is a basis of T_p^*M.

Proof. Let $c_j(t) = \varphi^{-1}(\varphi(p) + te_j)$ so that $[c_j]_p = \frac{\partial}{\partial x^j}\big|_p$. Let $\varphi : T_{f(p)}\mathbb{R} \cong \mathbb{R}$ be the canonical isomorphism. Then

$$\begin{aligned}
dx^i\big|_p \left(\frac{\partial}{\partial x^j}\big|_p \right) &= \varphi \circ T_p x^i [c_j] \\
&= \varphi[x^i \circ c_j]_p \\
&= \frac{d}{dt} \left(x^i \circ c_j \right) \big|_{t=0} \\
&= \frac{d}{dt} \left(\pi^i \circ \varphi \circ \varphi^{-1} \left(\varphi \left(p \right) + te_j \right) \right) \big|_{t=0} \\
&= \frac{d}{dt} \left(x^i \left(p \right) + t\delta^i_j \right) \big|_{t=0} \\
&= \delta^i_j,
\end{aligned}$$

This shows $dx^i|_p$ is a dual basis for $\frac{\partial}{\partial x^i}\big|_p$. □

Definition 2.2.7: 1-form

A **1-form** ω on M is a map

$$\begin{aligned}
\omega : M &\to \bigsqcup_{p \in M} T_p^*M \\
p &\mapsto \omega(p) \in T_p^*M
\end{aligned}$$

that satisfies $\pi \circ \omega = \mathrm{id}_M$. It is called a C^∞ **1-form** (or **smooth 1-form**) if for all $p \in M$, there is a chart $(U, \varphi = (x^1, \ldots, x^n))$ with $p \in U$ such that the coefficient functions $\omega_i : U \to \mathbb{R}$, defined by $\omega_i(q) = \omega(q) \left(\frac{\partial}{\partial x^i}\big|_p \right)$ are smooth.

The set of all smooth 1-forms on M is denoted by $\Omega^1(M)$.

Note. *For $q \in M$, we have $\omega(q) \in T_q^*M$ so that $\omega(q) : T_qM \to \mathbb{R}$. Thus, for any basis vector $\frac{\partial}{\partial x^i}|_q$, we have a real number $x = \omega(q) \left(\frac{\partial}{\partial x^i}|_q \right)$. The coordinate function $\omega_i : U \to \mathbb{R}$ is defined precisely so that*

$$\omega_i(q) = x = \omega(q) \left(\frac{\partial}{\partial x^i}\big|_q \right).$$

That is, the functions ω_i indicate which real number the basis vector $\frac{\partial}{\partial x^i}\big|_q$ gets mapped to by $\omega(q)$. This real number also coincides with the representation of $\omega(q)$ with respect to the dual basis $\{dx^i|_q\}$ of T_q^*M.

We also have the identity

$$\omega(q) = \sum_{i=1}^n \omega_i(q) dx^i\big|_q \qquad (q \in U). \tag{2.2.1}$$

Lemma 2.2.8

Let ω be a 1-form. The following are equivalent.

(i) $\omega \in \Omega^1(M)$.

(ii) For any chart $(V, \psi = (y^1, \dots, y^n))$, the corresponding coefficient functions $q \mapsto \omega(q)\left(\frac{\partial}{\partial y^i}|q\right)$ are smooth.

Proof. Kind of like Lemma 1.3.4 and Lemma 2.1.2. □

Many things we talked about for vector fields are completely analogous for 1-forms.

Remark 2.2.9. (i) We have $\omega \in \Omega^1(M) \iff \omega|_U \in \Omega^1(U)$ for all open $U \subset M$.

(ii) For any chart $(U, \varphi = (x^1, \dots, x^n))$, the **coordinate 1-forms**

$$\begin{aligned} dx^i : U &\to T^*M \\ p &\mapsto dx^i|_p \in T_p^*M \end{aligned}$$

are smooth 1-forms on U.

(iii) $\Omega^1(M)$ may be given a $C^\infty(M)$-module structure by the natural operations

$$\begin{aligned} \omega + \eta &: p \mapsto \omega(p) + \eta(p) \\ f \cdot \omega &: p \mapsto f(p) \cdot \omega(p), \end{aligned}$$

for $\omega, \eta \in \Omega^1(M)$, $f \in C^\infty(M)$. These operations are well-defined because T_p^*M is a vector space and thus an abelian group, and because $f(p) \in \mathbb{R}$ for all p.

(iv) The disjoint union

$$T^*M := \bigsqcup_{p \in M} T_p^*M$$

is the **cotangent bundle**. We can equip T^*M with a C^∞-structure such that the canonical projection $\pi : T^*M \to M$ is smooth and

$$\omega \in \Omega^1(M) \iff \omega \in C^\infty(M, T^*M) \text{ and } \pi \circ \omega = \mathrm{id}_M .$$

Example 2.2.10. For $f \in C^\infty(M)$, the map

$$\begin{aligned} df : M &\to T^*M \\ p &\mapsto df|_p \in T_p^*M \end{aligned}$$

is a smooth 1-form because for any chart $(U, \varphi = (x^1, \dots, x^n))$, the coefficient functions

$$p \mapsto df|_p \left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial f}{\partial x^i}(p) = \frac{\partial (f \circ \varphi^{-1})}{\partial x^i}(\varphi(p))$$

are smooth. The first equality follows from (1.4.9).

We call $df \in \Omega^1(M)$ the **differential** of f.

Remark 2.2.11. Let $\omega \in \Omega^1(M)$ and $X \in \mathfrak{X}(M)$. Then for any $p \in M$, we have $X(p) \in T_pM$ and $\omega(p) \in T_p^*M$; thus, $\omega(p)\big(X(p)\big) \in \mathbb{R}$. This means we may define a map

$$\begin{aligned} \omega(X) : M &\to \mathbb{R} \\ p &\mapsto \omega(p)(X(p)) \end{aligned}$$

If $\omega = df$ for some $f \in C^\infty(M)$, then for all $p \in M$,

$$df(X)(p) = df|_p\big(X(p)\big) = T_p f\big(X(p)\big) = X(f)(p).$$

This shows $df(X) = X(f)$.

Exercise 2.2.12. Let M be a smooth manifold and $(U, \varphi = (x^1, \dots, x^n))$ and $(V, \psi = (y^1, \dots, y^n))$ two charts with $U \cap V \neq \emptyset$. Find a formula which relates between the coordinate 1-forms $\{dx^i\}_{1 \leq i \leq n}$ and $\{dy^j\}_{1 \leq j \leq n}$.

Lemma

Let M be a smooth manifold and $(U, \varphi = (x^1, \dots, x^n))$ and $(V, \psi = (y^1, \dots, y^n))$ two charts with $U \cap V \neq \emptyset$. Then

$$dx^i = \sum_{j=1}^{n} \frac{\partial x^i}{\partial y^j} dy^j. \tag{2.2.2}$$

Proof. Let $p \in U \cap V$ and let $\varphi : T_p\mathbb{R} \cong \mathbb{R}$ be the natural isomorphism. Then for all $[c]_p \in T_pM$ and all $1 \leq i \leq n$, we have

$$\begin{aligned} dx^i|_p &= \frac{d}{dt}(x^i \circ c)\Big|_{t=0} \\ &= \frac{d}{dt}(x^i \circ \psi^{-1} \circ \psi \circ c)\Big|_{t=0} \\ &= D(x^i \circ \psi^{-1})|_{\psi(p)} \frac{d}{dt}(\psi \circ c)\Big|_{t=0} \\ &= \sum_{j=1}^{n} \frac{(\partial x^i \circ \psi^{-1})}{\partial y^j} (\psi(p)) \frac{d}{dt}(y^j \circ c)\Big|_{t=0} \\ &= \sum_{j=1}^{n} \frac{\partial x^i}{\partial y^j}(p) dy^j|_p. \end{aligned}$$

In particular, this shows that for all $p \in M$,

$$dx^i = \sum_{j=1}^{n} \frac{\partial x^i}{\partial y^j} dy^j. \qquad \square$$

Exercise 2.2.13. Let $M = \mathbb{R}^2$ and $U = \mathbb{R}^2 \setminus \{x \in \mathbb{R}^2 \,|\, x^2 = 0 \text{ and } x_1 \leq 0\} \subset \mathbb{R}^2$. Consider standard coordinates $(M, \varphi = (x^1, x^2)) = (M, \mathrm{id})$ and the polar coordinates $(U, \psi = (r, \theta))$ from Problem 13.

(i) Express dr and $d\theta$ in terms of dx^1, dx^2 and functions in x^1 and x^2. Conversely, express dx^1 and dx^2 in terms of $dr, d\theta$ and functions in r and θ

(ii) Let $f \in C^\infty(\mathbb{R}^2)$ be given by $f(x^1, x^2) = (x^1)^2 - (x^2)^2$. Express df in terms of the coordinate 1-forms of both charts.

Exercise 2.2.14. Let M be a smooth manifold and $\omega \in \Omega^1(M)$. Let $[a, b] \subset \mathbb{R}$ be a closed interval and $\gamma : [a, b] \to M$ be a smooth curve. We define the path integral of ω along γ by

$$\int_\gamma \omega = \int_a^b \omega(\gamma(t))(\gamma'(t)) \, dt$$

(i) Show that if $\varphi : [c, d] \to [a, b]$ is a diffeomorphism of intervals with $\varphi' > 0$, then

$$\int_{\gamma \circ \varphi} \omega = \int_\gamma \omega.$$

(ii) Show that for $f \in C^\infty(M)$, we have

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)).$$

2.3 Tensor fields

Recall that a map is **multilinear** if it is linear in each of its arguments.

Definition 2.3.1: (r, s)-tensors

Let V be an n-dimensional real vector space and V^* its dual space. A multilinear map

$$t : \underbrace{V^* \times \cdots \times V^*}_{r\text{-times}} \times \underbrace{V \times \cdots \times V}_{s\text{-times}} \to \mathbb{R}$$

is an **(r, s)-tensor** on V. We denote the set of all (r, s) tensors on V by V^r_s.

Under the natural addition and scalar multiplication of maps, V^r_s becomes a real vector space.

Example 2.3.2. (i) An inner product $\langle -, - \rangle : V \times V \to \mathbb{R}$ (see Section 3.1) is a $(0, 2)$-tensor.

(ii) The map

$$\begin{aligned} \det : \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R} \\ (v_1, \ldots, v_n) &\mapsto \det(v_1, \ldots, v_n) \end{aligned}$$

is a $(0, n)$-tensor on $\mathbb{R}^n$.

These examples motivate why we need to define tensors also on manifolds. We need them to study lengths, angles and volumes. It will turn out later that also curvature is described by tensors.

Definition 2.3.3: Tensor Product

Let V be an n-dimensional real vector space, $t_1 \in V^r_s$ and $t_2 \in V^{r'}_{s'}$. The **tensor product** of t_1 and t_2, denoted $t_1 \otimes t_2 \in V^{r+r'}_{s+s'}$, is defined by

$$\begin{aligned} &(t_1 \otimes t_2)(v_1^*, \ldots, v_r^*, w_1^*, \ldots, w_{r'}^*, v_1, \ldots, v_s, w_1, \ldots, w_{s'}) \\ :=\ & t_1(v_1^*, \ldots, v_r^*, v_1, \ldots, v_s) \cdot t_2(w_1^*, \ldots, w_{r'}^*, w_1, \ldots, w_{s'}). \end{aligned}$$

Remark 2.3.4. (i) The tensor product is associative but not commutative. That is, for $t_i \in V^{t_i}_{s_i}$, $i = 1, 2, 3$, we have

$$(t_1 \otimes t_2) \otimes t_3 = t_1 \otimes (t_2 \otimes t_3),$$

but in general we do not have $t_1 \otimes t_2 \neq t_2 \otimes t_1$.

(ii) Given vector spaces $V, V_1, \ldots, V_k$, denote by $\mathcal{M}(V_1 \times \cdots \times V_k, V)$ the set of multilinear maps $V_1 \times \cdots \times V_k \to V$. There is a natural identification

$$\mathcal{M}(V^s, V) \cong \mathcal{M}(V^* \times V^s, \mathbb{R}) = V^1_s.$$

Given $A \in \mathcal{M}(V^s, V)$ and $v_1, \ldots, v_s \in V$, we have $A(v_1, \ldots, v_s) \in V$; we may thus define

$$\begin{aligned} t : V^* \times V^s &\to \mathbb{R} \\ (v^*, v_1, \ldots, v_s) &\mapsto v^* \circ A(v_1, \ldots, v_s) \end{aligned}$$

As $v^* : V \to \mathbb{R}$ is linear and A is multilinear, we have $t \in V^1_s$. Conversely, given $t \in \mathcal{M}(V^* \times V^s, \mathbb{R})$, we see that for all fixed $v_1, \ldots, v_s \in V$, the function $t(-, v_1, \ldots, v_s) : V^* \to \mathbb{R}$ is linear. That is,

$$t(-, v_1, \ldots, v_s) \in \mathcal{L}(V^*, \mathbb{R}) = V^{**} \cong V.$$

In particular, we may view the map $t(-, v_1, \ldots, v_s)$ as an element of V by the natural isomorphism $V^{**} \cong V$. Since t is multilinear, the map

$$\begin{aligned} A : V^s &\to V \\ (v_1, \ldots, v_s) &\mapsto t(-, v_1, \ldots, v_s) \end{aligned}$$

is multilinear; thus $A \in \mathcal{M}(V^s, V)$.

It is clear from the definition that these maps are mutual inverses.

(iii) Since $\mathcal{M}(V,W) = \mathcal{L}(V,W)$ for any vector spaces V, W, we have

$$V_0^1 = \mathcal{L}(V^*, \mathbb{R}) = V^{**} \cong V$$
$$V_1^0 = \mathcal{L}(V, \mathbb{R}) = V^*.$$

We define $V_0^0 := \mathbb{R}$. (Note this means $\dim V_0^0 = 0$.)

Proposition 2.3.5

Let V be an n-dimensional real vector sapce, $\{e_1, \ldots, e_n\}$ a basis for V and $\{e^1, \ldots, e^n\}$ its dual basis. Then

$$\left\{e_{j_1}, \otimes \cdots \otimes e_{j_r} \otimes e^{i_1} \otimes \cdots \otimes e^{i_s} : 1 \leq i_1, \ldots, i_s, j_1, \ldots, j_r \leq n\right\}$$

is a basis for V_s^r and $\dim(V_s^r) = n^{r+s}$. In particular, this means every $t \in V_s^r$ can be uniquely expressed as

$$t = \sum_{\substack{j_1,\ldots,j_r=1\\ i_1,\ldots,i_s=1}}^{n} t_{i_1,\ldots,i_s}^{j_1,\ldots,j_r} \cdot e_{j_1} \otimes \cdots e_{j_r} \otimes e^{i_1} \otimes \cdots \otimes e^{i_s}. \tag{2.3.1}$$

Furthermore, the coefficients in (2.3.1) are (uniquely) determined by the identity

$$t_{i_1,\ldots,i_s}^{j_1,\ldots,j_r} = t\left(e^{j_1}, \ldots, e^{j_r}, e_{i_1}, \ldots, e_{i_s}\right).$$

Proof. This follows from the multilinearity of the elements of V_s^r, and the independence of the sets $\{e_1, \ldots, e_n\}$ and $\{e^1, \ldots, e^n\}$. The details are left as an exercise. □

Remark 2.3.6. We will now use the **Einstein Summation Convention**. This means we automatically sum over indices that appear in both upper and lower positions. With this convention, (2.3.1) becomes

$$t = t_{i_1,\ldots,i_s}^{j_1,\ldots,j_r} \cdot e_{j_1} \otimes \cdots e_{j_r} \otimes e^{i_1} \otimes \cdots \otimes e^{i_s}.$$

Now we are ready to define the corresponding objects on manifolds.

Definition 2.3.7: Coordinate Tensors & Tensor Fields

(i) For $p \in M$, the space $(T_pM)_s^r$ is the space of (r,s)-tensors at p. If (U, φ) is a chart of M with $p \in U$, the tensors

$$\partial_{j_1}|_p \otimes \cdots \otimes \partial_{j_r}|_p \otimes dx^{i_1}|_p \otimes \cdots \otimes dx^{i_s}|_p, \qquad (1 \leq i_1, \ldots, i_s, j_1, \ldots, j_r \leq n)$$

are called **coordinate** (r,s)**-tensors** with respect to (U, φ) at p.

(ii) A map t on M with $t(p) \in (T_pM)_s^r$ for all $p \in M$ is called **tensor field** on M. It is called a C^∞ (or **smooth**) tensor field if for all $p \in M$, there exists a chart (U, φ) with $p \in U$ such that the coefficient functions

$$\begin{aligned} t_{i_1,\ldots,i_s}^{j_1,\ldots,j_r} &: U \to \mathbb{R} \\ &q \mapsto t(q)(dx^{j_1}|_p, \ldots, dx^{j_r}|_p, \partial_{i_1}|_p, \ldots, \partial_{i_s}|_p) \end{aligned}$$

are smooth. We denote the set of smooth tensor fields on M by $\mathcal{T}_s^r(M)$.

(iii) The maps

$$\begin{aligned} \partial_{j_1} \otimes \cdots \otimes \partial_{j_r} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_s} &: U \to \bigsqcup_{q \in U} (T_qM)_s^r \\ &p \mapsto \partial_{j_1}|_p \otimes \cdots \otimes \partial_{j_r}|_p \otimes dx^{i_1}|_p \otimes \cdots \otimes dx^{i_s}|_p \end{aligned}$$

are smooth tensor fields on U, called the **coordinate tensor fields**.

Note. *(i) By Proposition 2.3.5, the coordinate (r,s)-tensors form a basis of $(T_pM)^r_s$.*

(ii) As with vector fields and 1-forms, a tensor field is smooth if and only if its coefficient functions with respect to any chart are smooth.

(iii) Given a tensor field t, Prop. 2.3.5 shows the coordinate functions t^J_I defined in Defn. 2.3.7(ii) satisfy

$$t(q) = t^J_I \cdot \partial_{j_1}|_q \otimes \cdots \otimes \partial_{j_r}|_q \otimes dx^{i_1}|_q \otimes \cdots \otimes dx^{i_s}|_q.$$

(iv) Finally, for $t, t_1, t_2 \in \mathcal{T}^r_s(M)$, and $s \in \mathcal{T}^{r'}_{s'}(M)$, we have natural operations

$$t_1 + t_2 : p \mapsto t_1(p) + t_2(p), \qquad f \cdot t \mapsto f(p) \cdot t(p), \qquad t \otimes s : p \mapsto t(p) \cdot s(p),$$

which yields smooth tensor fields $t_1 + t_2 \in \mathcal{T}^r_s(M)$, $f \cdot t \in \mathcal{T}^r_s(M)$ and $t \otimes s \in \mathcal{T}^{r+r'}_{s+s'}(M)$. Analogous to the cases of vector fields and 1-forms, $(\mathcal{T}^r_s(M), +, \cdot)$ is a $C^\infty(M)$-module.

Remark 2.3.8. (i) We have $t \in \mathcal{T}^r_s(M)$ if and only if $t|_U \in \mathcal{T}^r_s(U)$ for each open subset $U \subset M$.

(ii) The disjoint union $T^r_s M := \bigsqcup_{p \in M} (T_pM)^r_s$ can be equipped with a C^∞-structure such that the canonical projection $\pi : T^r_s M \to M$ is smooth and $t \in \mathcal{T}^r_s(M)$ if and only if $\pi \in C^\infty(M, T^r_s M)$ and $\pi \circ t = \mathrm{id}_M$. We call $T^r_s M$ the (r,s)**-tensor bundle** on M.

(iii) As special cases, we have $\mathcal{T}^1_0(M) = \mathfrak{X}(M)$ and $\mathcal{T}^0_1(M) = \Omega^1(M)$. Furthermore, because $(T_pM)^0_0$ for all $p \in M$, we get $\mathcal{T}^0_0(M) = C^\infty(M)$. (See Remark 2.3.4(iii).)

For our purposes, it will later be useful to have another (algebraic) characterization of the space of tensor fields.

Definition 2.3.9: $L^{r,s}_{C^\infty(M)}(M)$

We denote by $L^{r,s}_{C^\infty(M)}(M)$ the set of all maps

$$A : \underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{r-\text{times}} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{s-\text{times}} \to C^\infty(M),$$

which are $C^\infty(M)$-multilinear in each slot.

Defn. 2.3.9 means that for all $\omega_1, \dots, \omega_r, \eta \in \Omega^1(M)$, all $X_1, \dots, X_s, Y \in \mathfrak{X}(M)$, all $f \in C^\infty(M)$, and all $k, \ell \in \{1, \dots, n\}$, we have

$$\begin{aligned} A(\omega_1, \dots, \omega_k + f \cdot \eta, \dots, \omega_r, X_1, \dots, X_s) &= A(\omega_1, \dots, \omega_k, \dots, \omega_r, X_1, \dots, X_s) \\ &\quad + f \cdot A(\omega_1, \dots, \eta, \dots, \omega_r, X_1, \dots, X_s), \\ A(\omega_1, \dots, \omega_r, X_1, \dots, X_\ell + f \cdot Y, \dots, X_s) &= A(\omega_1, \dots, \omega_r, X_1, \dots, X_\ell, \dots, X_s) \\ &\quad + f \cdot A(\omega_1, \dots, \omega_r, X_1, \dots, Y, \dots, X_s). \end{aligned}$$

We claim that for each $t \in \mathcal{T}^r_s(M)$, we get an element $A_t \in L^{r,s}_{C^\infty(M)}(M)$ defined by

$$\begin{aligned} A_t : \Omega^1(M) \times \cdots \times \Omega^1(M) \times \mathfrak{X}(M) \times \cdots \mathfrak{X}(M) &\to C^\infty(M) \\ (\omega_1, \dots, \omega_r, X_1, \dots, X_s) &\mapsto A_t(\omega_1, \dots, \omega_r, X_1, \dots, X_s), \end{aligned}$$

where

$$\begin{aligned} A_t(\omega_1, \dots, \omega_r, X_1, \dots, X_s) : M &\to \mathbb{R} \\ p &\mapsto t(p)\big(\omega_1(p), \dots, \omega_r(p), X_1(p), \dots, X_s(p)\big) \end{aligned} \tag{2.3.2}$$

Given $\omega_1, \dots, \omega_r \in \Omega^1(M)$ and $X_1, \dots, X_s \in \mathfrak{X}(M)$, note that for all $p \in M$, we have

$$\begin{aligned} t(p) &\in (T_pM)^r_s = \mathcal{M}\left(\left(T^*_pM\right)^r \times (T_pM)^s, \mathbb{R}\right) \\ \omega_1(p), \dots, \omega_r(p) &\in T^*_pM \\ X_1(p), \dots, X_s(p) &\in T_pM. \end{aligned}$$

Keeping $\omega_1, \ldots, \omega_r, X_1, \ldots, X_s$ fixed, we may thus define the map (2.3.2). It remains to check that the map $f = A_t(\omega_1, \ldots, \omega_r, X_1, \ldots, X_s)$ is smooth. We will do so in local coordinates. Let $(U, \varphi = (x^1, \ldots, x^n))$ be a chart of p. For $1 \leq i_1, \ldots, i_s, j_1, \ldots, j_r \leq n$, we then get smooth functions

$$X^{i_1}, \ldots, X^{i_s}, \omega_{j_1}, \ldots, \omega_{j_r}, t^{j_1, \ldots, j_r}_{i_1, \ldots, i_s} : U \to \mathbb{R}$$

such that for $1 \leq k \leq s$ and $1 \leq \ell \leq r$,

$$\begin{aligned} X_k(q) &= X^{i_k}(q) \partial_{i_k}\big|_q, \\ \omega_\ell(q) &= \omega_{j_\ell}(q) dx^{j_\ell}\big|_q, \\ t^{j_1, \ldots, j_r}_{i_1, \ldots, i_s}(q) &= t(q)\Big(dx^{j_1}\big|_q, \ldots, dx^{j_r}\big|_q, \partial_{i_1}\big|_q, \ldots, \partial^{i_r}\big|_q\Big). \end{aligned}$$

(Note the use of the einstein summation convention in the first two terms.) Using the multilinearity of t, we then get that on U,

$$\begin{aligned} f(q) &= t(q)\big(\omega_1(q), \ldots, \omega_r(q), X_1(q), \ldots, X_s(q)\big) \\ &= t(q)\Big(\omega_{j_1}(q) dx^{j_1}\big|_q, \ldots, \omega_{j_r}(q) dx^{j_r}\big|_q, X^{i_1}(q)\partial_{i_1}\big|_q, \ldots, X^{i_s}(q)\partial_{i_s}\big|_q\Big) \\ &= \omega_{j_1}(q) \cdots \omega_{j_r}(q) \cdot X^{i_1}(q) \cdots X^{i_s}(q) \cdot t(q)\Big(dx^{j_1}\big|_q, \ldots, dx^{j_r}\big|_q, \partial_{i_1}\big|_q, \ldots, \partial_{i_s}\big|_q\Big) \\ &= \omega_{j_1}(q) \cdots \omega_{j_r}(q) \cdot X^{i_1}(q) \cdots X^{i_s}(q) \cdot t^{j_1, \ldots, j_r}_{i_1, \ldots, i_s}(q). \end{aligned}$$

This shows $f = A_t(\omega_1, \ldots, \omega_r, X_1, \ldots, X_s) \in C^\infty(M)$. Varying $\omega_1, \ldots, \omega_r, X_1, \ldots, X_s$, we may thus define the map

$$\begin{aligned} A_t : \Big(\Omega^1(M)\Big)^r \times (\mathfrak{X}(M))^s &\to C^\infty(M) \\ (\omega_1, \ldots, \omega_r, X_1, \ldots, X_s) &\mapsto A_t(\omega_1, \ldots, \omega_r, X_1, \ldots, X_s) \end{aligned}$$

The $C^\infty(M)$-multilinearity of A_t is a direct consequence of the $\mathbb{R}$-multilinearity of t.

We can thus regard any tensor field $t \in \mathcal{T}^r_s(M)$ in a natural way as an element of $L^{r,s}_{C^\infty(M)}(M)$. The following theorem shows that the converse is also true.

Theorem 2.3.10: Tensor field reconstruction theorem

For each $A \in L^{r,s}_{C^\infty(M)}(M)$, there is a tensor field $t \in \mathcal{T}^r_s(M)$ such that $A = A_t$.

Proof. In order to simplify the proof and avoid too many indices, we restrict to the case $r = s = 1$. The general proof is completely analogous. Now let $A \in L^{1,1}_{C^\infty(M)}(M)$, $\omega \in \Omega^1(M)$ and $X \in \mathfrak{X}(M)$. We divide the proof into four steps.

Step 1. *For an open subset $U \subset M$, the restriction $A(\omega, X)|_U \in C^\infty(U)$ only depends on $\omega|_U$ and $X|_U$.*

Let $\widetilde{\omega} \in \Omega^1(M)$ and $\widetilde{X} \in \mathfrak{X}(M)$ be so that $\widetilde{\omega}|_U = \omega|_U$ and $\widetilde{X}|_U = X|_U$. We will show $A(\omega, X)|_U = A(\widetilde{\omega}, \widetilde{X})|_U$. For $p \in U$, let V be an open neighborhood of p such that $p \in V \subset \overline{V} \subset U$ and choose a cutoff function $\chi \in C^\infty(M)$ such that

$$\chi|_V \equiv 1 \qquad \text{and} \qquad \operatorname{supp}(\chi) \subset U.$$

(Recall $\operatorname{supp}\chi = \overline{\{p \in M : \chi(p) \neq 0\}}$.) Since $X|_U = \widetilde{X}|_U$ and $\omega|_U = \widetilde{\omega}|_U$, and since $\chi|_{M \setminus U} \equiv 0$, we have $\chi \cdot \widetilde{\omega} = \chi \cdot \omega$ and $\chi \cdot \widetilde{X} = \chi \cdot X$. Using the $C^\infty(M)$-multilinearity of A, we compute

$$\begin{aligned} A(\omega, X)(p) &= (\chi(p))^2 A(\omega, X)(p) = A(\chi \cdot \omega, \chi \cdot X)(p) \\ &= A(\chi \cdot \widetilde{\omega}, \chi \cdot \widetilde{X}) = (\chi(p))^2 A(\widetilde{\omega}, \widetilde{X})(p) \\ &= A(\widetilde{\omega}, \widetilde{X})(p) \end{aligned}$$

and thus, $A(\omega, X)|_U = A(\widetilde{\omega}, \widetilde{X})|_U$.

Step 2. *The map A restricts to an element $A|_U \in L^{1,1}_{C^\infty(U)}(U)$ such that $A(\omega, X)|_U = A|_U(\omega|_U, X|_U)$.*

For $\omega \in \Omega^1(U)$ and $X \in \mathfrak{X}(U)$, we define $A|_U(\omega, X) \in C^\infty(U)$ as follows. For $p \in U$, pick a neighborhood V of p such that $\overline{V} \subset U$, then choose $\widetilde{\omega} \in \Omega^1(M)$ and $\widetilde{X} \in \mathfrak{X}(M)$ such that $\widetilde{\omega}|_V = \omega|_V$ and $\widetilde{X}|_V = X|_V$. (Such fields $\widetilde{\omega}$, $\widetilde{X}$ can be found by multiplying ω, X with the cutoff function χ from the proof of Step 1 and then extending by zero to all of the manifold.) Now set

$$A|_U(\omega, X)(p) = A(\widetilde{\omega}, \widetilde{X})(p).$$

By Step 1, the map $(A|_U(\omega, X))|_V \in C^\infty(V)$ is well-defined since it does not depend on the chosen extension of ω and X. As the above argument works for all $p \in U$, we get $A|_U(\omega, X) \in C^\infty(U)$.

We now prove $C^\infty(U)$-multilinearity. Let $\omega_1, \omega_2 \in \Omega^1(U)$, $X \in \mathfrak{X}(U)$ and $f \in C^\infty(M)$. Given $p \in U$, choose extensions $\widetilde{w}_1, \widetilde{w}_2 \in \Omega^1(M)$, $\widetilde{X} \in \mathfrak{X}(M)$ and $\widetilde{f} \in C^\infty(M)$ which coincide with the respective objects of U on a neighorhood V of p with $\overline{V} \subset U$. By the $C^\infty(M)$-multilinearity of A, we get

$$\begin{aligned} A|_U(\omega_1 + f\omega_2, X)(p) &= A(\widetilde{\omega}_1 + \widetilde{f}\widetilde{\omega}_2, \widetilde{X})(p) \\ &= A(\widetilde{\omega}_1, \widetilde{X})(p) + \widetilde{f}(p)A(\widetilde{\omega}_2, \widetilde{X})(p) \\ &= A|_U(\omega_1, X)(p) + f(p) \cdot A|_U(\omega_2, X)(p). \end{aligned}$$

As the above equalities hold for all $p \in U$, we have

$$A|_U(\omega_1 + f\omega_2, X) = A|_U(\omega_1, X) + f \cdot A|_U(\omega_2, X).$$

The $C^\infty(U)$-linearity in the second argument is completely analogous. The identity $A(\omega, X)|_U = A|_U(\omega|_U, X|_U)$ follows from the definition of $A|_U$ since ω and X are extensions of $\omega|_U$ and $X|_U$.

Step 3. *For $p \in M$, the real number $A(\omega, X)(p)$ only depends on ω and X evaluated at p.*

Choose a chart (U, φ) of M with $p \in U$. Write $\omega|_U = \omega_i dx^i$ and $X|_U = X^j \partial_j$. By $C^\infty(U)$-multilinearity,

$$A(\omega, X)|_U = A|_U(\omega|_U, X|_U) = A|_U(\omega_i dx^i, X^j \partial_j) = \omega_i \cdot X^j \cdot A|_U(dx^i, \partial_j)$$

In particular,

$$A(\omega, X)(p) = \omega_i(p) \cdot X^j(p) \cdot A|_U(dx^i, \partial_j)(p), \tag{2.3.3}$$

i.e. $A(\omega, X)(p)$ depends only on $\omega(p)$ and $X(p)$.

Step 4. *A defines a tensor field $t \in \mathcal{T}_s^r(M)$ such that $A = A_t$.*

Let $p \in M$, $\xi \in T_p^*M$ and $v \in T_pM$. Pick $\omega \in \Omega^1(M)$ and $X \in \mathfrak{X}(M)$ so that

$$\omega(p) = \xi \qquad \text{and} \qquad X(p) = v.$$

Then define

$$t(p)(\xi, v) = A(\omega, X)(p).$$

By Step 3, $t(p) \in (T_pM)^1_1$ is well-defined since $A(\omega, X)(p)$ depends only on $\omega(p) = \xi$ and $X(p) = v$. The multilinearity of $t(p)$ follows from the multilinearity of A. For proving smoothness, we note that with respect to any chart (U, φ),

$$t_i^j(p) = t(p)(dx^i|_p, \partial_j|_p) = A|_U(dx^i, \partial_j)(p) \qquad (p \in U).$$

As $A|_U(dx^i, \partial_j)$ is smooth, so is t. Lastly, note that the construction of t shows

$$A_t(\omega, X)(p) = t(p)(\omega(p), X(p)) = A(\omega, X)(p). \qquad \square$$

Remark 2.3.11. (i) Note that not every tensor field (in particular function, one-form or vector field), defined on an open subset $U \subset M$ can be extended to all of M as its coefficients with respect to a chart a priori diverge as we approach the boundary of U. This is the reason why we use a smaller neighborhood V and a cutoff function χ in the above proof.

(ii) The principle of Theorem 2.3.10 can be summarized as follows: A $C^\infty(M)$-multilinear structure on M always induces a pointwise structure at each $p \in M$.

(iii) Let $A \in L^{r,s}_{C^\infty(M)}(M)$. Then the proof of Theorem 2.3.10 shows that the local coefficients of the corresponding tensor field t with respect to a chart (U, φ) are given by

$$t^{j_1,\dots,j_r}_{i_1,\dots,i_s} = A|_U(dx^{j_1}, \dots, dx^{j_r}, \partial_{i_1}, \dots, \partial_{i_s}),$$

where $A|_U$ is the natural restriction from A to U according to Step 2 of the proof of Theorem 2.3.10.

(iv) In view of Remark 2.3.4 (ii), we can adapt the proof of Theorem 2.3.10 to get a natural identification

$$\begin{aligned}\mathcal{T}^1_s(M) &\cong L^s_{C^\infty(M)}(M, \mathfrak{X}(M)) \\ &:= \{A : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{s-\text{times}} \to \mathfrak{X}(M) \mid A \text{ is } C^\infty(M)\text{-multilinear}\}\end{aligned}$$

For $A \in L^s_{C^\infty(M)}(M, \mathfrak{X}(M))$, the coefficient functions of the corresponding tensor field $t \in \mathcal{T}^1_s(M)$ with respect to a chart (U, φ) are given by

$$t^i_{j_1,\dots,j_s} = dx^i(A|_U(\partial_{j_1}, \dots, \partial_{j_s})).$$

From now on, we will always use the identifications

$$\mathcal{T}^r_s(M) \cong L^{r,s}_{C^\infty(M)}(M), \qquad \mathcal{T}^1_s(M) \cong L^s_{C^\infty(M)}(M, \mathfrak{X}(M))$$

and regard tensor fields simultaneously as objects in both spaces.

Notation 2.3.12

For $t \in \mathcal{T}^r_s(M)$, we will from now on also simplify the notation as follows. For $U \subset M$ open and $\omega_i \in \Omega^1(U)$, $X_j \in \mathfrak{X}(U)$, we write

$$t(\omega_1, \dots, \omega_r, X_1, \dots, X_s) := t|_U(\omega_1, \dots, \omega_r, X_1, \dots, X_s) \in C^\infty(U).$$

Similarly, for $p \in M$, $\xi_i \in T^*_pM$ and $v_j \in T_pM$, we write

$$t(\xi_1, \dots, \xi_r, v_1, \dots, v_s) := t(p)(\xi_1, \dots, \xi_r, v_1, \dots, v_s) \in \mathbb{R}.$$

Depending on the context, it will be clear whether we consider the tensor field on all of the manifold, on an open subset or on a point. Thus we don't have to indicate formally that we restrict it to the respective set.

Exercise 2.3.13. Prove Proposition 2.3.5.

Exercise 2.3.14. Let M be a smooth manifold and $(U, \varphi = (x^1, \dots, x^n))$ and $(V, \psi = (y^1, \dots, y^n))$ be two charts such that $U \cap V \neq \emptyset$. Find a formula which relates the coordinate tensor fields of the two charts.

Exercise 2.3.15. Let x^1 and x^2 be the standard coordintes on $\mathbb{R}^2$ and $t = dx^1 \otimes dx^1 + dx^2 \otimes dx^2$ the standard inner product. Express t in terms of polar coordinates.

Chapter 3

Semi-Riemannian manifolds

We will now introduce a central concept which will allow us to define and measure lengths, angles, and curvature on manifolds later on.

3.1 Scalar products

Definition 3.1.1: Bilinear Forms

A **bilinear form** is a (0,2)-tensor $g \in V_2^0$. We say g is

(i) **symmetric** if $g(v, w) = g(w, v)$ for all $v, w \in V$;

(ii) **non-degenerate** if $g(v, -) = 0 \iff v = 0$; that is, for all $v \neq 0$, there exists $w \in V$ with $g(v, w) \neq 0$.

(iii) a **scalar product** if it is symmetric and non-degenerate.

Remark 3.1.2. (i) Let $\{e_i\}$ be a basis of V and $\{e^i\}$ its dual basis. Given $g \in V_2^0$, we have

$$g = g_{ij} e^i \otimes e^j,$$

where $g_{ij} = g(e_i, e_j) \in \mathbb{R}$. Then g is symmetric $\iff$ the matrix $\{g_{ij}\}$ is symmetric; g is non-degenerate $\iff$ the matrix $\{g_{ij}\}$ is invertible. If g is non-degenerate, we denote the inverse matrix by

$$\{g^{ij}\} := \{g_{ij}\}^{-1}.$$

We then have

$$g_{ij} g^{jk} = (\mathrm{id})_i^k = \delta_k^i = g^{kj} g_{ji}.$$

(Here we used the einstein summation convention.)

(ii) A scalar product g induces a map

$$v^\flat := g(v, -) \in V^*.$$

It then also induces the map

$$\begin{aligned} \flat : V &\to V^* \\ v &\mapsto v^\flat. \end{aligned}$$

Since g is non-degenerate, $\flat$ is injective and thus an isomorphism of V and V^*. In this manner, we obtain a (g-dependent) identification $V \cong V^*$. Let $v = v^i e_i \in V$ and $w = v^\flat \in V^*$. Write $w = w_j e^j$. We obtain a relation between the components of v and w as follows:

$$w_i = w(e_i) = v^\flat(e_i) = g(v, e_i) = g_{ji} v^j = g_{ij} v^j, \tag{3.1.1}$$

where the last equality follows from the symmetry of g. In this manner, we can say that $\{g_{ij}\}$ "lowers the indices" of v, which explains the notation $\flat$. The inverse of $\flat$ is denoted by

$$\sharp := (\flat)^{-1}$$

and is a map $V^* \to V$. With v and w as above, we have $v = w^\sharp$. Applying the inverse matrix $\{g^{ij}\}$ to both sides of (3.1.1) yields

$$g^{ki} w_i = g^{ki} g_{ij} v^j = \delta^k_j v^j = v^k.$$

In this manner, we say that $\{g^{ij}\}$ raises the indices (explaining the notation $\sharp$).

(iii) If $g \in V^0_2$ is symmetric, there is an associated quadratic form $q : V \to \mathbb{R}$ given by $q(v) = g(v, v)$. It contains the same information as g due to the **polarization identity**

$$g(v, w) = \frac{1}{2} \left[q(v + w) - q(v) - q(w) \right].$$

Definition 3.1.3: Positive/Negative (Semi-)definite

A scalar product g on V is called

(i) **positive definite** if $g(v, v) > 0$ for all $v \neq 0$. In this case, we say g is an **inner product**.

(ii) **positive semidefinite** if $g(v, v) \geq 0$.

(iii) **negative (semi-)definite** if $-g$ is positive (semi-)definite.

We further define the **index** of g, denoted $\operatorname{ind}(g)$, by

$$\operatorname{ind}(g) := \max\{\dim W : W \subset V \text{ is a subspace and } g|_{W \times W} \text{ is negative definite}\}.$$

Note $0 \leq \operatorname{ind} g \leq n$.

Example 3.1.4. Let $V = \mathbb{R}^2$ and $\{e_1, e_2\}$ its standard basis. Let $v = v^i e_i, w = w^i e_i \in \mathbb{R}^2$. Then

(i) $g_e(v, w) := v^1 w^1 + v^2 w^2$ is positive definite and hence $\operatorname{ind}(g_{\text{eucl}}) = 0$.

(ii) $g_{\min}(v, w) := -v^1 w^1 + v^2 w^2$ has $\operatorname{ind} g_{\min} = 1$.

Theorem 3.1.5: Sylvesters law of inertia

If g is a scalar product on index ν on V, there exists a basis $\{e_i\}$ of V such that

$$g_{ij} = g(e_i, e_j) = \varepsilon_i \delta_{ij} \qquad \text{and} \qquad \varepsilon_i = \begin{cases} -1 & \text{if } 1 \leq i \leq \nu, \\ 1 & \text{if } \nu + 1 \leq i \leq n. \end{cases}$$

Proof. Linear algebra. □

Remark 3.1.6. Such a basis is called a **pseudo-orthonormal basis** . With respect to a pseudo-orthonormal basis, any $v \in V$ can be uniquely written as

$$v = \sum_{i=1}^{n} \varepsilon_i g(v, e_i) e_i.$$

Note. $\operatorname{ind} g = 0 \iff$ *g is an inner product.*

It is clear $\operatorname{ind} g = 0$ *if g is an inner product, so we will show the converse. Suppose* $\operatorname{ind} g = 0$*; we will show* $g(v, v) > 0$ *for all* $v \neq 0$*. By Theorem 3.1.5, we can pick a pseudo-orthonormal basis* $\{e_i\}$*. Since* $g \geq 0$*, it will suffice to show* $g(e_i, e_i) > 0$ *for all i. By our choice of* $\{e_i\}$*, we then get* $g(e_i, e_i) = 1$*, as desired.*

From now on, let V be equipped with a scalar product.

Definition 3.1.7: Orthogonal

Two vectors $v, w \in V$ are **orthogonal** if $g(v, w) = 0$. In this case we write $v \perp w$. For $v \in V$, we call

$$v^\perp := \{w \in V : w \perp v\}$$

the **orthogonal complement** of v. For a subspace $W \subset V$, we call

$$W^\perp := \{v \in V : v \perp w \quad \forall w \in W\}$$

the **orthogonal complement** of W.

Lemma 3.1.8

Let $W \subset V$ be a subspace. Then

(i) $\dim W + \dim W^\perp = \dim V$.

(ii) $(W^\perp)^\perp = W$.

Proof. (i) Let $k = \dim W$ and $\{e_1, \ldots, e_k\}$ a basis of W. Extend this to a basis $\{e_1, \ldots, e_n\}$ of V. With respect to this basis, we have

$$v \in W^\perp \iff g_{ji} v^j = g(v, e_i) = 0 \qquad (1 \leq i \leq k).$$

As $\{g_{ij}\}$ is invertible, the solution space to this equation has dimension $n - k$. Therefore, $\dim W^\perp = n - k = \dim V - \dim W$.

(ii) By definition, we have $W \subset (W^\perp)^\perp$. On the other hand, (i) implies $\dim W = \dim (W^\perp)^\perp$. Thus, $W = (W^\perp)^\perp$. □

Example 3.1.9. (i) If $V = \mathbb{R}^2$ is equipped with g_{eucl} and $v = (v^1, v^2) \in \mathbb{R}^2$, then $v^\perp = \mathbb{R} \cdot (v^2, -v^1)$.

(ii) If $V = \mathbb{R}^2$ is equipped with $g_{\min}$ and $v = (v^1, v^2) \in \mathbb{R}^2$, then $v^\perp = \mathbb{R} \cdot (v^2, v^1)$. In that case, it can happen that $V = V^\perp$.

Definition 3.1.10: Non-degenerate subspaces

A subspace $W \subset V$ is **non-degenerate** if $g|_{W \times W}$ is non-degenerate.

Lemma 3.1.11

The following are equivalent.

(i) $W \subset V$ is non-degenerate.

(ii) $V = W \oplus W^\perp$.

(iii) $W^\perp$ is non-degenerate.

Furthermore, if any of the above hold, we have

$$\operatorname{ind} g = \operatorname{ind}(g|_{W \times W}) + \operatorname{ind}(g|_{W^\perp \times W^\perp}).$$

Proof. **(i) ⇔ (ii):** We have

$$\dim\left(W + W^\perp\right) + \dim\left(W \cap W^\perp\right) = \dim W + \dim W^\perp = \dim V.$$

This means

$$V = W \oplus W^\perp \iff W \cap W^\perp = \{0\}.$$

Since $W \cap W^\perp = \{v \in W : g(v, w) = 0 \quad \forall w \in W\}$, the result follows.

(i) ⇔ (iii): By (i) and Lemma 3.1.8, we have

$$W \text{ is non-degenerate} \iff V = W \oplus W^\perp = \left(W^\perp\right)^\perp \oplus W^\perp \iff W^\perp \text{ is non-degenerate.}$$

Lastly, take pseudo-normal bases $\{e_i\}$ and $\{f_j\}$ of W and $W^\perp$, respectively, ordered so that

$$g(e_i, e_i) = g(f_j, f_j) = -1 \iff 1 \le i \le \operatorname{ind}(g|_{W\times W}), 1 \le j \le \operatorname{ind}(g|_{W^\perp \times W^\perp}).$$

The union of these bases is a pseudo-normal basis of $W \oplus W^\perp = V$, with the right number of scalar products being equal to -1. □

3.2 Semi-Riemannian metrics

Definition 3.2.1: (semi-)Riemannian & Lorentzian metrics & manifolds

Let M be a smooth manifold and $g \in \mathcal{T}_2^0(M)$. If for all $p \in M$, the bilinear map $g(p) : T_pM \times T_pM \to \mathbb{R}$ is a scalar product (of index ν), we call g a **semi-Riemannian metric** (of index ν) on M and write $\operatorname{ind} g = \nu$.

(i) If $\operatorname{ind} g = 0$, we call g a **Riemannian metric**.

(ii) If $\operatorname{ind} g = 1$, we call g a **Lorentzian metric**.

The pair (M, g) is called a **semi-Riemannian manifold**, **Riemannian manifold**, or **Lorentzian manifold** in the respective cases.

Example 3.2.2. (i) Equip $M = \mathbb{R}^n$ with the standard coordinates, pick $\nu \in \{1, \dots, n\}$ and let

$$\varepsilon_i = \begin{cases} -1 & \text{if } 1 \le i \le \nu, \\ 1 & \text{if } \nu < i \le n. \end{cases}$$

Then

$$g_\nu := \sum_{i,j=1}^n \varepsilon_i \delta_i^j \, dx^i \otimes dx^j$$

is a semi-Riemannian metric of index ν on $\mathbb{R}^n$. We often abbreviate the pair $(\mathbb{R}^n, g_\nu)$ as $\mathbb{R}^{\nu, n-\nu}$.

- The metric $g_{\text{eucl}} = g_0$ is called the **Euclidean metric**, $\mathbb{R}^{0,n}$ is called **Euclidean space**.
- The metric $g_{\min} = g_1$ is called the **Minkowski metric**, $\mathbb{R}^{1,n-1}$ is called **Minowski space**.

(ii) Let (M, g) be a semi-Riemannian manifold and $N \subset M$ a submanifold. Define $g|_N \in \mathcal{T}_2^0(N)$ by

$$g_N(p) := g(p)\big|_{T_pN \times T_pN} : T_pN \times T_pN \to \mathbb{R} \qquad (p \in N).$$

Then the pair $(N, g|_N)$ is a semi-Riemannian manifold if and only if $T_pN \subset T_pM$ is a non-degenerate subspace for all $p \in N$. In that case, we call $(N, g|_N)$ a **semi-Riemannian submanifold** of (M, g).

Note that if (M, g) is a Riemannian manifold, every subspace of T_pM is non-degenerate since $g(p)$ is an inner product; thus, $(N, g|_N)$ is always a Riemannian manifold in this case.

(iii) If (M, g_M) and (N, g_N) are two semi-Riemannian manifolds, we can on the manifold $M \times N$ define the **product metric** $g_{M\times N} \in \mathcal{T}_2^0(M \times N)$ as

$$g_{M\times N}(p, q) := (g_M + g_N)(p, q) := \begin{pmatrix} g_M(p) & 0 \\ 0 & g_N(q) \end{pmatrix}$$

with respect to the decomposition $T_{(p,q)}(M \times N) = T_pM \oplus T_qN$. More explicitly, this means that if $v_M, w_M \in T_pM$ and $v_N, w_N \in T_qN$ so that $v = v_M + v_N, w = w_M + w_N \in T_pM \oplus T_qN$, then

$$\begin{aligned} g_{M\times N}(p, q)(v, w) &= g_{M\times N}(v_M + v_N, w_M + w_N) \\ &= g_{M\times N}(v_M, w_M) + g_{M\times N}(v_M, w_N) + g_{M\times N}(v_N, w_M) + g_{M\times N}(v_N, w_N) \\ &= g_M(p)(v_M, w_M) + g_N(q)(v_N, w_N). \end{aligned}$$

If we additionally have a positive function $f \in C^\infty(M)$, we can define a **warped product metric**

$$(g_M + f g_N)(p,q) := \begin{pmatrix} g_M(p) & 0 \\ 0 & f(p) \cdot g_N(q) \end{pmatrix}.$$

In this case, we have

$$g_{M\times N}(p,q)(v,w) = g_M(p)(v_M, w_M) + f(p) \cdot g_N(q)(v_N, w_N).$$

Note. *In the above remark, the matrix representations of $(g_M + g_N)(p,q)$ and $(g_M + f(p)g_N)(p,q)$ do not denote linear maps into $\mathbb{R}^2$, instead they refer to the matrix representations of scalar products as defined in Section 3.1.*

Definition 3.2.3: Isometry & Isometry Group

Let (M,g) and (N,h) be semi-Riemannian manifolds and $\varphi \in C^\infty(M,N)$ be a diffeomorphism. Then φ is called an **isometry** if

$$g(v,w) = h\left(T_p\varphi(v), T_p\varphi(w)\right) \qquad (v,w \in T_pM, p \in M).$$

The set

$$\operatorname{Iso}(M,g) := \{\varphi \in \operatorname{Diff}(M) : \varphi \text{ is an isometry}\}$$

is a subgroup of $\operatorname{Diff}(M)$ (see Remark 1.3.6), called the **isometry group**.

Remark 3.2.4. Let (M,g) be a semi-Riemannian manifold and (U,φ) a chart. Then we write

$$g|_U = g_{ij}dx^i \otimes dx^j, \qquad \text{where } g_{ij} := g\left(\partial_i, \partial_j\right).$$

The functions $g_{ij} \in C^\infty(U)$ are called the **coefficient functions** of the metric g with respect to the chart (U,φ). More explicitly, we have

$$\begin{aligned} g_{ij} &: U \to \mathbb{R} \\ p &\mapsto g\left(\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p\right). \end{aligned}$$

Quite often, especially in the physics literature, g is given given in such a local form. One often abbreviates

$$\left(dx^i\right)^2 = dx^i \otimes dx^i \qquad \text{and} \qquad 2dx^i dx^j = dx^i \otimes dx^j + dx^j + \otimes dx^i$$

so that

$$g|_U = \sum_i g_{ii}\left(dx^i\right)^2 + 2\sum_{i<j} dx^i dx^j.$$

Example 3.2.5. Let $M = \mathbb{S}^2 \subset \mathbb{R}^3$, $g_{\mathbb{S}^2} := g_{\text{eucl}}|_{\mathbb{S}^2}$ and the local parametrization

$$\begin{aligned} \psi &: (0,2\pi) \times (-\pi/2, \pi/2) \to M \\ (\phi,\theta) &\mapsto \begin{pmatrix} \cos\phi\cos\theta \\ \sin\phi\cos\theta \\ \sin\theta \end{pmatrix} \end{aligned}$$

from Example 2.1.5. Recall that

$$\frac{\partial}{\partial\phi} = \begin{pmatrix} -\sin\phi\cos\theta \\ \cos\phi\cos\theta \\ 0 \end{pmatrix} \qquad \text{and} \qquad \frac{\partial}{\partial\theta} = \begin{pmatrix} -\cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix}$$

are the coordinate vector fields form the chart $(U,\varphi) := (\operatorname{im}\psi, \psi^{-1})$. Now we compute

$$\begin{aligned} g_{\phi\phi} &= g_{\mathbb{S}^2}\left(\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\phi}\right) = g_{\mathbb{R}^3}\left(\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\phi}\right) = \cos^2(\theta)\left[\sin^2\phi + \cos^2\phi\right] = \cos^2\theta \\ g_{\theta\theta} &= g_{\mathbb{S}^2}\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right) = g_{\mathbb{R}^3}\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right) = \sin^2(\theta)\left[\sin^2\phi + \cos^2\phi\right] + \cos^2\theta = 1 \\ g_{\phi\theta} = g_{\theta\phi} &= g_{\mathbb{S}^2}\left(\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\theta}\right) = g_{\mathbb{R}^3}\left(\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\theta}\right) = 0, \end{aligned}$$

which means we can write the metric in these coordinates as

$$\begin{aligned} g|_U &= d\theta^2 + \cos^2(\theta) d\phi^2 \\ &= d\theta \otimes d\theta + \cos^2\theta \cdot d\phi \otimes d\phi. \end{aligned}$$

3.3 Gradient vector fields

Throughout this section, let (M, g) be a semi-Riemannian manifold.

Notation 3.3.1

Whenever the semi-Riemannian metric g is clear from context, we write M instead of (M, g). We also write

$$\langle v, w \rangle := g(v, w) \qquad \text{and} \qquad |v| := \sqrt{|\langle v, v \rangle|}.$$

(Here we are using the convention $g(v, w) = g(p)(v, w)$ for $v, w \in T_pM$; see Notn. 2.3.12.)

Definition 3.3.2: Gradient

The **gradient** of a function $f \in C^\infty(M)$ is the vector field $\operatorname{grad} f \in \mathfrak{X}(M)$ given implicitly by the equation

$$\langle \operatorname{grad} f, X \rangle = df(X) = X(f) \qquad (X \in \mathfrak{X}(M)).$$

More explicitly, this means that for all $X \in \mathfrak{X}(M)$ and all $p \in M$, we have

$$\begin{aligned} \langle (\operatorname{grad} f)(p), X(p) \rangle = g(p)\big((\operatorname{grad} f)(p), X(p)\big) &= df(X)(p) \\ &= df|_p(X(p)) \\ &= T_p f(X(p)). \end{aligned}$$

From now on we write $\operatorname{grad} f(p)$ instead of $(\operatorname{grad} f)(p)$.

Recall. *If $\langle -, - \rangle$ is a scalar product on a vector space V, then we have a g-dependent isomorphism*

$$\begin{aligned} \flat : V &\xrightarrow{\sim} V^* \\ v &\mapsto \langle v, - \rangle =: v^\flat \end{aligned}$$

with inverse

$$\begin{aligned} \sharp : V^* &\xrightarrow{\sim} V \\ f &\mapsto f^\sharp, \end{aligned}$$

where $f^\sharp \in V$ is such that $f(-) = \langle f^\sharp, - \rangle$; ie, such that

$$f(v) = \langle f^\sharp, v \rangle \qquad (v \in V).$$

(See Remark 3.1.2(ii).)

Going back to Defn. 3.3.2, we see that $\operatorname{grad} f$ is such that $\operatorname{grad} f(p) = \left(df|_p\right)^\sharp$ for all $p \in M$. If (U, φ) is a chart of M, we have

$$\operatorname{grad} f|_U = g^{ij} \partial_i f \partial_j,$$

where $\{g^{ij}\}$ is the inverse matrix of $\{g_{ij}\}$. To see this, write

$$\begin{aligned} g_{ij} &= g\left(\partial_i, \partial_j\right) \\ \operatorname{grad} f|_U &= X^i \partial_i \\ df &= \omega_j dx^j. \end{aligned}$$

Observe that

$$\omega_j(p) = df|_p(\partial_j) = \frac{\partial f}{\partial x^j}(p) = \partial_j f(p)$$

so that $\omega_j = \partial_j f$. By Remark 3.1.2(ii), $g^{ij}\omega_j = X^i$; hence,

$$\operatorname{grad} f|_U = X^i\partial_i = g^{ij}\omega_j\partial_i = g^{ij}\partial_j f\partial_i.$$

Lemma 3.3.3

Let $f \in C^\infty(M)$ and assume $\operatorname{grad} f(p) \neq 0$ for all $p \in M$.

(i) For all $c \in \mathbb{R}$, the level set $N_c := f^{-1}(c)$ is a submanifold of codimension 1 and $T_pN_c = (\operatorname{grad} f(p))^\perp$ for all $p \in N_c$.

(ii) The pair $(N_c, g|_{N_c})$ is a semi-Riemannian submanifold if and only if $|\operatorname{grad} f(p)| \neq 0$ for all $p \in N_c$.

(iii) If $(N_c, g|_{N_c})$ is a semi-Riemannian submanifold, then

$$\operatorname{ind} g|_{N_c} = \begin{cases} \operatorname{ind} g & \text{if } \langle \operatorname{grad} f(p), \operatorname{grad} f(p)\rangle > 0 \text{ on } N_c, \\ \operatorname{ind} g - 1 & \text{if } \langle \operatorname{grad} f(p), \operatorname{grad} f(p)\rangle < 0 \text{ on } N_c. \end{cases}$$

Proof.

(i) If $\operatorname{grad} f(p) \neq 0$ for all $p \in M$, the function $f - c$ is a submersion. Therefore, by Theorem 1.5.7, N_c is a submanifold of codimension 1 for every $c \in \mathbb{R}$. By Remark 1.5.10, the tangent space is given by

$$T_pN_c = \ker(df|_p) = \ker(\langle \operatorname{grad} f(p), -\rangle) = (\operatorname{grad} f(p))^\perp.$$

(ii) By Lemma 3.1.11(ii), $T_pN_c \subset T_pM$ is non-degenerate if and only if $(T_pN_c)^\perp = \mathbb{R}\cdot \operatorname{grad} f(p) \subset T_pM$ is non-degenerate. The latter holds precisely when $|\operatorname{grad} f(p)| \neq 0$.

(iii) By Lemma 3.1.11(iii), we have in this situation

$$\operatorname{ind} g(p) = \operatorname{ind} g(p)\big|_{T_pN_c\times T_pN_c} + \operatorname{ind} g(p)\big|_{(T_pN_c)^\perp\times(T_pN_c)^\perp}.$$

Because $(T_pN_c)^\perp = \mathbb{R}\cdot\operatorname{grad} f(p)$, we have

$$\operatorname{ind} g(p)\big|_{T_pN_c\times T_pN_c} = \begin{cases} \operatorname{ind} g & \text{if } \langle \operatorname{grad} f(p), \operatorname{grad} f(p)\rangle > 0 \text{ on } N_c, \\ \operatorname{ind} g - 1 & \text{if } \langle \operatorname{grad} f(p), \operatorname{grad} f(p)\rangle < 0 \text{ on } N_c. \end{cases}$$ □

Example 3.3.4. Consider $\mathbb{R}^{\nu,n-\nu}$ and the function $f_\nu \in C^\infty(\mathbb{R}^n)$, given by

$$f_\nu(x) := g_\nu(x,x) = \sum_{i=1}^n \varepsilon_i (x^i)^2 = -(x^1)^2 - \cdots - \left(x^{\nu-1}\right)^2 + (x^\nu)^2 + \cdots + (x^n)^2.$$

In this definition, we implicitly used the identification $T_x\mathbb{R}^n \cong \mathbb{R}^n$ and standard coordinates. Now again in standard coordinates, we have

$$df_\nu = \sum_{i=1}^n 2\varepsilon_i x^i dx^i$$

and the gradient is

$$\operatorname{grad} f_\nu = \sum_{i,j=1}^n 2\,(g_\nu)^{ij}\,\varepsilon_i x^i\partial_j = 2\sum_{i,j=1}^n \varepsilon_i\delta^{ij}\varepsilon_i x^i\partial_j = 2\sum_{j=1}^n x^j\partial_j$$

and the right side is not vanishing on $\mathbb{R}^n \setminus \{0\}$. Now we get

$$\langle \operatorname{grad} f_\nu(x), \operatorname{grad} f_\nu(x)\rangle = 4\sum_{j=1}^n \varepsilon_j \left(x^j\right)^2 = 4f_\nu(x).$$

Now we can apply Lemma 3.3.3 to the function $f_\nu|_{\mathbb{R}^n\setminus\{0\}}$ and its level sets $N_c = f^{-1}(c)$ which are subsets of $\mathbb{R}^n \setminus \{0\}$ for $c \neq 0$. Let $r > 0$. We obtain the following:

(i) $\mathbb{S}^{n-1}_\nu(r) := \left(N_{r^2}, g_\nu|_{N_{r^2}}\right)$ is a semi-Riemannian manifold of index ν, called **pseudo-sphere** of radius r and index ν.

(ii) $\mathbb{H}^{n-1}_{\nu-1}(r) : (N_{-r^2}, g_\nu|_{-r^2})$ is a semi-Riemannian manifold of index $\nu - 1$, called **pseudo-hyperbolic space** of radius r and index $\nu - 1$.

As some interesting special cases of these, we have the following:

(i) The Riemannian manifold $\mathbb{S}^2_0(1) \subset \mathbb{R}^{0,3}$ is the standard two-dimensional sphere.

(ii) The Riemannian manifold $\mathbb{H}^2_0(1) \subset \mathbb{R}^{0,3}$ is the hyperbolic plane.

(iii) The Lorentzian manifold $\mathbb{S}^4_1(1) \subset \mathbb{R}^{1,4}$ is the de-Sitter space. It is a model in cosmology.

(iv) The Lorentzian manifold $\mathbb{H}^4_1(1) \subset \mathbb{R}^{2,3}$ is the anti de-Sitter space. It is used in the so-called AdS-CFT correspondence.

3.4 Riemannian and Lorentzian Manifolds

Definition 3.4.1: Length of Curve

Let M be a Riemannian manifold and $c : [a, b] \to M$ be a C^∞ map. Then the **length** of c is

$$L(c) := \int_a^b |c'(t)|\, dt.$$

If M is connected, we define a function

$$\begin{aligned} d : M \times M &\to \mathbb{R} \\ (p, q) &\mapsto \inf \left\{L(c) : c \in C^\infty([0, 1], M), c(0) = p, c(1) = q\right\} \end{aligned}$$

(i) $L(c)$ is a reparametrization invariant quantity. That is, if $\varphi : [c, d] \to [a, b]$ is a diffeomorphism, then $L(c \circ \varphi) = L(c)$ (integral substitution).

(ii) One can show (M, d) is a metric space. It is complete if and only if the infimum in the definition of the d is always attained. For example, with the function d induced by g_{eucl}, $\mathbb{R}^2$ is a complete metric space but $M = \mathbb{R}^2 \setminus \{0\}$ is not.

Theorem 3.4.2

Every C^∞ manifold admits a Riemannian metric.

Definition 3.4.3: Partition of Unity

Let M be a C^∞ manifold and $\mathscr{A} = \{(U_i, \varphi_i) : i \in I\}$ an atlas. A family of nonnegative smooth functions $\{\chi_i\}_{i\in I}$, where each $\chi_i \in C^\infty(M)$, is called a **partition of unity** subordinate to $\mathscr{A}$ if

(i) $\operatorname{supp}(\chi_i) \subset U_i$ for all i,

(ii) For all $p \in M$, there exists a neighborhood V such that $V \cap \operatorname{supp} \chi_i \neq \emptyset$ for only finitely many i.

(iii) $\sum_{i\in I} \chi_i(p) = 1$.

Such a partition always exists because M is Hausdorff.

Proof of Theorem 3.4.2. Take an atlas $\mathscr{A} = \{(U_i, \varphi_i) : i \in I\}$ and a partition of unity $\{\chi_i\}_{i\in I}$ subordinate to $\mathscr{A}$. For all $i \in I$, define

$$g_i = \sum_{j=1}^n dx^j \otimes dx^j \in \mathscr{T}^0_2(U_i).$$

(Note that the definition of g_i above depends on the chart U_i.) Then g_i is a Riemannian metric on U_i. If we extend by 0 on $M \setminus U_i$, we see that $\chi_i g_i \in \mathcal{T}_2^0(M)$ is positive semidefinite on M. Set

$$g = \sum_{i \in I} \chi_i g_i.$$

We claim g is a Riemanninan metric. As $\{\chi_i\}$ is a partition of unity and $\chi_i g_i \in \mathcal{T}_2^0(M)$ for all i, we have $g \in \mathcal{T}_2^0(M)$. Since each g_i is a positive semi-definite scalar product on U_i and $\{\chi_i\}$ is a partition of unity, we have that $g(p)$ is a positive semi-definite scalar product for all $p \in M$; it remains to show $g(p)$ is positive definite. To that end, let $p \in M$. Note $\chi_i(p) \geq 0$ for all i with $\chi_{i_0}(p) > 0$ for some $i_0 \in I$ where $p \in U_i$. This implies that for all $v \in T_p M \setminus \{0\}$,

$$g(p)(v, v) = \sum_{i \in I} \chi_i(p) g_i(p)(v, v) \geq \chi_{i_0}(p) g_{i_0}(p)(v, v) > 0,$$

where the last inequality follows from $g_{i_0}(p)$ being positive definite on U_i. Hence, $g(p)$ is positive definite. As p is arbitrary, we see that g is a Riemannian metric. □

Definition 3.4.4: Lorentzian metric definitions

Let M be a Lorentzian manifold and $v \in TM$. Then v is called

(i) **timelike** if $\langle v, v \rangle < 0$.

(ii) **lightlike** if $\langle v, v \rangle = 0$ but $v \neq 0$.

(iii) **spacelike** if $\langle v, v \rangle > 0$ or $v = 0$.

(iv) **causal** if it is timelike or lightlike.

$c \in C^\infty(I, M)$ is **timelike/spacelike/lightlike/causal** if $c'(t)$ is timelike/spacelike/lightlike/causal for all $t \in I$.

Recall. *For $c \in C^\infty(I, M)$, we define $c'(t) = T_t c \left(\frac{\partial}{\partial t}\big|_t \right) = \sum_{i=1}^n \left(c^i\right)' \partial_i$.*

Remark 3.4.5 (Physical interpretation)**.** Lorentzian manifolds model spacetimes. Massive particles move along lightlike curves. Light moves around lightlike curves. Physical observers move along causal curves. We say

$$P(c) = \int_I |c'(t)| \, dt$$

is the **proper time** of the observer along the causal curve.

Example 3.4.6 (Physically relevant Lorentzian manifolds)**.** (i) $M = \mathbb{R} \times (2m, \infty) \times \mathbb{S}^2$ equipped with the **Schwarzschild metric** given by

$$-\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 g_{\mathbb{S}^2},$$

where $t \in \mathbb{R}, r \in (2m, \infty)$, models the exterior of a static black hole with mass m without charge or angular momentum.

(ii) Let (M, g) be a Riemannian manifold, $I \subset \mathbb{R}$ open interval, $f \in^\infty (I)$ positive. Then $(I \times M, dt^2 + f(t)g)$ is a Lorentzian manifold. These manifolds are called Friedman-Lemaitre-Robertson-Walker spacetimes in cosmology.

Theorem 3.4.7: Existence of Lorenzian metrics

Let M be a C^∞ manifold. The following are equivalent.

(i) M admits a Lorenzian metric.

(ii) There exists $X \in \mathfrak{X}(M)$ such that $X(p) \neq 0$ for all $p \in M$.

(iii) M is non-compact or M is compact and has Euler-characteristic $\chi(M) = 0$.

Proof. See the O'Neill text. □

Remark 3.4.8. $\mathbb{S}^2$ does not admit a Lorentzian metric since (ii) does not hold due to the Hairy ball theorem.

Chapter 4

The Covariant Derivative

Goal: We want to take the derivative of vector fields in the direction of other vector fields.

Naive approach: Write $X, Y \in \mathfrak{X}(M)$ in local coordinates as $X = X^i \partial_i, Y = Y^j \partial_j$ and write $\partial_X Y = X^i \partial_i Y^j \partial_j$. However, this is not independent of the choice of coordinates and hence not well-defined!

4.1 Definition of the covariant derivative

Throughout this section, let M be a fixed semi-Riemannian manifold.

Definition 4.1.1: Connections

A **connection** of M is a map

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ (X, Y) &\mapsto \nabla_X Y \end{aligned}$$

such that for all $X, X_i, Y, Y_i \in \mathfrak{X}(M), \alpha \in \mathbb{R}, f \in C^\infty(M)$,

(i) $\nabla_{(X_1 + f X_2)} Y = \nabla_{X_1} Y + f \nabla_{X_2} Y$,

(ii) $\nabla_X (Y_1 + \alpha Y_2) = \nabla_X Y_1 + \alpha \nabla_X Y_2$,

(iii) $\nabla_X (f Y) = X(f) Y + f \nabla_X Y$.

If, in addition

(iv)
$$\nabla_X Y - \nabla_Y X = [X, Y] \qquad (X, Y \in \mathfrak{X}(M)), \tag{4.1.1}$$
∇ is called **torsion-free**.

(v)
$$X(\langle Y, Z \rangle) = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \qquad (X, Y, Z \in \mathfrak{X}(M)), \tag{4.1.2}$$
∇ is called **metric**.

Conditions (i)-(iii) say ∇ is $C^\infty(M)$-linear in the first slot, $\mathbb{R}$-linear in the second slot, and satisfies a sort of product rule.

Recall. *$X(f)$ is the smooth function defined by $X(f)(p) = T_p f\,(X(p))$.*

Theorem 4.1.2: Existence and Uniqueness of Torson-free metric connections

There is exactly one torsion-free and metric connection on M. It is implicitly given by the **Koszul formula**

$$2\langle \nabla_X Y, Z \rangle = X(\langle Y, Z \rangle) + Y(\langle X, Z \rangle) - Z(\langle X, Y \rangle) + \langle Z, [X, Y] \rangle + \langle Y, [Z, X] \rangle - \langle X, [Y, Z] \rangle. \tag{4.1.3}$$

Proof. **Uniqueness:** By (4.1.2), we have

$$X(\langle Y,Z\rangle) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle. \tag{4.1.4}$$

By (4.1.1) and (4.1.2), we have

$$Y(\langle Z,X\rangle) = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_X Y\rangle - \langle Z, [X,Y]\rangle \tag{4.1.5}$$
$$Z(\langle X,Y\rangle) = \langle \nabla_X Z, Y\rangle + \langle [Z,X], Y\rangle + \langle X, \nabla_Y Z\rangle - \langle X, [Y,Z]\rangle. \tag{4.1.6}$$

The equation (4.1.4) + (4.1.5) − (4.1.6) implies (4.1.3). More explicitly, we have

$$\begin{aligned} X(\langle Y,Z\rangle) + Y(\langle Z,X\rangle) - Z(\langle X,Y\rangle) &= \left(\langle \nabla_X Y, Z\rangle + \cancel{\langle Y, \nabla_X Z\rangle}\right) + \left(\cancel{\langle \nabla_Y Z, X\rangle} + \langle Z, \nabla_X Y\rangle - \langle Z, [X,Y]\rangle\right) \\ &\quad - \left(\cancel{\langle \nabla_X Z, Y\rangle} + \langle [Z,X], Y\rangle + \cancel{\langle X, \nabla_Y Z\rangle} - \langle X, [Y,Z]\rangle\right) \\ &= \langle \nabla_X Y, Z\rangle + \langle Z, \nabla_X Y\rangle - \langle Z, [X,Y]\rangle - \langle [Z,X], Y\rangle + \langle X, [Y,Z]\rangle \\ &= 2\langle \nabla_X Y, Z\rangle - \langle Z, [X,Y]\rangle - \langle [Z,X], Y\rangle + \langle X, [Y,Z]\rangle. \end{aligned}$$

Thus,

$$X(\langle Y,Z\rangle) + Y(\langle Z,X\rangle) - Z(\langle X,Y\rangle) + \langle Z, [X,Y]\rangle + \langle [Z,X], Y\rangle - \langle X, [Y,Z]\rangle = 2\langle \nabla_X Y, Z\rangle,$$

which is equivalent to (4.1.3) since $\langle -,-\rangle$ is symmetric.

Existence: Define $\nabla_X Y$ by (4.1.3) and check all properties. □

Definition 4.1.3: Covariant derivative (Levi-Civita connection)

The torson-free and metric connection on M satisfying (4.1.3) is called the **covariant derivative** or the **Levi-Civita connection**.

Remark 4.1.4 (Locality). (i) The Koszul formula shows that ∇ is local in the sense that $(\nabla_X Y)|_U = \nabla_{X|_U} Y|_U$ for any open subset $U \subset M$. This holds because $\langle -,-\rangle$ and $[-,-]$ are also local.

(ii) If (U,φ) is a chart on M and $X|_U = X^i\partial_i$, $Y|_U = Y^j\partial_j$, then

$$\begin{aligned} \nabla_X Y|_U = \nabla_{X|_U} Y|_U &= \nabla_{X^i\partial_i}\left(Y^j\partial_j\right) \\ &= X^i \nabla_{\partial_i}\left(Y^j\partial_j\right), && \text{by Defn. 4.1.1(i)} \\ &= X^i\partial_i\left(Y^j\right)\partial_j + X^i Y^j \nabla_{\partial_i}\partial_j, && \text{by Defn. 4.1.1(ii) \& (iii).} \end{aligned}$$

If we define functions $\Gamma_{ij}^k : U \to \mathbb{R}$ so that $\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k\partial_k \in \mathfrak{X}(M)$, this reads

$$\nabla_X Y|_U = \left(X^i\partial_i Y^k + X^i Y^j \Gamma_{ij}^k\right)\partial_k. \tag{4.1.7}$$

(iii) Let $c : I \to M$ is a smooth curve satisfying $c(0) = p$ and $c'(0) = X(p)$. Then the chain rule (Prop. 1.4.9(ii)) shows

$$\begin{aligned} X^i(p)\partial_i Y^k\big|_p = X^i(p) T_p Y^k\left(\partial_i\big|_p\right) &= T_p Y^k\left(X^i(p)\,\partial_i\big|_p\right) = T_p Y^k\left(c'(0)\right) \\ &= \left(Y^k \circ c\right)'(0). \end{aligned}$$

Combining the above with (4.1.7), we obtain

$$\nabla_X Y(p) = \left[\left(Y^k\circ c\right)'(0) + \left(X^i\circ c\right)(0)\left(Y^j\circ c\right)(0)\Gamma_{ij}^k\left(c(0)\right)\right]\partial_k\big|_{c(0)}.$$

Thus, (4.1.7) implies that $\nabla_X Y(p)$ only depends on $X(p)$ and the values of Y along a curve c satisfying $c(0) = p$ and $c'(0) = X(p)$.

Definition 4.1.5: Christoffel symbols (Γ_{ij}^k)

The functions $\Gamma_{ij}^k : U \to \mathbb{R}$ in Remark 4.1.4(ii) are called the **Christoffel symbols** with respect to the chart (U, φ).

Lemma 4.1.6

Let (U, φ) be a chart of M and $g_{ij} : U \to \mathbb{R}$ be the coefficient functions of the metric. Then we have

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\ell} \left(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij} \right). \tag{4.1.8}$$

Proof. Since $\left[\partial_i, \partial_j\right] \equiv 0$ for coordinate vector fields, (4.1.1) shows $\nabla_{\partial_i} \partial_j = \nabla_{\partial_j} \partial_i$. Since ∇ is metric and $\langle -, - \rangle$ is symmetric, we get

$$\begin{aligned}
2 \left\langle \nabla_{\partial_i} \partial_j, \partial_\ell \right\rangle + \partial_\ell \left\langle \partial_i, \partial_j \right\rangle &= \left\langle \nabla_{\partial_i} \partial_j, \partial_\ell \right\rangle + \left\langle \nabla_{\partial_i} \partial_j, \partial_\ell \right\rangle + \left\langle \nabla_{\partial_\ell} \partial_i, \partial_j \right\rangle + \left\langle \partial_i, \nabla_{\partial_\ell} \partial_j \right\rangle \\
&= \left\langle \nabla_{\partial_i} \partial_j, \partial_\ell \right\rangle + \left\langle \nabla_{\partial_j} \partial_i, \partial_\ell \right\rangle + \left\langle \partial_j, \nabla_{\partial_i} \partial_\ell \right\rangle + \left\langle \partial_i, \nabla_{\partial_j} \partial_\ell \right\rangle \\
&= \partial_i \left\langle \partial_j, \partial_\ell \right\rangle + \partial_j \left\langle \partial_i, \partial_\ell \right\rangle .
\end{aligned}$$

We have thus shown $2 \left\langle \nabla_{\partial_i} \partial_j, \partial_\ell \right\rangle + \partial_\ell g_{ij} = \partial_i g_{j\ell} + \partial_j g_{i\ell}$. Inserting $X = \partial_i$, $Y = \partial_j$, $Z = \partial_\ell$ in (4.1.3) yields

$$2 \Gamma_{ij}^m g_{m\ell} = 2 \left\langle \Gamma_{ij}^m \partial_m, \partial_\ell \right\rangle = 2 \left\langle \nabla_{\partial_i} \partial_j, \partial_\ell \right\rangle = \partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}.$$

Multiplying with $g^{k\ell}$ yields

$$2 \Gamma_{ij}^k = 2 \delta_m^k \Gamma_{ij}^m = 2 g^{k\ell} g_{m\ell} \Gamma_{ij}^m = g^{k\ell} \left(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij} \right). \qquad \square$$

Example 4.1.7. The Christoffel symbols of $\mathbb{R}^{\nu, n-\nu}$ with respect to the standard chart are $\Gamma_{ij}^k \equiv 0$ for all i, j, k This is because $g_{ij} \equiv \text{const}$ for all i, j so that by (4.1.7),

$$\nabla_X Y(x) = X^i (\partial_i Y^j) \partial_j = DY|_x(X).$$

4.2 The Covariant Derivative of Submanifolds

Throughout, let $\left(\overline{M}, \overline{g}\right)$ be a semi-Riemannian manifold and (M, g) a semi-Riemannian submanifold of $\left(\overline{M}, \overline{g}\right)$. The goal will be to compare the Levi-Civita connections $\overline{\nabla}, \nabla$ of these manifolds at $p \in M \subset \overline{M}$. For $p \in M$, $T_p M \subset T_p \overline{M}$ is a non-degenerate subspace. Thus by Lemma 3.1.11(i),

$$T_p \overline{M} = T_p M \oplus \left(T_p M\right)^\perp . \tag{4.2.1}$$

This means that if $v \in T_p \overline{M}$, then v may be decomposed as $v = (v^\top, v^\perp)$

Definition 4.2.1: Vector fields along M, normal fields, & extensions

(i) The space of **vector fields in $\overline{M}$ along M** is

$$\mathfrak{X}\left(M,\overline{M}\right) := \left\{X \in C^\infty\left(M,T\overline{M}\right) : X(p) \in T_p\overline{M} \text{ for all } p \in M\right\}$$

(ii) The space of **normal fields on M** is

$$\mathfrak{X}\left(M\right)^\perp := \left\{X \in C^\infty\left(M,T\overline{M}\right) : X\left(p\right) \in \left(T_pM\right)^\perp \text{ for all } p \in M\right\}$$

(iii) Let $X \in \mathfrak{X}\left(M,\overline{M}\right)$. A vector field $\overline{X} \in \mathfrak{X}\left(\overline{M}\right)$ is called an **extension** of X if $\overline{X}|_M = X$. Such an extension always exists (use local coordinates and partition of unity).

Note that (4.2.1) induces a splitting

$$\begin{aligned}\mathfrak{X}\left(M,\overline{M}\right) &= \mathfrak{X}\left(M\right) \oplus \mathfrak{X}\left(M\right)^\perp \\ X &\mapsto \left(X^\top, X^\perp\right)\end{aligned}$$

Definition 4.2.2: $\overline{\nabla}_X Y$

Let $X \in \mathfrak{X}\left(M\right), Y \in \mathfrak{X}\left(M,\overline{M}\right)$. Then we define $\overline{\nabla}_X Y \in \mathfrak{X}\left(M,\overline{M}\right)$ by

$$\overline{\nabla}_X Y := \overline{\nabla}_{\overline{X}}\overline{Y}\big|_M,$$

where $\overline{X},\overline{Y} \in \mathfrak{X}\left(\overline{M}\right)$ are extensions of X and Y.

By Remark 4.1.4(iii), $\overline{\nabla}_X Y$ does not depend on the chosen extension.

Lemma 4.2.3

For all $X, X_1, X_2 \in \mathfrak{X}\left(M\right), Y, Y_1, Y_2, Z \in \mathfrak{X}\left(M,\overline{M}\right), \alpha \in \mathbb{R}, f \in C^\infty(M)$, the map in Defn. 4.2.2 satisfies

(i) $\overline{\nabla}_{X_1+fX_2} Y = \overline{\nabla}_{X_1} Y + f\overline{\nabla}_{X_2} Y$,

(ii) $\overline{\nabla}_X\left(Y_1 + \alpha Y_2\right) = \overline{\nabla}_X Y_1 + \alpha\overline{\nabla}_X Y_2$,

(iii) $\overline{\nabla}_X(fY) = X(f)Y + f\overline{\nabla}_X Y$,

(iv) $X\left(\overline{g}\left(Y,Z\right)\right) = \overline{g}\left(\overline{\nabla}_X Y, Z\right) + \overline{g}\left(Y, \overline{\nabla}_X Z\right)$.

Proof. Choose extensions of the vector fields, use the rules of the Levi-Civita connection of $\overline{g}$, and restrict to M. □

Lemma 4.2.4

Let $X,Y \in \mathfrak{X}(M)$ and $\overline{X},\overline{Y} \in \mathcal{X}\left(\overline{M}\right)$ be extensions of X,Y respectively. Then $[\overline{X},\overline{Y}]|_M = [X,Y]$; that is, $[\overline{X},\overline{Y}] \in \mathfrak{X}(\overline{M})$ is an extension of $[X,Y] \in \mathfrak{X}(M)$ and $[\overline{X},\overline{Y}]|_M \in \mathfrak{X}(M) \subset \mathfrak{X}\left(M,\overline{M}\right)$.

Proof. Exercise. □

Lemma 4.2.5

Let $X, Y \in \mathfrak{X}(M)$ and decompose $\overline{\nabla}_X Y = \left(\overline{\nabla}_X Y\right)^\top + \left(\overline{\nabla}_X Y\right)^\perp$. Then

(i) $\left(\overline{\nabla}_X Y\right)^\top = \nabla_X Y$ (the Levi-Civita connection on M).

(ii) $(X, Y) \mapsto \left(\overline{\nabla}_X Y\right)^\perp$ is $C^\infty(M)$-bilinear and symmetric.

Proof. (i) We will show that

$$\begin{aligned} \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ (X, Y) &\mapsto \left(\overline{\nabla}_X Y\right)^\top \end{aligned}$$

is a torsion-free and metric connection on M; this will imply $\left(\overline{\nabla}_X Y\right)^\top = \nabla_X Y$ by uniqueness. Apply $(-)^\top$ to Lemma 4.2.3(i-iii) to see that the map is a connection. If $\overline{X}, \overline{Y} \in \mathfrak{X}\left(\overline{M}\right)$ are extensions of $X, Y \in \mathfrak{X}(M)$, we have by Lemma 4.2.4,

$$\overline{\nabla}_X Y - \overline{\nabla}_Y X = \left(\overline{\nabla}_{\overline{X}} \overline{Y} - \overline{\nabla}_{\overline{Y}} \overline{X}\right)\big|_M = \left[\overline{X}, \overline{Y}\right]\big|_M = [X, Y].$$

Thus,

$$\left(\overline{\nabla}_X Y\right)^\top - \left(\overline{\nabla}_Y X\right)^\top = [X, Y]^\top = [X, Y].$$

This shows the connection is torsion-free. If in addition, $\overline{Z} \in \mathfrak{X}\left(\overline{M}\right)$ is an extension of $Z \in \mathfrak{X}(M)$, then

$$\begin{aligned} X\left(\overline{g}(Y, Z)\right) = \overline{X}\left(\overline{g}\left(\overline{Y}, \overline{Z}\right)\right)\big|_M &= \overline{g}\left(\overline{\nabla}_{\overline{X}} \overline{Y}, \overline{Z}\right)\big|_M + \overline{g}\left(\overline{Y}, \overline{\nabla}_{\overline{X}} \overline{Z}\right)\big|_M \\ &= \overline{g}\left(\overline{\nabla}_X Y, Z\right) + \overline{g}\left(Y, \overline{\nabla}_X Z\right) \\ &= \overline{g}\left(\left(\overline{\nabla}_X Y\right)^\top + \left(\overline{\nabla}_X Y\right)^\perp, Z\right) + \overline{g}\left(Y, \left(\overline{\nabla}_X Z\right)^\top + \left(\overline{\nabla}_X Z\right)^\perp\right) \\ &= \overline{g}\left(\left(\overline{\nabla}_X Y\right)^\top, Z\right) + \overline{g}\left(Y, \left(\overline{\nabla}_X Z\right)^\top\right). \end{aligned}$$

This shows the connection is metric.

(ii) By (i), we have $\overline{\nabla}_X Y - \overline{\nabla}_Y X = [X, Y]$ so that

$$\left(\overline{\nabla}_X Y\right)^\perp - \left(\overline{\nabla}_Y X\right)^\perp = [X, Y]^\perp = 0;$$

this shows symmetry. By Lemma 4.2.3(i), the map $X \mapsto \left(\overline{\nabla}_X Y\right)^\perp$ is C^∞-linear. By the symmetry we have just shown, the map $Y \mapsto \left(\overline{\nabla}_X Y\right)^\perp$ is also C^∞-linear. □

Definition 4.2.6: Second Fundamental Form

The symmetric $C^\infty(M)$-bilinear map

$$\begin{aligned} \Pi : \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M)^\perp \\ (X, Y) &\mapsto \left(\overline{\nabla}_X Y\right)^\perp \end{aligned}$$

is called the **second fundamental form**.

By $C^\infty(M)$-multilinearity we see that for all $p \in M$, it induces a pointwise map

$$\Pi(p) : T_pM \times T_pM \to (T_pM)^\perp .$$

(The proof is similar to the proof of Theorem 2.3.10.) We will see that it measures how M is "curved" in $\overline{M}$.

Note. *Combining Defn. 4.2.6 with the ccontent of Lemma 4.2.5, we see that* $\overline{\nabla}_X Y$ *decomposes as*

$$\overline{\nabla}_X Y = \nabla_X Y + \Pi(X, Y).$$

Remark 4.2.7. Lemma 4.2.5 a posteriori explains the definition of the Levi-Civita connection: If $M \subset \mathbb{R}^n$ is a submanifold equipped with the metric $g = g_{\mathrm{eucl}}|_M$, we could have defined the Levi-Civita connection for $X, Y \in \mathfrak{X}(M)$ as

$${}^g\nabla_X Y(p) := \left({}^{\mathbb{R}^n}\overline{\nabla}_{\overline{X}}\overline{Y}\right)^\top(p) = \left[D\overline{Y}|_p\left(\overline{X}(p)\right)\right]^\top,$$

where $\overline{X}, \overline{Y} \in \mathfrak{X}(\mathbb{R}^n)$ are extensions of X, Y respectively. (See Example 4.1.7.) For example, if $p \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$, we have $v^\top(p) = v - \langle v, p\rangle p \in T_p\mathbb{S}^{n-1}$ for $v \in \mathbb{R}^n$. In particular, if $v = {}^{\mathbb{R}^n}\overline{\nabla}_{\overline{X}}\overline{Y}$, we get

$${}^{\mathbb{S}^{n-1}}\nabla_X Y(p) = \left({}^{\mathbb{R}^n}\overline{\nabla}_{\overline{X}}\overline{Y}\right)^\top(p) = D\overline{Y}|_p\left(\overline{X}(p)\right) - \left\langle D\overline{Y}|_p\left(\overline{X}(p)\right), p\right\rangle p.$$

This was the first definition of the Levi-Civita connection. Our (axiomatic) definition has the advantage that it does not need an embedding into $\mathbb{R}^n$.

Definition 4.2.8: Weingarten map (Shape Operator)

For a normal field $\xi \in \mathfrak{X}(M)^\perp$, the **Weingarten map** (or **shape operator**) $S^\xi \in \mathcal{T}_1^{\,1}(M)$ is defined by the equation

$$\overline{g}(\Pi(X, Y), \xi) = g\left(S^\xi(X), Y\right).$$

(Here, S^ξ is considered as a map $S^\xi : \mathfrak{X}(M) \to \mathfrak{X}(M)$.)

Proposition 4.2.9

The Weingarten map fulfills the **Weingarten equation**: For $X \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{X}(M)^\perp$, we have

$$\left(\overline{\nabla}_X \xi\right)^\top = -S^\xi(X). \tag{4.2.2}$$

Proof. Let $Y \in \mathfrak{X}(M)$ be arbitrary. Then

$$\begin{aligned}
g\left(\left(\overline{\nabla}_X \xi\right)^\top, Y\right) &= \overline{g}\left(\overline{\nabla}_X \xi, Y\right), && \text{since } Y \text{ is tangential} \\
&= X\left(\overline{g}(\xi, Y)\right) - \overline{g}\left(\xi, \overline{\nabla}_X Y\right), && \text{since } \overline{\nabla} \text{ is metric} \\
&= -\overline{g}\left(\xi, \overline{\nabla}_X Y\right), && \text{since } \xi \perp Y \\
&= -\overline{g}\left(\xi, \left(\overline{\nabla}_X Y\right)^\perp\right), && \text{since } \xi \in (\mathfrak{X}(M))^\perp \\
&= -g\left(S^\xi(X), Y\right).
\end{aligned}$$

□

Definition 4.2.10: Semi-Riemannian Hypersurfaces

A subset $M \subset \overline{M}$ is called a **semi-Riemannian hypersurface** if $\dim \overline{M} = \dim M + 1$.

Lemma 4.2.11

If $M \subset \overline{M}$ is a semi-Riemannian hypersurface, then for each $p \in M$, there exists a neighborhood $U \subset M$ of p and up to sign a unique $\xi \in \mathfrak{X}(U)^{\perp}$ such that

(i) $|\overline{g}(\xi, \xi)| = 1$.

Such a ξ is called a **unit normal field**. It also satisfies

(ii) $\left(\overline{\nabla}_X \xi\right)^{\perp} = 0$ for all $X \in \mathfrak{X}(U)$,

(iii) $S^{\xi}(X) = -\overline{\nabla}_X \xi$ for all $X \in \mathfrak{X}(U)$,

(iv) $\Pi(X, Y) = -\overline{g}(\xi, \xi) \cdot g\left(\overline{\nabla}_X \xi, Y\right) \cdot \xi$ for all $X, Y \in \mathfrak{X}(U)$.

Proof. (i) Let $\overline{U} \subset \overline{M}$ be a neighborhood of p such that there exists a locally defining function $f \in C^{\infty}\left(\overline{U}\right)$ with $U := \overline{U} \cap M = f^{-1}(0)$. We claim grad $f(q) \neq 0$ for all $q \in \overline{U}$. Since f is a locally defining function, Remark 1.5.10 shows

$$T_p M = \ker df|_q = \ker\left(\overline{g}\left(-, \operatorname{grad} f(q)\right)\right) = \left(\operatorname{grad} f(q)\right)^{\perp}.$$

In particular, this means $\dim\left(\operatorname{grad} f(q)\right)^{\perp} = \dim T_p M < \dim T_p \overline{M}$ so that grad $f(q) \neq 0$.

We may thus apply Lemma 3.3.3(i). Since grad $f \in \mathfrak{X}(U)^{\perp}$, we see that

$$\xi := \frac{\operatorname{grad} f}{|\operatorname{grad} f|}$$

fulfills (i). Uniqueness follows from the fact that $\dim\left(T_p M\right)^{\perp} = 1$ for all $p \in M$.

(ii) For all $X \in \mathfrak{X}(U)$, we may apply the product rule to get that on U:

$$0 = X(\underbrace{\overline{g}(\xi, \xi)}_{\pm 1}) = 2 \cdot \overline{g}\left(\overline{\nabla}_X \xi, \xi\right).$$

This implies

$$\overline{\nabla}_X \xi(q) \in (\xi(q))^{\perp} = T_q M \qquad (q \in U).$$

Thus, $\left(\overline{\nabla}_X \xi\right)^{\perp} = 0$.

(iii) By (ii) and Prop. 4.2.9, we have

$$\overline{\nabla}_X \xi = \left(\overline{\nabla}_X \xi\right)^{\top} = -S^{\xi}(X).$$

(iv) The set $\{\xi(q)\}$ is a pseudo-orthonormal basis of $\left(T_q M\right)^{\perp}$ for all $q \in U$. This implies that we can write any normal field η as

$$\eta = \overline{g}(\xi, \xi) \cdot \overline{g}(\eta, \xi)\, \xi \qquad (\eta \in \mathfrak{X}(U)^{\perp}).$$

In particular, the above equality holds for normal fields of the form $\eta = \Pi(X, Y)$, where $X, Y \in \mathfrak{X}(U)$. Then

$$\begin{aligned}
\Pi(X, Y) &= \overline{g}(\xi, \xi) \cdot \overline{g}\left(\Pi(X, Y), \xi\right) \cdot \xi \\
&= \overline{g}(\xi, \xi) \cdot g\left(S^{\xi}(X), Y\right) \cdot \xi, && \text{by Defn. 4.2.8} \\
&= -\overline{g}(\xi, \xi) \cdot g\left(\overline{\nabla}_X \xi, Y\right) \cdot \xi, && \text{by Lemma 4.2.11(iii).}
\end{aligned}$$

□

Definition 4.2.12: Totally Geodesic & Totally Umbilic hypersurfaces

A semi-Riemannian hypersurface is called

(i) **totally geodesic** if $\Pi = 0$.

(ii) **totally umbilic** if there exists $\alpha \in \mathbb{R}$ such that $\Pi(X,Y) = \alpha g(X,Y) \cdot \xi$ for some (and hence every) choice of unit normal field.

(If ξ is a unit normal field, then (ii) holds for the only other unit normal field $(-\xi)$ by replacing α with $-\alpha$.)

Example 4.2.13. (i) Let $\overline{M} = \mathbb{R}^n$, equipped with g_{eucl} and $M = v^\perp$ for $v \in \mathbb{R}^n \setminus \{0\}$. Then $\xi := \frac{v}{\|v\|} \in \mathfrak{X}(\mathbb{R}^n)$ is a (constant) vector field such that $\xi = \overline{\xi}|_M$ is a unit normal. This implies

$$\overline{\nabla}_X \xi\,(x) = D\overline{\xi}|_x\,(X\,(x)) = 0 \qquad (x \in \mathbb{R}^n, X \in \mathfrak{X}(M)).$$

By Lemma 4.2.11(iv), we have

$$\Pi(X,Y) = -\overline{g}\,(\xi,\xi) \cdot \underbrace{\overline{g}\left(\overline{\nabla}_X \xi, Y\right)}_{=0} \cdot \xi = 0$$

and hence M is totally geodesic.

(ii) The spaces $\mathbb{S}^{n-1}_\nu(r) \subset \mathbb{R}^{\nu,n-\nu}$ and $\mathbb{H}^{n-1}_{\nu-1}(r) \subset \mathbb{R}^{\nu,n-\nu}$ are totally umbilic. (Exercise.)

Remark 4.2.14. Recall that Π is symmetric and $\overline{g}\,(\Pi\,(X,Y)\,,\xi) = g\left(S^\xi\,(X)\,,Y\right)$ (Lemma 4.2.5(ii), Defn. 4.2.6, & Defn. 4.2.8). This means that for all normal fields $\xi \in \mathfrak{X}(M)^\perp$, the map

$$S^\xi : \mathfrak{X}(M) \to \mathfrak{X}(M)$$

is self-adjoint. (This means that $g\left(S^\xi\,(X)\,,Y\right) = g\left(X, S^\xi\,(Y)\right)$ for all $X,Y \in \mathfrak{X}(M)$.) More precisely, we have

$$g\left(S^\xi\,(X)\,,Y\right) = \overline{g}\,(\Pi\,(X,Y)\,,\xi) = \overline{g}\,(\Pi\,(Y,X)\,,\xi) = g\left(S^\xi\,(Y)\,,X\right) = g\left(X, S^\xi\,(Y)\right).$$

Since $S^\xi(p)$ is self-adjoint, the spectral theorem implies that $S^\xi\,(p)$ admits eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$ (with $n = \dim M$) and T_pM admits a pseudo-orthonormal basis of eigenvectors of $S^\xi\,(p)$.

Definition 4.2.15: Principal Curvatures

Let $M \subset \overline{M}$ be a semi-Riemannian hypersurface, let $p \in M$, and let ξ be a unit normal field near p. Then the eigenvalues of $S^\xi\,(p)$ are called **principal curvatures** of M at p.

The eigenvectors are called **principal curvature directions**.

Note that the principal curvatures of M at p are well-defined up to sign since $S^{-\xi}\,(p) = -S^\xi\,(p)$.

Example 4.2.16. Let $f \in C^\infty(\mathbb{R}^n)$ be given by $f(x) = \frac{1}{2}\sum_{i=1}^n \lambda_i \left(x^i\right)^2$ and

$$M = \Gamma(f) = \{(x, f\,(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$$

(equipped with g_{eucl}). Let $F : \mathbb{R}^{n+1} \to \mathbb{R}$ be the locally defining function

$$F\left(x^1, \ldots, x^{n+1}\right) = x^{n+1} - f\left(x^1, \ldots, x^n\right).$$

Then $M = F^{-1}(0)$. By the proof of Lemma 4.2.11(i), $\xi = \operatorname{grad} F / |\operatorname{grad} F|$ is a unit normal field. Observe,

$$\xi = \frac{\operatorname{grad} F}{|\operatorname{grad} F|} = \frac{\left(-\lambda_1 x^1, \ldots, -\lambda_n x^n, 1\right)}{\sqrt{1 + \sum_{i=1}^n \left(\lambda_i x^i\right)^2}}.$$

Fix $p = 0$. For $1 \le i \le n$, by Lemma 4.2.11(iii), we have

$$\begin{aligned} S^{\xi}(\partial_i)(0) &= -\overline{\nabla}_{\partial_i}\xi(0) = -D\xi|_0(\partial_i) \\ &= -\left[\partial_i\left(\frac{1}{|\operatorname{grad} F|}\right)\bigg|_0 \cdot \operatorname{grad} F(0) + \frac{1}{|\operatorname{grad} F(0)|}\overline{\nabla}_{\partial_i}\operatorname{grad} F(0)\right] \\ &= -\left[\underbrace{\partial_i\left(\frac{1}{|\operatorname{grad} F|}\right)\bigg|_0 \cdot (0,0,\ldots,1)}_{=0} + \frac{1}{|1|}\lambda_i\partial_i\right] \\ &= \lambda_i\partial_i. \end{aligned}$$

This means the principal curvatures are given by λ_i and the principal curvature directions are ∂_i.

Note that if $M \subset \mathbb{R}^{n+1}$ is an arbitrary hypersurface and $p \in M$, it is, up to rotation and translation, of the above form (with $f(x) = \frac{1}{2}\sum_{i=1}^{n}\lambda_i\left(x^i\right)^2 + O\left(|x|^3\right)$.

Chapter 5

Geodesics

Intuition: Geodesics are parametrized curves with no acceleration. On $\mathbb{R}^n$, this means that $c'' = 0$ for $c : I \to \mathbb{R}^n$. Thus, $c(t) = at + b$ for $a, b \in \mathbb{R}^n$.

On a manifold, we already defined c' for $c \in C^\infty(I, M)$, but not c''. Thus, we need to differentiate vector fields which are only defined along a parametrized curve c (in particular c').

From now on, let M be a semi-Riemannian manifold.

5.1 The Covariant derivative along curves

Definition 5.1.1: $\mathfrak{X}(M)_c$: Vector fields along curves

Let $I \subset \mathbb{R}$ be an interval and $c \in C^\infty(I, M)$. Define

$$\mathfrak{X}(M)_c := \left\{ X \in C^\infty(I, TM) : X(t) \in T_{c(t)}M \quad \forall t \in I \right\}.$$

Then $X \in \mathfrak{X}(M)_c$ is called a **vector field along** c.

Remark 5.1.2. We have $c' \in \mathfrak{X}(M)_c$. (See Remark 1.4.11(iii).) For $Z \in \mathfrak{X}(M)$, we have $Z \circ c \in \mathfrak{X}(M)_c$. However, not every $X \in \mathfrak{X}(M)_c$ is of this form.

Theorem 5.1.3

Let $I \subset \mathbb{R}$ be an interval and $c \in C^\infty(I, M)$. Then there exists a unique map

$$\begin{aligned} \frac{\nabla}{dt} : \mathfrak{X}(M)_c &\to \mathfrak{X}(M)_c \\ X &\mapsto \frac{\nabla}{dt} X \end{aligned}$$

such that for all $X, Y \in \mathfrak{X}(M)_c$, $Z \in \mathfrak{X}(M)$, and all $f \in C^\infty(I)$,

(i) $\frac{\nabla}{dt}(X + Y) = \frac{\nabla}{dt} X + \frac{\nabla}{dt} Y$;

(ii) $\frac{\nabla}{dt}(f \cdot X) = f' X + f \cdot \frac{\nabla}{dt} X$;

(iii) $\frac{\nabla}{dt}(Z \circ c) = \nabla_{c'} Z$.

We call $\frac{\nabla}{dt}$ the **covariant derivative along** c. It also satisfies

(iv) $\frac{d}{dt}\langle X, Y \rangle = \langle \frac{\nabla}{dt} X, Y \rangle + \langle X, \frac{\nabla}{dt} Y \rangle$.

Remark. The expression $\nabla_{c'} Z$ in Theorem 5.1.3(iii) can be interpeted as an element in $\mathfrak{X}(M)_c$ in the following way. The map

$$\begin{aligned}\mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M)\\ (W, Z) &\mapsto \nabla_W Z\end{aligned}$$

is $C^\infty(M)$-linear in the first slot. Thus for fixed $Z \in \mathfrak{X}(M)$, the vector $\nabla_W Z(p) \in T_pM$ only depends on $W(p)$. Therefore, it makes sense to define $\nabla_v Z := \nabla_W Z$, where W is any vector field satisfying $W(p) = v$. As a consequence we obtain a well-defined linear map

$$\begin{aligned}T_pM &\to T_pM\\ v &\mapsto \nabla_v Z.\end{aligned}$$

We may extend this to a $C^\infty(I)$-linear map

$$\begin{aligned}\mathfrak{X}(M)_c &\to \mathfrak{X}(M)_c\\ X &\mapsto \nabla_X Z\end{aligned}$$

by setting $\nabla_X Z(t) = \nabla_{X(t)} Z \in T_{c(t)}M$. Setting $X = c'$ in the above, we see that $\nabla_{c'} Z$ is a $C^\infty(I)$-linear map.

Proof. **Uniqueness:** A cutoff function argument (such as the one given in the proof of Theorem 2.3.10) shows that $\frac{\nabla}{dt}$ is local, i.e. $\left(\frac{\nabla}{dt}X\right)\big|_J = \frac{\nabla}{dt}(X|_J)$ for an open subinterval $J \subset I$. (For a more detailed proof of this, see following Remark.)

Take J so small that $c(J) \subset U$ for a chart (U, φ) of M. Write $X|_J = X^i \partial_i \circ c$ and $\varphi \circ c(t) = (c^1(t), \ldots, c^n(t))$ for $t \in J$, where $\partial_i \circ c(t) = \partial_i|_{c(t)}$. Then $c'|_J = (c^j)' \partial_j \circ c$ (see (1.4.8)). Then

$$\begin{aligned}\left(\frac{\nabla}{dt}X\right)\Big|_J = \frac{\nabla}{dt}(X|_J) = \frac{\nabla}{dt}\left(X^i \partial_i \circ c\right) \qquad&\\
= \left(X^i\right)' \partial_i \circ c + X^i \frac{\nabla}{dt}\left(\partial_i \circ c\right), &\qquad \text{by (i) \& (ii)}\\
= \left(X^i\right)' \partial_i \circ c + X^i \nabla_{c'} \partial_i, &\qquad \text{by (iii)}\\
= \left(X^i\right)' \partial_i \circ c + X^i \nabla_{(c^j)' \partial_j \circ c} \partial_i &\\
= \left(X^i\right)' \partial_i \circ c + X^i \cdot \left(c^j\right)' \nabla_{\partial_j \circ c} \partial_i &\\
= \left(X^i\right)' \partial_i \circ c + X^i \cdot \left(c^j\right)' \cdot \left(\Gamma^k_{ji} \circ c\right) \cdot \left(\partial_k \circ c\right), &\qquad \text{by (i)}\\
= \left[\left(X^k\right)' + X^i \cdot \left(c^j\right)' \cdot \left(\Gamma^k_{ij} \circ c\right)\right]\left(\partial_k \circ c\right). &\end{aligned}$$

This implies uniqueness. In particular, we have

$$\left(\frac{\nabla}{dt}X\right)\Big|_J = \left(X^i\right)' \partial_i \circ c + X^i \nabla_{(c^j)' \partial_j \circ c}\left(\partial_i \circ c\right) = \left[\left(X^k\right)' + X^i \left(c^j\right)' \Gamma^k_{ij} \circ c\right]\left(\partial_k \circ c\right). \tag{5.1.1}$$

Existence: Define $\frac{\nabla}{dt}X$ locally by (5.1.1) and check the properties (i)-(iv). By uniqueness, this defintion does not depend on the chosen chart. □

Remark. Here we give a more detailed proof of the locality property of $\frac{\nabla}{dt}$.

Step 1. *The restriction $\left(\frac{\nabla}{dt}X\right)\big|_J$ depends only on $X|_J$.*

Let $t \in J$ and pick an open interval K with $t \in K$ and $\overline{K} \subset J$. Pick a cutoff function $\chi \in C^\infty(I)$ with $\operatorname{supp}\chi \subset J$ and $\chi|_K \equiv 1$. Suppose $\widetilde{X} \in \mathfrak{X}(M)_c$ satisfies $X|_J = \widetilde{X}|_J$. Since $\chi'(s) = 0$ for all $s \in K$ and since $\chi \cdot X = \chi \cdot \widetilde{X}$, we have for all $s \in K$,

$$\begin{aligned}\left(\frac{\nabla}{dt}X\right)(s) = \chi'(t)\, X(t) + \chi(t)\left(\frac{\nabla}{dt}X\right)(s) &= \left(\frac{\nabla}{dt}(\chi \cdot X)\right)(s)\\
= \left(\frac{\nabla}{dt}\left(\chi \cdot \widetilde{X}\right)\right)(s) &= \left(\frac{\nabla}{dt}\widetilde{X}\right)(s).\end{aligned}$$

This shows $\left(\frac{\nabla}{dt}X\right)\big|_K = \left(\frac{\nabla}{dt}\widetilde{X}\right)\big|_K$ whenever $X|_J = \widetilde{X}|_J$ for any $t \in J$ and any such neighborhood K of t.

Step 2. *The map $\frac{\nabla}{dt}$ restricts to a map $\frac{\nabla}{dt}\big|_J : \mathfrak{X}(M)_{c|_J} \to \mathfrak{X}(M)_{c|_J}$ in such a way that $\left(\frac{\nabla}{dt}X\right)\Big|_J = \frac{\nabla}{dt}\big|_J (X|_J)$.*

Given $\frac{\nabla}{dt} : \mathfrak{X}(M)_c \to \mathfrak{X}(M)_c$, define

$$\begin{aligned} \frac{\nabla}{dt}\Big|_J : \mathfrak{X}(M)_{c|_J} &\to \mathfrak{X}(M)_{c|_J} \\ X &\mapsto \left(\frac{\nabla}{dt}\widetilde{X}\right)\Big|_J, \end{aligned}$$

where $\widetilde{X}$ is an extension of X. By Step 1, this map does not depend on the chosen extension. It remains to show $\frac{\nabla}{dt}\big|_J$ satisfies (i)-(iv) of Theorem 5.1.3. To show (i), observe that for $X, Y \in \mathfrak{X}(M)_{c|_J}$ with extensions $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(M)_c$,

$$\frac{\nabla}{dt}\Big|_J (X+Y) = \frac{\nabla}{dt}\left(\widetilde{X}+\widetilde{Y}\right) = \frac{\nabla}{dt}\widetilde{X} + \frac{\nabla}{dt}\widetilde{Y} = \frac{\nabla}{dt}\Big|_J (X) + \frac{\nabla}{dt}\Big|_J (Y).$$

The verification of (ii)-(iv) is similar. For all $X \in \mathfrak{X}(M)_c$, since X is an extension of $X|_J$, see that

$$\frac{\nabla}{dt}\Big|_J (X|_J) = \left(\frac{\nabla}{dt}X\right)\Big|_J,$$

as desired.

Definition 5.1.4: Parallel Vector Fields

Let $c \in C^\infty(I, M)$. A vector field $X \in \mathfrak{X}(M)_c$ is called **parallel** if $\frac{\nabla}{dt}X = 0$.

Theorem 5.1.5

Let $c \in C^\infty([a,b], M)$, $v \in T_{c(a)}M$. Then there exists exactly one parallel vector field $X \in \mathfrak{X}(M)_c$ such that $X(a) = v$.

Proof. **Case 1:** Suppose first that there exists a chart (U, φ) such that $c([a,b]) \subset U$. (For example, such a chart exists if a, b are chosen so that $b - a$ is small.) Then for $X \in \mathfrak{X}(M)_c$, $X = X^i \partial_i \circ c$, the condition $\frac{\nabla}{dt}X = 0$ is equivalent to the ODE system

$$\left(X^k\right)' + X^i \left(c^j\right)' \Gamma^k_{ij} = 0 \qquad (1 \le i \le n). \tag{5.1.2}$$

(See (5.1.1).)

Write $v = v^i \partial_i|_{c(a)} \in T_{c(a)}M$. By ODE theory, there exists a unique solution $\left\{X^i \in C^\infty([a,b], \mathbb{R})\right\}_{1 \le i \le n}$ of (5.1.2) with $X^i(a) = v^i$. This implies $X = X^i \partial_i \circ c \in \mathfrak{X}(M)_c$ is the unique parallel vector field on c such that $X(a) = v$.

Case 2: For the general case, the compactness of $c([a,b])$ lets us choose finitely may (U_j, φ_j) $(1 \le j \le N)$ and $a = t_0 < t_1 < \cdots < t_N = b$ such that

$$c\left([t_{j-1}, t_j]\right) \subset U_j.$$

The result follows from N applications of the argument in Case 1. □

Definition 5.1.6: Parallel Transport

For $c \in C^\infty([a,b], M)$, we define a map

$$\begin{aligned} P_a^b : T_{c(a)}M &\to T_{c(b)}M \\ v &\mapsto X(b), \end{aligned}$$

where $X \in \mathfrak{X}(M)_c$ is the parallel vector field such that $X(a) = v$. We call P_a^b the **parallel transport**.

Lemma 5.1.7

The map $P_a^b : T_{c(a)}M \to T_{c(b)}M$ is a linear isometry.

Proof. **Linearity:** The relation $P_a^b(v + \alpha w) = P_a^b(v) + \alpha P_a^b(w)$ follows from the fact that the ODE-system (5.1.2) is linear.

Isometry: Let $v, w \in T_{c(a)}M$ and $X, Y \in \mathfrak{X}(M)_c$ be the parallel vector fields with $X(a) = v, Y(a) = w$. Then

$$\frac{d}{dt}\langle X, Y\rangle = \langle \frac{\nabla}{dt}X, Y\rangle + \langle X, \frac{\nabla}{dt}Y\rangle = 0,$$

so

$$\langle P_a^b(v), P_a^b(w)\rangle = \langle X(b), Y(b)\rangle = \langle X(a), Y(a)\rangle = \langle v, w\rangle.$$

□

5.2 Geodesics

Definition 5.2.1: Geodesic

A curve $c \in C^\infty(I, M)$ is called **geodesic** if $\frac{\nabla}{dt}c' = 0$. That is, if c' is parallel.

Lemma 5.2.2

Let $a \in \mathbb{R}$. For each $p \in M$ and $v \in T_pM$, there exists an interval I with $a \in I$ and a unique geodesic $c : I \to M$ with $c(a) = p$ and $c'(a) = v$.

Proof. Let (U, φ) be a chart around p. By (5.1.2), $c : I \to U$ is a geodesic if and only if the functions $c^i : I \to \mathbb{R}$ defined by $\varphi \circ c(t) = (c^1(t), \dots, c^n(t))$ satisfy

$$\left(c^k\right)'' + \left(c^i\right)' \left(c^j\right)' \Gamma_{ij}^k \circ c = 0 \qquad (1 \le k \le n). \tag{5.2.1}$$

(See (5.1.2).) Standard theory of 2nd order ODE-systems yield a unique solution $\widetilde{c}(t) = \left(c^1(t), \dots, c^n(t)\right)$ for given initial data $\left(c^1(a), \dots, c^n(a)\right) := \varphi(p)$ and $\left((c^1)'(a), \dots, (c^n)'(a)\right) := T_p\varphi(v)$, defined on an interval I around a. This implies $c := \varphi^{-1} \circ \widetilde{c}$ is the desired unique geodesic. □

Example 5.2.3. (i) On $\mathbb{R}^{\nu, n-\nu}$, we have $\Gamma_{ij}^k = 0$ for all i, j, k in the standard chart. This means that for $x \in \mathbb{R}^{\nu, n-\nu}$ and $v \in T_x\mathbb{R}^{\nu, n-\nu} \cong \mathbb{R}^n$, the geodesic with $c(0) = x$ and $c'(0) = v$ is $c : t \mapsto x + tv$.

(ii) The cylinder $M = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - 1 = 0\right\}$ is covered by charts of the form

$$\varphi : (\cos\theta, \sin\theta, z) \mapsto (\theta, z) \qquad (z \in \mathbb{R}, \theta \in (a, a + 2\pi)).$$

In each of these charts, $\partial_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\partial_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$. If we equip M with the Riemannian metric $g := g_{\mathrm{eucl}}|_M$, we get

$$\left\{g_{ij}\right\}_{i,j\in\{z,\theta\}} = \left\{g_{\mathrm{eucl}}\left(\partial_i, \partial_j\right)\right\}_{i,j\in\{z,\theta\}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \implies \Gamma_{ij}^k = 0.$$

This means $c : I \to M$ is a geodesic $\iff$ In each of these charts, the components (c^θ, c^z) satisfy $(c^\theta)'' = 0 = (c^z)''$. This means c^θ, c^z are of the form $a + tb$.

Thus, if $c(0) = (\cos\theta_0, \sin\theta_0, z_0)$ and $c'(0) = a\partial_\theta|_{c(0)} + b\partial_z|_{c(0)}$ (in a chart of the above form around $c(0)$), then

$$c(t) = (\cos(\theta_0 + at), \sin(\theta_0 + at), z_0 + bt).$$

(iii) For an arbitrary semi-Riemannian manifold, the geodesic $c : I \to M$ with $c(0) = p$ and $c'(0) = 0 \in T_pM$ is the constant curve $c(t) = p$ for all $t \in I$. We call all other geodesics **nonconstant**

Lemma 5.2.4

Let $c_1, c_2 : I \to M$ be geodesics such that $c_1(a) = c_2(a)$ and $c_1'(a) = c_2'(a)$ for some $a \in I$. Then $c_1 = c_2$.

Proof. Let $J := \{t \in I : c_1(t) = c_2(t), c_1'(t) = c_2'(t)\}$. Then the following are true.

(i) $J \neq \emptyset$ since $a \in J$.

(ii) J is closed since it is defined by a continuous equation.

(iii) J is open. To see this, let $b \in J$, then by Lemma 5.2.2, there exists $\varepsilon > 0$ such that $c_1(t) = c_2(t)$ for all $t \in (b-\varepsilon, b+\varepsilon)$. This means $c_1'(t) = c_2'(t)$ for all $t \in (b-\varepsilon, b+\varepsilon)$ so that $(b-\varepsilon, b+\varepsilon) \subset J$ and J is open.

As I is connected, we have $J = I$. □

As a consequence, we obtain

Proposition 5.2.5

For $p \in M$ and $v \in T_pM$, there exists a unique geodesic $c_v : I \to M$ such that

(i) $c_v(0) = p$, $c_v'(0) = v$,

(ii) The domain of c_v is maximal, i.e. if $c : J \to M$ with $0 \in J$ is another geodesic with $c(0) = p$ and $c'(0) = v$, then $J \subset I$ and $c_v|_J = c$.

Definition 5.2.6: Maximal Geodesics & Geodesically Complete

We call the geodesic c_v from Prop. 5.2.5 **maximal**. If for each $p \in M, v \in T_pM$, the geodesic c_v is defined on $\mathbb{R}$, then M is called **geodesically complete**.

Example 5.2.7. (i) $\mathbb{R}^{\nu,n-\nu}$ is geodesically complete.

(ii) $\mathbb{R}^{\nu,n-\nu} \setminus \{0\}$ is *not* geodesically complete.

Lemma 5.2.8

let $c : I \to M$ be a nonconstant geodesic and $h : J \to I$ a diffeomorphism of intervals. Then the following are equivalent

(i) $c \circ h$ is a geodesic.

(ii) $h(t) = at + b$ for $a, b \in \mathbb{R}$.

Proof. **(ii) ⇒ (i):** Chain rule & (5.2.1).

(i) ⇒ (ii): Let $t_1 \in J$ and $s_1 := h(t_1) \in I$. Then $(c \circ h)(t_1) = c(s_1)$ and $(c \circ h)'(t_1) = c'(s_1) \cdot h'(t_1)$. Set $a = h'(t_1), b = s_1 - at_1$ and $\widetilde{h}(t) = at + b$. Then $c \circ \widetilde{h}$ is a geodesic by the implication (ii) ⇒ (i) with

$$\left(c \circ \widetilde{h}\right)(t_1) = c(s_1) = (c \circ h)(t_1),$$
$$\left(c \circ \widetilde{h}\right)'(t_1) = c'(s_1) \cdot a = c'\left(h\left(t_1\right)\right) \cdot h'\left(t_1\right) = (c \circ h)'\left(t_1\right).$$

By uniqueness, we have $c \circ \widetilde{h} = c \circ h$. As c is nonconstant, this means $\widetilde{h} = h$. □

Lemma 5.2.9

Let $p \in M$ and $v \in T_pM \subset TM$. Then there exists an open neighborhood U of v in TM and an interval I around 0 such that the mapping

$$\begin{aligned} U \times I &\to M \\ (w, s) &\mapsto c_w(s) \end{aligned}$$

is C^∞.

Proof. ODE-theory (smooth dependence of ODE's with C^∞-coefficients on initial data). □

5.3 The Exponential Map

Definition 5.3.1: Exponential map ($\exp_p$)

For $p \in M$, set $D_p = \left\{v \in T_pM : c_v \text{ is well-defined on } [0,1]\right\}$. We call the map

$$\begin{aligned} \exp_p : D_p &\to M \\ v &\mapsto c_v(1) \end{aligned} \tag{5.3.1}$$

the **exponential map** at p.

Remark 5.3.2. (i) D_p is the maximal domain of $\exp_p$. If M is geodesically complete, then $D_p = T_pM$ for all $p \in M$.

(ii) For $v \in T_pM, t \in \mathbb{R}$, the map $\mapsto c_v(t \cdot s)$ is a geodesic (by Lemma 5.2.8) with initial point p and initial velocity $t \cdot c_v'(0) = t \cdot v$. This implies $c_v(t \cdot s) = c_{t \cdot v}(s)$ if both sides of the equation are defined. This gives the relation

$$\exp_p(t \cdot v) = c_{tv}(1) = c_v(t). \tag{5.3.2}$$

(iii) D_p contains an open neighborhood of $0 \in T_pM$. To see this, let $U \subset TM$ be the neighborhood of $0 \in T_pM$ in Lemma 5.2.9. Then there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subset I$ (with I as in Lemma 5.2.9). The set $U_p := U \cap T_pM$ is an open neighborhood of $0 \in T_pM$. For all $v \in U_p, c_v$ is defined up to $(-\varepsilon, \varepsilon)$. Thus, for $\frac{\varepsilon}{2} \cdot v, v \in U_p, c_{\varepsilon/2v} : s \mapsto c_{\varepsilon/2 \cdot v}(s) = c_v(\varepsilon/2, s)$ is defined up to $(-2, 2)$ so that $\left\{\frac{\varepsilon}{2} v : v \in U_p\right\}$ is a neighborhood of $0 \in T_pM$ contained in D_p.

Theorem 5.3.3

Let $p \in M$. Then there exist open neighborhoods $V \subset T_pM$ of 0 and $U \subset M$ of p such that $\exp_p : V \to U$ is a diffeomorphism.

Proof. As T_pM is a vector space, there is a canonical identifcation $T_pM \cong T_v(T_pM)$, given by

$$\begin{aligned} T_pM &\to T_v(T_pM) \\ w &\mapsto [c : t \mapsto v + tw]_v \in T_v(T_pM). \end{aligned}$$

Now we compute

$$T_0 \exp_p : T_0(T_pM) \cong T_pM \to T_{\exp_p(0)}M = T_pM.$$

Let $v \in T_pM$ and $c : t \mapsto t \cdot v$ a straight line in T_pM. Then by the chain rule,

$$\begin{aligned} T_0 \exp_p(v) = T_0 \exp_p(c'(0)) = \left(\exp_p \circ c\right)'(0) &= \left.\frac{d}{dt}\right|_{t=0} \exp_p(tv) \\ &= \left.\frac{d}{dt}\right|_{t=0} c_v(t) = c_v'(0) = v. \end{aligned}$$

Thus, $T_0 \exp_p = \mathrm{id}_{T_pM}$ and thus is a linear isomorphism. The assertion now follows from Lemma 1.5.3. □

Definition 5.3.4: Geodesic Chart

Let $p \in M$ and $\mathscr{B} = \{e_1, \ldots, e_n\}$ be an ordered basis of T_pM. Then we have an isomorphism $\varphi_{\mathscr{B}} : T_pM \to \mathbb{R}^n, v = v^i e_i \mapsto (v^1, \ldots, v^n)$. With U and V as in Theorem 5.3.3, we call (U, φ), with $\varphi = \varphi_{\mathscr{B}} = (\exp_p|_V)^{-1} : U \to \mathbb{R}^n$ a **geodesic chart** of M centered at p with respect to the basis $\mathscr{B}$.

Proposition 5.3.5

Let $p \in M$ and $\mathscr{B} = \{e_1, \ldots, e_n\}$ be a pseudo-orthonormal basis of T_pM (ie $\langle e_i, e_j\rangle = \varepsilon_i \delta_{ij}, \varepsilon_i \in \{\pm 1\}$). Then in the geodesic chart at p with respect to $\mathscr{B}$, we have for all i, j, k,

(i) $g_{ij}(p) = \varepsilon_i \delta_{ij}$,

(ii) $\Gamma^k_{ij}(p) = 0$,

(iii) $\partial_i g_{jk}(p) = 0$

This implies $g_{ij} \circ \varphi^{-1}(x) = \varepsilon_i \delta_{ij} + O\left(|x|^2\right)$.

Proof. (i) Let $\{f_1, \ldots, f_n\}$ be the standard basis of $\mathbb{R}^n$. Then $T_0 \varphi_{\mathscr{B}}(e_i) = D_{\varphi_{\mathscr{B}}}|_0(e_i) = f_i$ for all i. Since $\left(T_0 \exp_p\right)^{-1} = \mathrm{id}$, we get

$$T_p\varphi(e_i) = T_0 \varphi_{\mathscr{B}} \circ \left(T_0 \exp_p\right)^{-1}(e_i) = T_p \varphi_{\mathscr{B}}(e_i) = f_i.$$

This means $\partial_i|_p = (T_p\varphi)^{-1}(f_i) = e_i$ so that

$$g_{ij}(p) = \langle \partial_i|_p, \partial_j|_p\rangle = \langle e_i, e_j\rangle = \varepsilon_i \delta_{ij}.$$

(ii) For $v = v^i e_i \in T_pM$, the corresponding geodesic $c_v : t \mapsto \exp_p(tv)$ has local components $(c^1(t), \ldots, c^n(t)) = \varphi \circ c(t) = (tv^1, \ldots, tv^n)$. By (5.2.1) with $t = 0$, we get $0 = v^i v^j \Gamma^k_{ij}(p)$ for all k. Then the symmetric bilinear form $Q^k : (v, w) \mapsto v^k w^j \Gamma^k_{ij}(p)$ satisfies $Q^k(v, v) = 0$ for all $v \in T_pM$. By polarization, we get $Q^k = 0$ so that $\Gamma^k_{ij}(p) = 0$ for all i, j, k.

(iii) Observe

$$\begin{aligned}
\partial_i g_{jk}(p) &= \left(\partial_i \langle \partial_j, \partial_k \rangle\right)(p) \\
&= \langle \nabla_{\partial_i} \partial_j, \partial_k \rangle(p) + \langle \partial_j, \nabla_{\partial_i} \partial_j \rangle, && \text{since } \nabla \text{ is metric} \\
&= \left(\Gamma^{\ell}_{ij} g_{\ell k}\right)(p) + \left(\Gamma^{\ell}_{ik} g_{j\ell}\right)(p) \\
&= 0, && \text{by (ii).}
\end{aligned}$$

□

Remark 5.3.6. The properties of geodesic charts in Prop. 5.3.5 are sometimes very useful for calculational purposes.

Example 5.3.7. On $\mathbb{R}^{\nu,n-\nu}$ and $x \in \mathbb{R}^{\nu,n-\nu}$, the exponential map is given by

$$\exp_x : v \mapsto c_v(1) = x + 1 \cdot v = x + v.$$

5.4 Geodesics in Submanifolds

Let M be a semi-Riemannian submanifold of the semi-Riemannian manifold $\overline{M}$. We denote the covariant derivatie along curves in M and $\overline{M}$ by $\frac{\nabla}{dt}$ and $\frac{\overline{\nabla}}{dt}$, respectively.

Lemma 5.4.1

Let $c : I \to M \subset \overline{M}$ be smooth and $X \in \mathfrak{X}(M)_c \subset \mathfrak{X}(\overline{M})_c$. Then

$$\frac{\overline{\nabla}}{dt} X = \frac{\nabla}{dt} X + \Pi(X, c'). \tag{5.4.1}$$

In particular,

$$\frac{\overline{\nabla}}{dt} c' = \frac{\nabla}{dt} c' + \Pi(c', c'). \tag{5.4.2}$$

Proof. Let $(U, \varphi = (x^1, \ldots, x^n, \ldots, x^m))$ be a submanifold chart Then on $J \subset I$ with $c(J) \subset U$, we have

$$\begin{aligned}
\frac{\overline{\nabla}}{dt} X &= \frac{\overline{\nabla}}{dt} \left(\sum_{i=1}^{n} X^i \partial_i \circ c \right) \\
&= \sum_{i=1}^{n} \left[(X^c)' \, \partial_i \circ c + \sum_{j=1}^{n} X^i (c^j)' \overline{\nabla}_{\partial_i} \partial_j \circ c \right], && \text{by Theorem 5.1.3} \\
&= \sum_{i=1}^{n} (X^i)' \, \partial_i \circ c + \sum_{i,j=1}^{n} X^i (c^j)' \left[\nabla_{\partial_i} \partial_j + \Pi\left(\partial_i, \partial_j\right) \right] \circ c, && \text{by Lem. 4.2.5 and Defn. 4.2.6} \\
&= \cdots = \frac{\nabla}{dt} X + \Pi(X, c').
\end{aligned}$$

□

Corollary 5.4.2. *Let $c \in C^\infty(I, M) \subset C^\infty(I, \overline{M})$. Then c is a geodesic in M if and only if $\frac{\overline{\nabla}}{dt} c'$ is orthogonal to M.*

Proof. c is a geodesic in M if and only if $\frac{\nabla}{dt} c' = 0$; by (5.4.2), this is equivalent to $\frac{\overline{\nabla}}{dt} c' = \Pi(c', c') \perp M$. □

Corollary 5.4.3. *The following are equivalent:*

(i) $M \subset \overline{M}$ is a totally geodesic submanifold.

(ii) Every $c \in C^\infty(I, M) \subset C^\infty(I, \overline{M})$ which is a geodesic in M is also a geodesic in $\overline{M}$.

Proof. **(i) ⇒ (ii):** If c is a geodesic in M then

$$\frac{\overline{\nabla}}{dt}c' = \underbrace{\frac{\nabla}{dt}c'}_{=0} + \underbrace{\Pi\left(', c'\right)}_{\text{by (i)}} = 0$$

so that c is a geodesic in $\overline{M}$.

(ii) ⇒ (i): Let $p \in M$ and $v \in T_pM$ be arbitrary and let c_v be the geodesic in M satisfying $c_v(0) = p$ and $c_v'(0) = v$. By (ii), c is a geodesic in $\overline{M}$ so

$$0 = \frac{\overline{\nabla}}{dt}c'(0) = \underbrace{\frac{\nabla}{dt}c'(0)}_{=0} + \Pi(v, v);$$

thus $\Pi(v, v) = 0$ for all $v \in T_pM$ and all $p \in M$. By the symmetry of Π and polarization, we get $\Pi(v, w) = 0$ for all $v, w \in T_pM$, $p \in M$. This shows (i) holds. □

Example 5.4.4. Let $\overline{M} = (\mathbb{R}^n, g_{\text{eucl}})$ and $M = \left(\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}} = g_{\text{eucl}}|_{\mathbb{S}^{n-1}}\right)$. Let $p \in \mathbb{S}^{n-1}$ and $v \in T_p\mathbb{S}^{n-1} = p^{\perp}, v \neq 0$. Then the maximal geodesic $c_v \in \mathbb{S}^{n-1}$ with $c_v(0) = p, c_v'(0) = v$ is given by

$$\begin{aligned} c_v : \mathbb{R} &\to \mathbb{S}^{n-1} \\ t &\mapsto \cos\left(|v|\, t\right) \cdot p + \frac{v}{|v|} \sin\left(|v|\, t\right). \end{aligned}$$

Then

- $\langle c_v(t), c_v(t)\rangle = \cdots = 1$ implies $c_v \in C^{\infty}\left(\mathbb{R}, \mathbb{S}^{n-1}\right)$.
- Check $c_v(0) = p, c_v'(0) = v$.
- $\frac{\overline{\nabla}}{dt}c_v'(t) = c_v''(t) = -|v|^2 \cdot c_v(t) \in \mathbb{R}c_v(t) = \left(T_{c_v(t)}M\right)^{\perp}$ so that c_v is a geodesic.

Chapter 6

Curvature

As usual, let M be a fixed semi-Riemannian manifold with Levi-Civita connction ∇.

6.1 The Riemannian Curvature Tensor

Definition 6.1.1: Hessian

The **Hessian** of a function $f \in C^\infty(M)$ is

$$\begin{aligned} \nabla^2 f : \mathfrak{X}(M) \times \mathfrak{X}(M) &\to C^\infty(M) \\ (X, Y) &\mapsto \nabla^2_{X,Y} f = X(Y(f)) - (\nabla_X Y)(f). \end{aligned}$$

Lemma 6.1.2

(i) $\nabla^2 f \in \mathscr{T}_2^0(M)$.

(ii) $\nabla^2_{X,Y} f = \nabla^2_{Y,X} f$ for all $X, Y \in \mathfrak{X}(M)$.

(iii) $\nabla^2_{X,Y} f = \langle \nabla_X \operatorname{grad} f, Y \rangle$ for all $X, Y \in \mathfrak{X}(M)$.

Proof. (i) It follows from the definition that

$$\nabla_{hX,Y} f = h\nabla^2_{X,Y} f \qquad (h \in C^\infty(M), X, Y \in \mathfrak{X}(M)).$$

Then

$$\begin{aligned} \nabla^2_{X,hY} f &= X(h \cdot Y(f)) - (\nabla_X (h \cdot Y))(f) \\ &= \cancel{X(h) \cdot Y(f)} + h \cdot Y(f) - [\cancel{X(h) \cdot Y(f)} + h \cdot (\nabla_X Y)(f)] \\ &= h \cdot \nabla^2_{X,Y} f. \end{aligned}$$

This shows $\nabla^2 f$ is $C^\infty(M)$-bilinear, hence a tensor field.

(ii) Observe that

$$\begin{aligned} \nabla^2_{X,Y} f - \nabla^2_{Y,X} f &= X(Y(f)) - (\nabla_X Y)(f) - Y(X(f)) + (\nabla_Y X)(f) \\ &= [X, Y](f) - \underbrace{[(\nabla_X Y) - (\nabla_Y X)]}_{=[X,Y]}(f) \\ &= 0. \end{aligned}$$

(iii) $\nabla^2_{X,Y} f = X(Y(f)) - (\nabla_X Y)(f) = X(\langle \operatorname{grad} f, Y \rangle) - \langle \operatorname{grad} f, \nabla_X Y \rangle = \langle \nabla_X \operatorname{grad} f, Y \rangle.$ □

Remark 6.1.3. Lemma 6.1.2(ii) asserts that the Schwartz thereom also holds in manifolds, i.e. covariant derviatives commute when applied to functions. This is no longer true if f is replaced by a vector field Z. Our first notion of curvature measures the failure of commutativity.

Definition 6.1.4: Riemannian Curvature Tensor

The map

$$\begin{aligned} R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ (X, Y, Z) \mapsto R_{X,Y}Z &:= \nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z \\ &:= \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y}Z - \left(\nabla_Y(\nabla_X Z) - \nabla_{\nabla_Y X}Z\right) \\ &= \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z \end{aligned} \tag{6.1.1}$$

is called the **Riemannian curvature tensor**.

Lemma 6.1.5

R is $C^\infty(M)$-trilinear, hence defines a $(1,3)$-tensor field.

Proof. The $C^\infty(M)$-bilinearity of $(X, Y) \mapsto \nabla^2_{X,Y}Z = \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y}Z$ is shown as in Lemma 6.1.2(i). Thus, $(X, Y) \mapsto R_{X,Y}Z$ is $C^\infty(M)$-bilinear. It remains to show $C^\infty(M)$-linearity in Z. Observe that

$$\begin{aligned} \nabla^2_{X,Y}(fZ) &= \nabla_X(\nabla_Y(fZ)) - \nabla_{\nabla_X Y}(fZ) \\ &= \nabla_X\left(Y(f) + f\nabla_Y Z\right) - (\nabla_X Y)(f) \cdot Z - f\nabla_{\nabla_X Y}Z \\ &= X(Y(f)) + Y(f)\nabla_X Z + X(f)\nabla_Y Z + \nabla_X(\nabla_Y Z) - (\nabla_X Y)(f) \cdot Z - f\nabla_{\nabla_X Y}Z \\ &= \nabla^2_{X,Y}f \cdot Z + Y(f)\nabla_X Z + X(f)\nabla_Y Z + f\nabla^2_{X,Y}Z. \end{aligned}$$

Antisymmetrizing in X, Y and Lemma 6.1.2(ii) yields $R_{X,Y}(fZ) = fR_{X,Y}Z$. □

Lemma 6.1.6

Let (U, φ) be a chart of M and write

$$R|_U = R^i_{jk\ell}\partial_i \otimes dx^j \otimes dx^k \otimes dx^\ell$$

Then $R^i_{jk\ell} = \partial_j\Gamma^i_{k\ell} - \partial_k\Gamma^i_{j\ell} + \Gamma^m_{k\ell}\Gamma^i_{jm} - \Gamma^m_{j\ell}\Gamma^i_{km}$.

Proof. We compute

$$\begin{aligned} R^i_{jk\ell} &= dx^i\left(R_{\partial_j,\partial_k}\partial_\ell\right) \\ &= dx^i\left(\nabla_{\partial_j}\left(\nabla_{\partial_k}\partial_\ell\right) - \nabla_{\partial_k}\left(\nabla_{\partial_j}\partial_\ell\right)\right), \qquad \text{since } \left[\partial_j, \partial_k\right] = 0 \\ &= dx^i\left(\nabla_{\partial_j}\left(\Gamma^m_{k\ell}\partial_m\right) - \nabla_{\partial_k}\left(\Gamma^m_{j\ell}\partial_m\right)\right) \\ &= dx^i\left(\left(\partial_j\Gamma^m_{k\ell}\right)\partial_m + \Gamma^m_{k\ell}\nabla_{\partial_j}\partial_m - \left(\partial_k\Gamma^m_{j\ell}\right)\partial_m - \Gamma^m_{j\ell}\nabla_{\partial_k}\partial_m\right) \\ &= dx^i\left(\partial_j\Gamma^m_{k\ell} \cdot \partial_m + \Gamma^m_{k\ell}\Gamma^p_{jm}\partial_p - \partial_k\Gamma^m_{j\ell} - \Gamma^m_{j\ell}\Gamma^p_{km}\partial_p\right) \\ &= \partial_j\Gamma^i_{k\ell} + \Gamma^m_{k\ell}\Gamma^i_{jm} - \partial_k\Gamma^i_{j\ell} - \Gamma^m_{j\ell}\Gamma^i_{km}. \end{aligned}$$

□

Theorem 6.1.7

For all $X, Y, Z, W \in \mathfrak{X}(M)$, the Riemannian curvature tensor satisfies

(i) $R_{X,Y}Z = -R_{Y,X}Z$,

(ii) $R_{X,Y}Z + R_{Y,Z}X + R_{Z,X}Y = 0$ (**First Bianchi identity**),

(iii) $\langle R_{X,Y}Z, W\rangle = -\langle R_{X,Y}W, Z\rangle$,

(iv) $\langle R_{X,Y}Z, W\rangle = \langle R_{Z,W}X, Y\rangle$ (pair symmetry).

Proof.

(i) Immediate from the definition.

(ii) Observe that

$$\begin{aligned} R_{X,Y}Z + R_{Y,Z}X + R_{Z,X}Y &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \\ &\quad + \nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X - \nabla_{[Y,Z]}X \\ &\quad + \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z,X]}Y \\ &= \nabla_X[Y,Z] + \nabla_Y[Z,X] + \nabla_Z[X,Y] - \nabla_{[X,Y]}Z - \nabla_{[Y,Z]}X - \nabla_{[Z,X]}Y \\ &= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]], \end{aligned}$$

where the last two equalities follow from ∇ being torsion-free. Then the result follows from Lemma 2.1.9.

(iii) By polarization, we have that (iii) holds if and only if $\langle R_{X,Y}Z, Z\rangle = 0$ for all $X, Y, Z \in \mathfrak{X}(M)$. Since ∇ is metric, we have

$$\begin{aligned} \langle R_{X,Y}Z, Z\rangle &= \langle \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, Z\rangle \\ &= X\left(\langle\nabla_Y Z, Z\rangle\right) - \langle\nabla_Y Z, \nabla_X Z\rangle - Y\left(\langle\nabla_X Z, Z\rangle\right) + \langle\nabla_X Z, \nabla_Y Z\rangle - \frac{1}{2}[X,Y]\left(\langle Z, Z\rangle\right) \\ &= \frac{1}{2}X\left(Y\left(\langle Z, Z\rangle\right)\right) - \frac{1}{2}Y\left(X\left(\langle Z, Z\rangle\right)\right) - \frac{1}{2}[X,Y]\left(\langle Z, Z\rangle\right) \\ &= 0. \end{aligned}$$

(iv) By (ii),

$$\begin{aligned} \underbrace{\langle R_{X,Y}Z, W\rangle}_{(1)} + \underbrace{\langle R_{Y,Z}X, W\rangle}_{(2')} + \underbrace{\langle R_{Z,X}Y, W\rangle}_{(5)} &= 0 \\ \underbrace{\langle R_{Y,Z}W, X\rangle}_{(2)} + \underbrace{\langle R_{Z,W}Y, X\rangle}_{(3')} + \underbrace{\langle R_{W,Y}Z, X\rangle}_{(6)} &= 0 \\ \underbrace{\langle R_{Z,W}X, Y\rangle}_{(3)} + \underbrace{\langle R_{W,X}Z, Y\rangle}_{(4')} + \underbrace{\langle R_{X,Z}W, Y\rangle}_{(5')} &= 0 \\ \underbrace{\langle R_{W,X}Y, Z\rangle}_{(4)} + \underbrace{\langle R_{X,Y}W, Z\rangle}_{(1')} + \underbrace{\langle R_{Y,W}X, Z\rangle}_{(6')} &= 0. \end{aligned}$$

Then (iii) implies $(k) + (k') = 0$ for $k = 1, 2, 3, 4$. By (i) and (iii), we have $(k) = (k')$ for $k = 5, 6$. Adding up the four equations above yields

$$\begin{aligned} 0 &= \langle R_{Z,X}Y, W\rangle + \langle R_{W,Y}Z, X\rangle \\ &= \langle R_{Z,X}Y, W\rangle - \langle R_{Y,W}Z, X\rangle, \qquad \text{by (i).} \end{aligned}$$

This implies (iv). $\square$

Remark 6.1.8. Often, we consider the Riemannian curvature tensor as a $(0,4)$-tensor field by setting $R(X, Y, Z, W) = \langle R_{X,Y}Z, W\rangle$. Both tensor fields contain the same information and it will be clear from the context whether R is considered as a $(1,3)$ or a $(0,4)$ tensor field.

Definition 6.1.9: Flat Manifolds

A semi-Riemannian manifold is **flat** if $R \equiv 0$.

Example 6.1.10. $\mathbb{R}^{\nu,n-\nu}$ is flat: In standard coordinates on $\mathbb{R}^n$, we have $\Gamma^k_{ij} = 0$ for all i, j, k; by Lemma 6.1.5, we have $R^i_{jk\ell} \equiv 0$ for all i, j, k, ℓ so that $R \equiv 0$.

Remark 6.1.11. One can show that in geodesic coordinates centered at $p \in M$, we have

$$g_{ij} \circ \varphi^{-1}(x) = \varepsilon_i \delta_{ij} + \frac{1}{3} R_{ikj\ell}(p) x^k x^\ell + O\left(|x|^3\right).$$

This implies that R measures the deviation of the metric from being the standard (flat) semi-Riemannian metric g_ν on $\mathbb{R}^n$.

6.2 Sectional Curvature

Let V be an n-dimensional real vector space equipped with a scalar product $\langle -, - \rangle$, $\sigma \subset V$ be a 2-dimensional subspace (2-plane) and $\{u, v\} \subset \sigma$ be a basis of σ. Define

$$Q(u, v) = \langle u, u \rangle \langle v, v \rangle - \langle u, v \rangle^2.$$

If $\langle -, - \rangle$ is positive definite, then $Q(u, v)$ measures the area of the parallelogram spanned by u and v.

Recall. *$\sigma \subset V$ is non-degenerate if and only if $\langle -, - \rangle|_\sigma$ is a scalar product, i.e. still non-degenerate.*

Lemma 6.2.1

A subspace $\sigma \subset V$ is non-degenerate if and only if $Q(u, v) \neq 0$ for any basis $\{u, v\}$ of σ.

Proof. With respect to a basis $\{u, v\}$, $\langle -, - \rangle|_\sigma$ has the matrix representation

$$A := \begin{pmatrix} \langle u, u \rangle & \langle u, v \rangle \\ \langle u, v \rangle & \langle v, v \rangle \end{pmatrix}.$$

By linear algebra, we have σ is non-degenerate if and only if $Q(u, v) = \det A \neq 0$. □

Definition 6.2.2: Sectional Curvature

For $p \in M$, the **sectional curvature** of a non-degenerate 2-plane $\sigma \subset T_pM$ is

$$K(\sigma) := K(u, v) := \frac{R(u, v, v, u)}{Q(u, v)},$$

where $\{u, v\}$ is a basis of σ.

Lemma 6.2.3

$K(\sigma)$ is well-defined, i.e. independent of the chosen basis $\{u, v\}$ of σ.

Proof Sketch. Let $\{\widetilde{u}, \widetilde{v}\}$ be another basis of σ and write $\widetilde{u} = au + bv, \widetilde{v} = cu + dv$. Then by the symmetries of R, we have $R(\widetilde{u}, \widetilde{v}, \widetilde{v}, \widetilde{u}) = (ad - cb)^2 R(u, v, v, u)$. Similarly, $Q(\widetilde{u}, \widetilde{v}) = (ad - cb)^2 Q(u, v)$. □

Definition 6.2.4: Gauß Curvature

For a two-dimensional semi-Riemannian manifold M, we call $K \in C^\infty(M)$, given by $K(p) := K(T_pM)$ the **Gauß curvature**.

Remark 6.2.5 (Interpretation)**.** If M is a *Riemannian* manifold and u, v an orthonormal basis of a 2-plane $\sigma \subset T_pM$ for some $p \in M$, then one can show that

$$d\big(\exp_p(tu), \exp_p(tv)\big)^2 = t^2 \|u - v\|^2 - \frac{1}{3}K(\sigma)t^4 + O\left(t^5\right).$$

Proposition 6.2.6

The Riemannian curvature tensor and the sectional curvatures are equivalent. More precisely, the following holds for all $p \in M$:

(i) $R(p)$ determines $K(\sigma)$ for all 2-planes $\sigma \subset T_pM$.

(ii) $K(\sigma)$ of all 2-planes determines $R(p)$.

Proof.

(i) Follows from the definition.

(ii)

Step 1. *The sectional curvatures determine $R(u, v, v, u)$ for all $u, v \in T_pM$.*

$R(u, v, v, u) = K(u, v) \cdot Q(u, v)$ for all linear independent u, v, which span a non-degenerate 2-plane. The set of such pairs (u, v) is dense in $T_pM \times T_pM$, so $R(u, v, v, u)$ is determined by continuity.

Step 2. *The expressions $R(u, v, v, u)$ determine $R(u, v, v, w)$ for all $u, v, w \in T_pM$.*

By Theorem 6.1.7, $B_v(u, w) \mapsto R(u, v, v, w)$ is a symmetric bilinear form, through polarization determined by the associated quadratic form which we know from Step 1.

Step 3. *The expressions $R(u, v, v, w)$ determine $R(u, v, z, w) + R(u, z, v, w)$ for all $u, v, z, w \in T_pM$.*

By Theorem 6.1.7, $B_{u,w}(z, v) \mapsto R(u, v, z, w) + R(u, z, v, w)$ is a symmetric bilinear form determined by the associated quadratic form which we know from Step 2.

Step 4. *The expressions $R(u, v, z, w) + R(u, z, v, w)$ determine $R(u, v, z, w)$ for all $u, v, z, w \in T_pM$.*

Observe that

$$\begin{aligned} &R(u, v, z, w) + R(u, z, v, w) - R(v, u, z, w) - R(v, z, u, w) \\ =&R(u, v, z, w) + R(u, v, z, w) \underbrace{- R(z, u, v, w) - R(v, z, u, w)}_{=R(u,v,z,w) \text{ by Theorem 6.1.7(ii)}} \\ =&3R(u, v, z, w). \end{aligned}$$

□

Corollary 6.2.7. *Let $p \in M$ and $\kappa \in \mathbb{R}$. Then the following are equivalent.*

(i) $K(\sigma) = \kappa$ *for all non-degenerate 2-planes* $\sigma \subset T_pM$.

(ii) $R(u, v, w, z) = \kappa\left(\langle u, z\rangle\langle v, w\rangle - \langle u, w\rangle\langle v, z\rangle\right)$.

Proof. Exercise. □

Definition 6.2.8: Constant Curvature

A semi-Riemannian manifold is of **constant curvature** if there exists $\kappa \in \mathbb{R}$ (independent of $p \in M$) such that Corollary 6.2.7(i) and (ii) hold for all $p \in M$.

6.3 Curvature of Submanifolds

Let $(\overline{M}, \overline{g})$ be a semi-Riemannian manifold with curvature tensor $\overline{R}$ and (M, g) a semi-Riemannian submanifold of $(\overline{M}, \overline{g})$ with curvature tensor R.

Proposition 6.3.1

For all $X, Y, Z, W \in \mathfrak{X}(M)$, we have

$$\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) + \overline{g}(\Pi(X,Z), \Pi(Y,W)) - \overline{g}(\Pi(Y,Z), \Pi(X,W)).$$

Proof. By Lemma 4.2.5 and Defn. 4.2.6, we have

$$\begin{aligned}
\overline{g}\left(\overline{\nabla}_X \overline{\nabla}_Y Z, W\right) &= \overline{g}\left(\overline{\nabla}_X (\nabla_Y Z + \Pi(Y,Z)), W\right) \\
&= \overline{g}\left(\nabla_X \nabla_Y Z + \underbrace{\Pi(X, \nabla_Y Z)}_{\perp M}, W\right) + \overline{g}\left(\overline{\nabla}_X (\Pi(Y,Z)), W\right) \\
&= \overline{g}(\nabla_X \nabla_Y Z, W) - \overline{g}\left(S^{\Pi(Y,Z)}(X), W\right), && \text{by Lemma 4.2.11(iii)} \\
&= g(\nabla_X \nabla_Y Z, W) - \overline{g}(\Pi(Y,Z), \Pi(X,W)), && \text{by Defn. 4.2.8.}
\end{aligned}$$

Analogously, we have

$$-\overline{g}\left(\overline{\nabla}_X \overline{\nabla}_Y Z, W\right) = -g(\nabla_Y \nabla_X Z, W) + \overline{g}(\Pi(Y,W), \Pi(X,Z)).$$

Finally, by Lemma 4.2.5 and Defn. 4.2.6 we get

$$-\overline{g}\left(\overline{\nabla}_{[X,Y]} Z, W\right) = -\overline{g}\left(\nabla_{[X,Y]} Z + \underbrace{\Pi([X,Y], Z)}_{\perp M}, W\right) = -g\left(\nabla_{[X,Y]} Z, W\right)$$

and adding up yields the result. □

Corollary 6.3.2. *The sectional curvatures $\overline{K}(v,w)$ and $K(v,w)$ of a basis $\{v,w\}$ of a non-degenerate 2-plane $\sigma \subset T_pM \subset T_p\overline{M}$ are related by*

$$\overline{K}(v,w) = K(v,w) - \frac{\overline{g}(\Pi(v,v), \Pi(w,w)) - \overline{g}(\Pi(v,w), \Pi(v,w))}{g(v,v)g(w,w) - g(v,w)^2}.$$

Proof. Immediate from Prop. 6.3.1. □

Example 6.3.3. (i) Let $(\overline{M}, \overline{g}) = (\mathbb{R}^3, g_{\text{eucl}})$ and (M, g) be a 2-dimensional (Riemannian) hypersurface. Let $p \in M \subset \mathbb{R}^3$, ξ a unit normal field around p, and $\{e_1, e_2\}$ be an orthonormal basis of T_pM. Then by

Corollary 6.3.2,

$$\begin{aligned}
0 = \overline{K}(e_1,e_2) &= K(e_1,e_2) - \overline{g}\left(\Pi\left(e_1,e_2\right),\Pi\left(e_2,e_2\right)\right) + \overline{g}\left(\Pi\left(e_1,e_2\right),\Pi\left(e_1,e_2\right)\right) \\
&= K(e_1,e_2) - \overline{g}\left(g\left(S^{\xi}\left(e_1\right),e_1\right)\cdot\xi, g\left(S^{\xi}\left(e_2\right),e_2\right)\right) \\
&\quad + \overline{g}\left(g\left(S^{\xi}\left(e_1\right),e_2\right)\cdot\xi, g\left(S^{\xi}\left(e_1\right),e_2\right)\right), && \text{by Defn. 4.2.8 and } \xi \text{ unit normal} \\
&= K(e_1,e_2) - g\left(S^{\xi}\left(e_1\right),e_1\right)\cdot g\left(S^{\xi}\left(e_2\right),e_2\right) + g\left(S^{\xi}\left(e_1\right),e_2\right)^2 \\
&= K(p) - \det\left(S^{\xi}\right), && \text{by definition of Gauß curvature.}
\end{aligned}$$

This implies

$$K(p) = \det\left(S^{\xi}\right) = \lambda_1\cdot\lambda_2,$$

if λ_1,λ_2 are the principal curvatures of M at p.

Next, suppose $M = \left\{(x,f(x)) : x\in\mathbb{R}^2\right\} \subset \mathbb{R}^3$ with $f(x) = \frac{1}{2}\left(\lambda_1\left(x^1\right)^2 + \lambda_2\left(x^2\right)^2\right)$. Then by Example 4.2.16, $K(0) = \lambda_1\cdot\lambda_2$.

(ii) Let $(\overline{M},\overline{g}) = (\mathbb{R}^n, g_\nu)$ and $(M,g) = \left(f_\nu^{-1}(c), g_\nu|_{f_\nu^{-1}(c)}\right)$, with $f_\nu(x) = g_\nu(x,x)$ for $c\neq 0$. That is,

$$(M,g) = \begin{cases} \mathbb{S}^{n-1}_\nu\left(\sqrt{c}\right) & \text{if } c>0 \\ \mathbb{H}^{n-1}_{\nu-1}\left(\sqrt{-c}\right) & \text{if } c<0. \end{cases}$$

(See Example 3.3.4.) By Problem 24, we have $\Pi(X,Y) = \frac{1}{\sqrt{|c|}}g_\nu(X,Y)\cdot\xi$, where $\xi = \frac{\operatorname{grad} f_\nu}{|\operatorname{grad} f_\nu|} = \frac{\operatorname{grad} f_\nu}{2\sqrt{|c|}}$. For any non-degenerate 2-plane $\sigma\subset TM$, Corollary 6.3.2 implies

$$0 = \overline{K}(\sigma) = K(\sigma) - \left(\frac{1}{\sqrt{|c|}}\right)^2\overline{g}\left(\xi,\xi\right) = K(\sigma) - \frac{1}{|c|}\operatorname{sgn}(c).$$

Thus, (M,g) has constant curvature $\frac{1}{|c|}\operatorname{sgn}\left(c\right) = \frac{1}{c}$. In particular, $\mathbb{S}^{n-1}_\nu(r)$ has constant curvature $\frac{1}{r^2}$ and $\mathbb{H}^{n-1}_\nu(r)$ has constant curvature $-\frac{1}{r^2}$.

6.4 Ricci and scalar curvature

Definition 6.4.1: Ric **and Ricci tensors**

For $p\in M$ and $v,w\in T_pM$, we define

$$\operatorname{Ric}(v,w) := \operatorname{tr}\left(\underbrace{u\mapsto R_{u,v}w}_{\in (T_pM)^1_1}\right) = \operatorname{tr}_g\left(\underbrace{(u,z)\mapsto R\left(u,v,w,z\right)}_{\in (T_pM)^2_0}\right).$$

The function $\operatorname{Ric}\in\mathscr{T}^0_2(M)$ is called the **Ricci tensor**.

(See Problem 17.) If $\{e_i\}$ is an orthonormal basis of T_pM, we have

$$R(v,w) = \sum_{i=1}^n \varepsilon_i\langle R_{e_i,v}w, e_i\rangle = \sum_{i=1}^n \varepsilon_i R\left(e_i,v,w,e_i\right).$$

Lemma 6.4.2

Let $p \in M$. Then we have

(i) $\operatorname{Ric}(v, w) = \operatorname{Ric}(w, v)$ for all $v, w \in T_pM$.

(ii) If $v \in T_pM$ is such that $\langle v, v\rangle \neq 0$ and $\{e_2, \ldots, e_n\}$ is an orthonormal basis of $v^\perp$, then $R(v, v) = \langle v, v\rangle \sum_{j=2}^n K(v, e_i)$.

Proof. Let $\{e_1, \ldots, e_n\}$ be a pseudo-orthonormal basis of T_pM.

(i) Observe that

$$\begin{aligned} \operatorname{Ric}(v, w) &= \sum_{i=1}^n \varepsilon_i \langle R_{e_i, v} w, e_i\rangle \\ &= \sum_{i=1}^n \varepsilon_i \langle R_{e_i, w} v, e_i\rangle, \qquad \text{by Theorem 6.1.7} \\ &= \operatorname{Ric}(w, v). \end{aligned}$$

(ii) Without loss of generality, we may assume that $g(v, v) = \pm 1$ and $\{e_1, \ldots, e_n\}$ is chosen such that $e_1 = v$. Then

$$\begin{aligned} \operatorname{Ric}(v, v) &= \sum_{i=1}^n \varepsilon_i \langle R_{e_i, v} v, e_i\rangle \\ &= \sum_{i=2}^n \varepsilon_i \langle R_{e_i, v} v, e_i\rangle, \qquad \text{by Theorem 6.1.7} \\ &= \sum_{i=2}^n \varepsilon_i K(v, e_i) \left[\langle v, v\rangle \underbrace{\langle e_i, e_i\rangle}_{=\varepsilon_i} - \underbrace{\langle v, e_i\rangle}_{=0}\right] \\ &= \langle v, v\rangle \sum_{i=2}^n K(v, e_i) \cdot \underbrace{(\varepsilon_i)^2}_{=1}. \end{aligned}$$

□

Remark 6.4.3 (Interpretation of the Ricci curvature). (i) Lemma 6.4.2(ii) asserts that $\operatorname{Ric}(v, v)$ is the "mean" of all sectional curvatures of 2-planes containing v.

(ii) If M is a Riemannian manifold, $p \in M$, $v \in T_pM$, then $\operatorname{Ric}(v, v)$ measures volume distortion of a cone at p pointing in the direction of v.

Definition 6.4.4: Scalar curvature

The **scalar curvature** is the function $\operatorname{scal} = \operatorname{tr}_g \operatorname{Ric} \in C^\infty(M)$.

With respect to a pseudo-orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM, we have

$$\operatorname{scal}(p) = \sum_{i=1}^n \varepsilon_i \operatorname{Ric}(e_i, e_i) = \sum_{i,j=1}^n \varepsilon_i \varepsilon_j R(e_j, e_i, e_i, e_j).$$

Remark 6.4.5 (Interpretation). (i) Since

$$\operatorname{scal}(p) = \sum_{i,j=1}^n \varepsilon_i \varepsilon_j R(e_j, e_i, e_i, e_j) = \sum_{i \neq j} K(e_i, e_j),$$

we have that $\operatorname{scal}(p)$ is the "mean" over all $i \neq j$ sectional curvatures of non-degenerate 2-planes in T_pM.

(ii) If M is a Riemannian manifold, scal measures the volume growth of small geodesic balls. That is, for $B_r(0) \subset T_pM$ and r small, $\exp_p(B_r(0)) = \{q \in M : d(p,q) < r\} =: B_r(p)$ and

$$\operatorname{vol}(B_p(p)) = \left(1 - \frac{\operatorname{scal}}{6(n-2)} r^2 + O\left(r^4\right)\right) \cdot \operatorname{vol}(B_r(0)).$$

Remark 6.4.6. In general, Ric and scal contain less information than R and K.

(i) In 1-dimensional spaces, $R = K = \operatorname{Ric} = \operatorname{scal} = 0$ (since all vectors are linearly dependent and so $R_{u,v}w = 0$ by Theorem 6.1.7).

(ii) In 2-dimensional spaces, we have by Corollary 6.2.7 that $R(X,Y,Z,W) = K[\langle Y,Z\rangle\langle X,W\rangle - \langle X,Z\rangle\langle Y,W\rangle]$, where K is the Gauß curvature. This implies $\operatorname{Ric}(X,Y) = K \cdot \langle X,Y\rangle$ and $\operatorname{scal} = 2K$. Thus, all quantities contain the same information.

(iii) In 3-dimensional spaces, one can compute R (and hence K) out of Ric.

Summarizing,

dim M	2	3	≥ 4
	R $\Updownarrow$ K $\Updownarrow$ Ric $\Updownarrow$ scal	R $\Updownarrow$ K $\Updownarrow$ Ric $\Downarrow$ scal	R $\Updownarrow$ K $\Downarrow$ Ric $\Downarrow$ scal

Definition 6.4.7: Einstein's equations (with cosmological constant Λ)

The equations are

$$\underbrace{\operatorname{Ric} - \frac{1}{2}\operatorname{scal} \cdot g + \Lambda \cdot g}_{\text{geometry}} = \underbrace{\frac{8\pi G}{c^4} T}_{\text{matter}}, \tag{6.4.1}$$

where T is the energy momentum tensor, G is the gravitational constant, and c is the speed of light.

Lorentzian manifolds solving (6.4.1) for a suitable matter model T are models of our universe.

Example 6.4.8. (i) If M^n is a Lorentzian manifold of constant curvature κ, then by Lemma 6.4.2(ii), $\operatorname{Ric} = (n-1)\kappa g$ so that $\operatorname{scal} = n(n-1)\kappa$. Thus, (6.4.1) holds with $T = 0$ and $\Lambda = \frac{1}{2}(n-2)(n-1)\kappa$.

In Minkowski space, de-Sitter space $\mathbb{S}^4_1(1)$ and anti de-Sitter space $\mathbb{H}^1_1(1)$ are solutions (see Example 6.1.10 and Example 6.3.3).

(ii) The Schwartzchild metric in Example 3.4.6(i) is a solution of (6.4.1) with $\Lambda = 0$ and $T = 0$.

Remark 6.4.9. Einstein's equation form in local coordinates a highly complicated system of partial differential equations in $\frac{n(n+1)}{2}$ variables. Even in the (most relevant) case $n = 4$, we have 10 variables involved. The equations are far from being well understood and their analysis forms a very active area of current research.

Chapter 7

Differential Forms and Stokes' Theorem

7.1 Alternating Tensors

Throughout this section, let V be an n-dimensional real vector space.

Definition 7.1.1: Alternating Tensors

A tensor $t \in V_k^0$ is **alternating** if

$$t\left(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k\right) = -t\left(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k\right) \qquad (v_1, \ldots, v_k \in V, 1 \le i < j \le k).$$

The set $\Lambda^k V := \left\{t \in V_k^0 : t \text{ is alternating}\right\}$ is a subspace of V_k^0.

Lemma 7.1.2

Let $t \in V_k^0$. Then the following are equivalent.

(i) t is alternating;

(ii) $t\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) = \operatorname{sgn}(\sigma) \cdot t\left(v_1, \ldots, v_k\right)$ for every permutation $\sigma \in \mathfrak{S}_k$;

(iii) $t(-, \ldots, -, v, -, \ldots, -, v, -, \ldots, -) = 0$ for all $v \in V$ and all $1 \le i < j \le k$, where the v's appear in the ith and jth positions.

(ii) ⇒ (i): (i) asserts that (ii) holds for each transposition.

(i) ⇒ (ii): Write $\sigma \in \mathfrak{S}_k$ as a product of transpositions $\sigma = \sigma_1 \circ \cdots \circ \sigma_\ell$. Then $\operatorname{sgn}(\sigma) = (-1)^\ell$ and (ii) follows by applying (i) ℓ times.

(i) ⇒ (iii): By setting $v = v_i = v_j$ in Defn. 7.1.1, we see that

$$t(-, \ldots, -, v, -, \ldots, -, v, -, \ldots, -) = -t(-, \ldots, -, v, -, \ldots, -, v, -, \ldots, -)$$

so that $t(-, \ldots, -, v, -, \ldots, -, v, -, \ldots, -) = 0$.

(iii) ⇒ (i): Polarization: by (iii),

$$\begin{aligned} 0 &= t\left(-, \ldots, v_i + v_j, \ldots, v_i + v_j, \ldots, -\right) \\ &= \underbrace{t\left(-, \ldots, v_i, \ldots, v_i, \ldots -\right)}_{=0} + t\left(-, \ldots, v_i, \ldots, v_j, \ldots, -\right) \\ &\quad + t\left(-, \ldots, v_j, \ldots, v_i, \ldots, -\right) + \underbrace{t\left(-, \ldots, v_j, \ldots, v_j, \ldots -\right)}_{=0}. \end{aligned}$$

Remark 7.1.3. We set $\Lambda^0 V = V_0^0 = \mathbb{R}$ and $\Lambda^1 V = V_1^0 = V^*$.

Definition 7.1.4: Wedge Product

The **wedge product** of $v_1^*, \ldots, v_k^* \in V^* = \Lambda^1 V$ is a map

$$\begin{aligned} v_1^* \wedge \cdots \wedge v_k^* \, (u_1, \ldots, u_k) &= \det\left(\left(v_i^* \, (u_j) \right)_{1 \le i,j \le k} \right) \\ &= \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) \, v_{\sigma(1)}^* \, (u_1) \cdots v_{\sigma(k)}^* \, (u_k) \\ &= \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) \, v_{\sigma(1)}^* \otimes \cdots \otimes v_{\sigma(k)}^* \, (u_1, \ldots, u_k) \, . \end{aligned}$$

Remark 7.1.5. By the properties of the determinant, $v_1^* \wedge \cdots \wedge v_k^* \in \Lambda^k V$ and $v_{\sigma(1)}^* \wedge \cdots \wedge v_{\sigma(k)}^* = \operatorname{sgn}(\sigma) v_1^* \wedge \cdots \wedge v_k^*$ for each $\sigma \in \mathfrak{S}_k$.

Proposition 7.1.6

Let $\{e_1, \ldots, e_n\}$ be a basis of V and $\left\{e^1, \ldots, e^n\right\}$ its dual basis. Then $\left\{e^{i_1} \wedge \cdots \wedge e^{i_k}\right\}_{1 \le i_1 < \cdots < i_k \le n}$ is a basis of $\Lambda^k V$. In particular, $\dim \Lambda^k V = \binom{n}{k}$ and $\Lambda^k V = \{0\}$ for $k > n$. For $\omega \in \Lambda^k V$, written as

$$\omega = \sum_{1 \le i_1 < \cdots i_k \le n} \omega_{i_1, \ldots, i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} \tag{7.1.1}$$

the coefficients are given by

$$\omega_{i_1, \ldots, i_k} = \omega\left(e_{i_1}, \ldots, e_{i_k}\right).$$

Proof. Exercise. □

Proposition 7.1.7

Let $e^1, \ldots, e^k \in V^*$ and $f^1, \ldots, f^k \in V^*$ such that $f^i = a^i_j e^j$ for some matrix $A = \left\{a^i_j\right\}$. Then

$$f^1 \wedge \cdots \wedge f^k = \det(A) e^1 \wedge \cdots \wedge e^k.$$

Proof. For $v_1, \ldots, v_k \in V$ arbitrary, we have (by the multiplicity of the determinant)

$$\begin{aligned} f^1 \wedge \cdots \wedge f^k \, (v_1, \ldots, v_k) &= \det\left(\left(f^i \, (v_j) \right)_{1 \le i,j \le k} \right) \\ &= \det\left(\left(a^i_\ell e^\ell \, (v_j) \right)_{1 \le i,j \le k} \right) \\ &= \det\left(\left(a^i_\ell \right)_{1 \le i,\ell \le n} \right) \cdot \det\left(\left(e^\ell \, (v_j) \right)_{1 \le \ell,j \le n} \right) \\ &= \det(A) e^1 \wedge \cdots \wedge e^k \, (v_1, \ldots, v_k) \, . \end{aligned}$$ □

Theorem 7.1.8

There is exactly one bilinear map

$$\wedge : \Lambda^k V \times \Lambda^\ell V \to \Lambda^{k+\ell} V$$

such that

$$\left(v^1 \wedge \cdots \wedge v^k\right) \wedge \left(v^{k+1} \wedge \cdots \wedge v^{k+\ell}\right) = v^1 \wedge \cdots \wedge v^{k+\ell} \qquad (v^1, \ldots, v^{k+\ell} \in V^*).$$

Proof. Let $e^1, \dots, e^n$ be a basis of V^*. For

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1,\dots,i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} \quad \text{and} \quad \eta = \sum_{1 \le i_1 < \cdots < i_k \le n} \eta_{i_1,\dots,i_k} e^{i_1} \wedge \cdots \wedge e^{i_k},$$

set

$$\omega \wedge \eta = \sum_{\substack{1 \le i_1 < \cdots < i_k \le n \\ 1 \le j_1 < \cdots < j_\ell \le n}} \omega_{i_1,\dots,i_k} \cdot \eta_{j_1,\dots,j_\ell} e^{i_1} \wedge \cdots e^{i_k} \wedge e^{j_1} \wedge \cdots \wedge e^{j_\ell}. \tag{7.1.2}$$

With this definition, $\wedge$ is bilinear and satisifes the desired property. It is unique as it is determined by how it acts on the bases $\left\{e^{i_1} \wedge \cdots \wedge e^{i_k}\right\}_{1 \le i_1 < \cdots < i_k \le n}$ and $\left\{e^{j_1} \wedge \cdots \wedge e^{j_\ell}\right\}_{1 \le j_1 < \cdots < j_\ell \le n}$. □

Lemma 7.1.9

For $\omega \in \Lambda^k V, \eta \in \Lambda^\ell V$ and $\lambda \in \Lambda^m V$, we have

(i) $(\omega \wedge \eta) \wedge \lambda = \omega \wedge (\eta \wedge \lambda)$

(ii) $\omega \wedge \eta = (-1)^{k \cdot \ell} \eta \wedge \omega$.

Proof. Straightforward from (7.1.2). □

7.2 Differential forms and the exterior derivative

Throughout, let M^n be a C^∞-manifold (but not necessarily semi-Riemannian).

Definition 7.2.1: Differential k-form

A tensor field $\omega \in \mathscr{T}^0_k(M)$ is called a **differential k-form** (or k-form for short) if $\omega(p) \in \Lambda^k T_p M$ for all $p \in M$. We denote by $\Omega^k(M)$ the $C^\infty(M)$-module of differential k-forms on M.

Remark 7.2.2. (Compare with Lemma 7.1.2.) Let $\omega \in \mathscr{T}^0_k(M)$. Then the following are equivalent.

(i) $\omega \in \Omega^k(M)$;

(ii) $\omega\left(-, \dots, X_i, \dots, X_j, \dots, -\right) = -\omega\left(-, \dots, X_j, \dots, X_i, \dots, -\right)$ for all $X_i, X_j \in \mathfrak{X}(M), 1 \le i < j < n$;

(iii) $\omega\left(X_{\sigma(1)}, \dots, X_{\sigma(k)}\right) = \operatorname{sgn}(\sigma)\omega\left(X_1, \dots, X_k\right)$ for all $X_1, \dots, X_k \in \mathfrak{X}(M)$ and all $\sigma \in \mathfrak{S}_k$;

(iv) $\omega(-, \dots, X, -, \dots, -, X, \dots, -) = 0$ for all $X \in \mathfrak{X}(M)$.

Remark 7.2.3. (i) We have an operation $\wedge : \Omega^k(M) \times \Omega^\ell(M) \to \Omega^{k+\ell}(M)$ by setting

$$(\omega \wedge \eta)(p) = \omega(p) \wedge \eta(p).$$

(ii) With respect to a chart $(U, \varphi = (x^1, \dots, x^n))$, we can express $\omega \in \Omega^k(M)$ as

$$\omega|_U = \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1,\dots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where (see 7.1.1), $\omega_{i_1,\dots,i_k} = \omega\left(\partial_{x^{i_1}}, \dots, \partial_{x^{i_k}}\right) \in C^\infty(U)$.

(iii) $\Omega^0(M) := C^\infty(M)$ and $\Omega^1(M) = \mathscr{T}^0_1(M)$. (This is why they are called 1-forms.)

Definition 7.2.4: Pullback

Let M, N be C^∞-manifolds and $\varphi \in C^\infty(M, N)$. Then we have a linear map $\varphi^* : \Omega^k(N) \to \Omega^k(M)$ defined by

$$(\varphi^*\omega)(p)(v_1, \ldots, v_k) := \omega(\varphi(p))\left(T_p\varphi(v_1), \ldots, T_p\varphi(v_k)\right) \qquad \left(p \in M, v_1, \ldots, v_k \in T_pM\right).$$

The map $\varphi^*\omega$ is called the **pullback** of ω under φ.

Remark 7.2.5. (i) For $f \in C^\infty(N) = \Omega^0(N)$, we have $\varphi^* f = f \circ \varphi \in C^\infty(M)$.

(ii) For $\omega \in \Omega^k(M), \eta \in \Omega^\ell(M)$,

$$\varphi^*(\omega \wedge \eta) = (\varphi^*\omega) \wedge (\varphi^*\eta).$$

Notation

Given arbitrary sets A, B, elements $a_0, \ldots, a_m \in A$, and a map $\varphi : \prod_{j=1}^m A \to B$, we will use the shorthand

$$\varphi(a_0, \ldots, \widehat{a_i}, \ldots, a_m) = \varphi(a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m).$$

In otherwords, the term $\widehat{a_i}$ denotes "removing the input a_i".

Definition 7.2.6: Exterior Derivative

The **exterior derivative** d is the map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ defined by

$$\begin{aligned} d\omega(X_0, \ldots, X_k) = &\sum_{j=0}^{k}(-1)^j X_j\left(\omega\left(X_0, \ldots, \widehat{X}_j, \ldots, X_k\right)\right) \\ &+ \sum_{0 \le i < j \le k}(-1)^{i+j}\omega\left(\left[X_i, X_j\right], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k\right). \end{aligned}$$

Lemma 7.2.7

d is well-defined.

Proof. Check that $d\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$ is $C^\infty(M)$-multilinear (so that $d\omega \in \mathcal{T}^0_{k+1}(M)$) and antisymmetric in all variables (so that $d\omega \in \Omega^{k+1}(M)$)). Then the result follows from an exercise. □

Remark 7.2.8. For $k = 0$, $d : \Omega^0(M) = C^\infty(M) \to \Omega^1(M) = \mathcal{T}_1^{\,0}(M)$ coincides with the differential of functions. For $f \in C^\infty(M), df(X) = X(f)$.

Lemma 7.2.9

Let $f \in C^\infty(M)$, N another C^∞-manifold and $\varphi \in C^\infty(N, M)$. Then

(i) $d(\varphi^* f) = \varphi^* df$;

(ii) $d(df) = 0$.

Proof. (i) For $p \in N$, $g \in C^\infty(N)$, $dg(p)(v) = v(g) = T_pg(v)$ for all $v \in T_pN$. Thus,

$$\begin{aligned} d(\varphi^* f)(p)(v) &= T_p(\varphi^* f)(v) = T_{\varphi(p)}(f \circ \varphi)(v) = T_pf(T_p\varphi)(v) = df(\varphi(p))\left(T_p\varphi(v)\right) \\ &= (\varphi^* df)(p)(v) \end{aligned}$$

(ii) If $\omega = df$ then

$$\begin{aligned} d\omega(X,Y) &= X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) \\ &= X(df(Y)) - Y(df(X)) - df([X,Y]) \\ &= X(Y(f)) - Y(X(f)) - [X,Y](f) = 0. \end{aligned}$$

□

Lemma 7.2.10

Let $\omega \in \Omega^k(M)$ and (U,φ) be a chart of M. Then if

$$\omega|_U = \sum_{1\le i_1<\cdots<i_k\le n} \omega_{i_1,\ldots,i_k} dx^{i_1}\wedge\cdots\wedge dx^{i_k},$$

we have

$$d\omega|_U = \sum_{1\le i_1<\cdots<i_k\le n} d\omega_{i_1,\ldots,i_k}\wedge dx^{i_1}\cdots\wedge dx^{i_k}.$$

Proof.

$$\begin{aligned} d\omega_{i_1,\ldots,i_k}\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k} &= \sum_{\substack{i_0=1 \\ i_0\notin\{i_1,\ldots,i_k\}}}^{n} \partial_{i_0}\omega_{i_1,\ldots,i_k} dx^{i_0}\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k} \\ &= \sum_{i_0<i_1}\left(\partial_{i_0}\omega_{i_1,\ldots,i_k}\right) dx^{i_0}\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k} \\ &\quad - \sum_{i_1<i_0<i_2}\left(\partial_{i_0}\omega_{i_1,\ldots,i_k}\right) dx^{i_1}\wedge dx^{i_0}\wedge\cdots\wedge dx^{i_k} \\ &\quad + \sum_{i_2<i_0<i_3}\ldots - \sum_{i_3<i_0<i_4}\ldots + \cdots \\ &= \sum_{1\le i_1<\cdots<i_k\le n} d\omega_{i_1,\ldots,i_k}\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}, \qquad \text{by changing indices} \\ &= \sum_{1\le i_0<i_1<\cdots<i_k\le n} \underbrace{\sum_{m=0}^{k}(-1)^m\partial_{i_m}\omega_{i_0,\ldots,\widehat{i_m},\ldots,i_k}}_{=d\omega\left(\partial_{i_0},\ldots,\partial_{i_k}\right) \text{ by def of } d} dx^{i_0}\wedge\cdots\wedge dx^{i_k} \\ &= d\omega|_U. \end{aligned}$$

□

Lemma 7.2.11

Let $\omega \in \Omega^k(M)$, $\eta \in \Omega^\ell(M)$ and $\varphi \in C^\infty(N,M)$. Then

(i) $d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^k\omega\wedge d\eta$;

(ii) $d^2\omega = d(d\omega) = 0$;

(iii) $\varphi^* d\omega = d(\varphi^*\omega)$.

Proof. Because d and φ^* are $\mathbb{R}$-linear operations, it suffices to check these identities for forms supported in a coordinate neighborhood. (Otherwise, use a partition of unity and write $\omega = \sum_i \chi_i\omega$.)

(i) Write

$$\omega\wedge\eta = \sum_{\substack{i_1<\cdots<i_k \\ j_1<\cdots<j_\ell}} \omega_{i_1,\ldots,i_k}\cdot\eta_{j_1,\ldots,j_\ell} dx^{i_1}\wedge\cdots\wedge dx^{i_k}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_\ell}.$$

Then

$$\begin{aligned} d(\omega\wedge\eta) &= \sum_{\substack{i_1<\cdots<i_k\\ j_1<\cdots<j_\ell}} \left(d\omega_{i_1,\dots,i_k}\cdot\eta_{j_1,\dots,j_\ell}+\omega_{i_1,\dots,i_k}\cdot d\eta_{j_1,\dots,j_\ell}\right) dx^{i_1}\wedge\cdots\wedge dx^{i_k}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_\ell}\\ &= \sum_{\substack{i_1<\cdots<i_k\\ j_1<\cdots<j_\ell}} \Big[\left(d\omega_{i_1,\dots,i_k}\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}\right)\wedge\left(\eta_{j_1,\dots,j_\ell}dx^{j_1}\wedge\cdots\wedge dx^{j_\ell}\right)\\ &\qquad\qquad + (-1)^k\left(\omega_{i_1,\dots,i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k}\right)\wedge d\eta_{j_1,\dots,j_\ell}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_\ell}\Big]\\ &= d\omega\wedge\eta+(-1)^k\omega\wedge d\eta. \end{aligned}$$

(ii) By Lemma 7.2.9(ii), we see that (ii) holds for $k=0$. Then

$$\begin{aligned} d(d\omega) &= d\left(\sum_{i_1<\cdots<i_k} d\omega_{i_1,\dots,i_k}\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}\right)\\ &= \sum_{i_1<\cdots<i_k}\left[\underbrace{d^2\omega_{i_1,\dots,i_k}}_{=0}\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}+\sum_{m=1}^{k}(-1)^m d\omega_{i_1,\dots,i_k}dx^{i_1}\wedge\cdots\wedge\underbrace{d^2x^{i_m}}_{=0}\wedge\cdots\wedge dx^{i_k}\right], \qquad \text{by (i)}\\ &= 0. \end{aligned}$$

(iii) By Lemma 7.2.9(i), we see that (iii) holds for $k=0$. Then

$$\begin{aligned} 2\varphi^*d\omega &= \sum_{i_1<\cdots<i_k}\varphi^*\left(d\omega_{i_1,\dots,i_k}\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}\right)\\ &= \sum_{i_1<\cdots<i_k}\left(\varphi^*d\omega_{i_1,\dots,i_k}\right)\wedge\left(\varphi^*dx^{i_1}\right)\wedge\cdots\wedge\left(\varphi^*dx^{i_k}\right), && \text{by Remark 7.2.5(ii)}\\ &= \sum_{i_1<\cdots<i_k} d\left(\varphi^*\omega_{i_1,\dots,i_k}\right)\wedge d\left(\varphi^*x^{i_1}\right)\wedge\cdots\wedge d\left(\varphi^*x^{i_k}\right), && \text{by (iii) for } k=0\\ &= d\left(\sum_{i_1<\cdots<i_k}\varphi^*\omega_{i_1,\dots,i_k}\cdot d\left(\varphi^*x^{i_1}\right)\wedge\cdots\wedge d\left(\varphi^*x^{i_k}\right)\right), && \text{by (i) \& (ii)}\\ &= d\left(\varphi^*\left(\sum_{i_1<\cdots<i_k}\omega_{i_1,\dots,i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k}\right)\right), && \text{by (iii) for } k=0 \text{ and Remark 7.2.5(ii)}\\ &= d\left(\varphi^*\omega\right). \end{aligned}$$

□

Lemma 7.2.12

There are identifications $\Omega^1(\mathbb{R}^3)\cong\mathfrak{X}(\mathbb{R}^3), \Omega^2(\mathbb{R}^3)\cong\mathfrak{X}(\mathbb{R}^3), \Omega^3(\mathbb{R}^3)\cong C^\infty(\mathbb{R}^3)$ such that we have a commutative diagram

$$\begin{tikzcd} \Omega^0\left(\mathbb{R}^3\right) \arrow[r, "d"] \arrow[d, equal] & \Omega^1\left(\mathbb{R}^3\right) \arrow[r, "d"] \arrow[d, "\sim"] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"] \arrow[d, "\sim"] & \Omega^3(\mathbb{R}^3) \arrow[d, "\sim"] \\ C^\infty\left(\mathbb{R}^3\right) \arrow[r, "\mathrm{grad}"] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\mathrm{curl}"] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\mathrm{div}"] & C^\infty(\mathbb{R}^3) \end{tikzcd}.$$

Proof. Exercise. □

Definition 7.2.13: Closed & Exact k-forms, k-th de-Rham Cohomology, k-th Betti number, & Euler Characteristic

(i) The space of **closed k-forms** is

$$C_k(M) := \left\{\omega \in \Omega^k(M) : d\omega = 0\right\} = d^{-1}(\{0\}).$$

The space of **exact k-forms** is the set

$$E_k(M) := \left\{\omega \in \Omega^k(M) : \omega = d\eta \text{ for some } \eta \in \Omega^{k-1}(M)\right\} = d\left(\Omega^{k-1}(M)\right).$$

(ii) The **k-th de-Rham cohomology** is the set $H_k(M) = C_k(M)/E_k(M)$.

(iii) The **k-th Betti number** is $b_k(M) := \dim H_k(M)$.

The **Euler characteristic** is

$$\chi(M^n) := \sum_{k=0}^{n} (-1)^k\, b_k(M).$$

Note that by Lemma 7.2.11(ii), we have $E_k(M) \subset C_k(M)$ so the quotient $C_k(M)/E_k(M)$ is well-defined.

Remark 7.2.14. (i) If $\varphi : M \to N$ is a diffeomorphism, then by Lemma 7.2.11(iii), φ^* restricts to an isomorphism

$$\varphi^* : C_k(N) \xrightarrow{\sim} C_k(M), \qquad \varphi^* : E_k(N) \xrightarrow{\sim} E_k(M), \qquad \text{and} \qquad \varphi^* : H_k(N) \xrightarrow{\sim} H_k(M).$$

Thus, $b_k(M) = b_k(N)$ and $\chi(M) = \chi(N)$. These equalities hold even for homeomorphisms.

(ii) If M is compact, then $b_k(M) < \infty$ and $\chi(M) < \infty$.

7.3 Integration on Manifolds

We first need the concept of **orientation**. Let V be an n-dimensional real vectore space and $\omega \in \Lambda^n V \setminus \{0\}$. Then for each basis $e_1, \ldots, e_n$ of V, $\omega_{1,\ldots,n} := \omega(e_1, \ldots, e_n) \neq 0$ (as $\omega = \omega_{1,\ldots,n} e^1 \wedge \cdots \wedge e^n$, where $\left\{e^i\right\}$ is the dual basis). Then ω induces an orientation on V as follows:

(i) A basis $\{e_1, \ldots, e_n\}$ of V is called **positively (negatively) oriented** if $\omega(e_1, \ldots, e_n) > 0$ (< 0).

(ii) Two bases $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ are **equally oriented** $\iff$ the matrix $A = \left(a_i^j\right)_{1 \le i,j \le n}$ with $f_j = a^i_j e_i$ satisfies $\det A > 0$. Note that by Prop. 7.1.7, we have $\omega(f_1, \ldots, f_n) = \det A\, \omega(e_1, \ldots, e_n)$.

From now on, let M^n be a C^∞-manifold.

Definition 7.3.1: Orientable, Oriented Manifolds, & Orientation-preserving diffeomorphisms

(i) M is called **orientable** if there exists $\omega \in \Omega^n(M)$ such that $\omega(p) \neq 0$ for all $p \in M$. The pair (M, ω) is an **oriented manifold**.

(ii) Let (M, ω_M) and (N, ω_N) be oriented manifolds and $\varphi \in C^\infty(M, N)$ a diffeomorphism. Then φ is **orientation-preserving** if there exists a positive function $f \in C^\infty(M)$ such that $\varphi^*\omega_N = f \cdot \omega_M$.

Notation

We often just write M instead of (M, ω) when the map ω is clear from context.

Lemma 7.3.2

The following are equivalent.

(i) M is orientable.

(ii) There exists an atlas $\mathscr{A} = \{(U_i, \varphi_i) : i \in I\}$ of M such that

$$\det D\left(\varphi_i \circ \varphi_j^{-1}\right)\big|_x > 0 \qquad \left(x \in \varphi_j\left(U_i \cap U_j\right), i, j \in I\right).$$

Proof. Exercise. □

Example 7.3.3. (i) $\mathbb{S}^1 \cong I$ is orientable. Mobius strips are nonorientable.

(ii) Any open subset $U \subset \mathbb{R}^n$ admits a **canonical orientation**, induced by

$$dx^1 \wedge \cdots \wedge dx^n \in \Omega^n(U),$$

where $(x^1, \dots, x^n)$ are the standard coordinates. From now on, open subsets $U \subset \mathbb{R}^n$ are always equipped with this orientation unless stated otehrwise.

Lemma 7.3.4

Let $U, V \subset \mathbb{R}^n$ be open and $\varphi : V \to U$ a diffeomorphism. Then φ is orientation-preserving $\iff$ $\det(D\varphi|_y) > 0$ for all $y \in V$.

Proof. Let $x^1, \dots, x^n$ be the coordinates on U and $y^1, \dots, y^n$ the coordinates on V. Then with $\varphi = (x^1 \circ \varphi, \dots, x^n \circ \varphi) = (\varphi^1, \dots, \varphi^n)$, we have

$$\begin{aligned}
\varphi^*\left(dx^1 \wedge \cdots \wedge dx^n\right) &= \varphi^* dx^1 \wedge \cdots \wedge \varphi^* dx^n \\
&= d\left(\varphi^* x^1\right) \wedge \cdots \wedge d\left(\varphi^* x^n\right) \\
&= d\varphi^1 \wedge \cdots \wedge d\varphi^n \\
&= \partial_{i_1}\varphi^1 dy^{i_1} \wedge \cdots \wedge \partial_{i_n}\varphi^n dy^{i_n} \\
&= \det(D\varphi)\, dy^1 \wedge \cdots \wedge dy^n, && \text{by Prop. 7.1.7}
\end{aligned}$$

and the result follows from Defn. 7.3.1 since $\det(D\varphi) > 0$. □

Definition 7.3.5: Integral of $\omega \in \Omega^n(U)$ with compact support ($U \subset \mathbb{R}^n$)

Let $U \subset \mathbb{R}^n$ be an open subset. Then for $\omega \in \Omega^n(U)$ with compact support, define the **integral of** ω as

$$\int_U \omega = \int_U \omega_{1,\dots,n} dx^1 \cdots dx^n,$$

where $\omega = \omega_{1,\dots,n} dx^1 \wedge \cdots \wedge dx^n$.

Proposition 7.3.6

Let $U, V \subset \mathbb{R}^n$ be open and $\varphi : V \to U$ an orientation-preserving diffeomorphism. Then $\int_V \varphi^*\omega = \int_U \omega$ for any $\omega \in \Omega^n(U)$ with compact support.

Proof. Let $x^1, \dots, x^n$ and $y^1, \dots, y^n$ be the coordinates on U and V, respectively. Write $\omega = \omega_{1,\dots,n} dx^1 \wedge \cdots \wedge dx^n$. By the proof of Lemma 7.3.4, we have

$$\begin{aligned}\varphi^*\left(\omega_{1,\dots,n} dx^1 \wedge \cdots \wedge dx^n\right) &= \omega_{1,\dots,n} \circ \varphi \cdot \det(D\varphi) dy^1 \wedge \cdots \wedge dy^n \\ &= \omega_{1,\dots,n} \circ \varphi \cdot |\det(D\varphi)|\, dy^1 \wedge \cdots \wedge dy^n, && \text{by assumption and Lemma 7.3.4.}\end{aligned}$$

Thus,

$$\begin{aligned}\int_V \varphi^*\omega &= \int_V \omega_{1,\dots,n} \circ \varphi\, |\det(D\varphi)|\, dy^1 \cdots dy^n \\ &= \int_U \omega_{1,\dots,n} dx^1 \cdots dx^n, && \text{by the multi-dimensional chain rule} \\ &= \int_U \omega.\end{aligned}$$

□

Definition 7.3.7: Integral of $\omega \in \Omega^n(M^n)$ with compact support

Let

(i) M^n be an oriented manifold,

(ii) $\mathscr{A} = \{(U_i, \varphi_i) : i \in I\}$ a **positively oriented atlas** (i.e. all $\varphi_i : U_i \to \varphi_i(U_i)$ are orientation-preserving),

(iii) $\{\chi_i\}_{i \in I}$ a partition of unity subordinate to $\mathscr{A}$.

Then for $\omega \in \Omega^n(M)$ with compact support, we define

$$\int_M \omega = \sum_{i \in I} \int_{\varphi_i(U_i)} \left(\varphi_i^{-1}\right)^* (\chi_i \omega),$$

where the integrals on the right are as in Defn. 7.3.5.

Lemma 7.3.8

$\int_M \omega$ is independent of the choices of positively oriented atlas and partition of unity.

Proof. Let $\mathscr{B} = \left\{(V_j, \psi_j) : j \in J\right\}$ be another positively oriented atlas and $\left\{\rho_j\right\}_{j \in J}$ a partition of unity subordinate to $\mathscr{B}$. Observe that all $\varphi_i \circ \psi_j^{-1} : \psi_j(U_i \cap V_j) \to \varphi_i(U_i \cap V_j)$ are orientation-preserving diffeomorphisms. Thus,

$$\begin{aligned}\sum_{j \in J} \int_{\psi_j(V_j)} \left(\psi_j^{-1}\right)^* (\rho_j \omega) &= \sum_{j \in J} \int_{\psi_j(V_j)} \left(\psi_j^{-1}\right)^* \left(\Big(\overbrace{\sum_{i \in I} \chi_i}^{=1}\Big) \rho_j \omega\right) \\ &= \sum_{\substack{i \in I \\ j \in J}} \int_{\psi_j(V_j \cap U_i)} \underbrace{\left(\varphi_i^{-1} \circ \varphi_i \circ \psi_j^{-1}\right)^*}_{=\left(\varphi_i \circ \psi_j^{-1}\right)^* \left(\varphi_i^{-1}\right)^*} (\chi_i \rho_j \omega) \\ &= \sum_{\substack{i \in I \\ j \in J}} \int_{\varphi_i(V_j \cap U_i)} \left(\varphi_i^{-1}\right)^* (\chi_i \rho_j \omega), && \text{by Prop. 7.3.6} \\ &= \cdots = \sum_{i \in I} \int_{\varphi_i(U_i)} \left(\varphi_i^{-1}\right)^* (\chi_i \omega).\end{aligned}$$

□

7.4 Manifolds with Boundary & Stokes' theorem

Stokes' theorem will relate integrals on M and ∂M. In the simplest case, $M = [a,b]$ and $\partial M = \{a,b\}$. Then

$$\int_{[a,b]} \underbrace{f\,dx}_{\in\Omega^1(M)} = \underbrace{F(b) - F(a)}_{\text{“=” integral on } \{a,b\}},$$

where $F' = f$.

Standard manifolds (i.e. without boundary) are modeled over open sets of $\mathbb{R}^n$. That is, $\mathbb{R}^n$ is the "model space" of the manifold. Manifolds with boundary are modeled over the half-space.

Definition 7.4.1: Half-space

The **half-space** is $\mathbb{R}^n_- := \left\{(x^1, \dots, x^n) \in \mathbb{R}^n : x^1 \le 0\right\} \subset \mathbb{R}^n$ with the subspace topology.

Let $V \subset \mathbb{R}^n_-$ be open. Then $f : V \to \mathbb{R}^m$ is **smooth (or C^∞)** on V if there exists $U \subset \mathbb{R}^n$ open with $V = U \cap \mathbb{R}^n_-$ and a function $\widehat{f} \in C^\infty(U, \mathbb{R}^m)$ such that $\widehat{f}|_V = f$. For $x \in V$, set $Df|_x = D\widehat{f}|_x$.

(i) By definition of the subspace topology on $\mathbb{R}^n_-$, we have $V \subset \mathbb{R}^n_-$ is open if there exists $U \subset \mathbb{R}^n$ open such that $V = U \cap \mathbb{R}^n_-$. Furthermore,

$$\underbrace{\partial\mathbb{R}^n_- = \{0\} \times \mathbb{R}^{n-1}}_{\text{boundary of } \mathbb{R}^n_-} \qquad \text{and} \qquad \underbrace{\mathring{\mathbb{R}}^n_- = \mathbb{R}^n_- \setminus \partial\mathbb{R}^n_-}_{\text{interior of } \mathbb{R}^n_-}.$$

(ii) The derivative $Df|_x$ of $f : V \to \mathbb{R}^m$ is well-defined: For $x \in V$, $D\widehat{f}|_x$ depends only on $\widehat{f}|_V$.

Definition 7.4.2: Manifolds with boundary & Boundary points

(i) A **topological manifold** with boundary of dimension $n \in \mathbb{Z}_{\ge 0}$ is a second countable, Hausdorff topological space such that for all $p \in M$, there is a neighborhood U of p, an open subset $V \subset \mathbb{R}^n_-$, and a homeomorphism $\varphi : U \to V$. The pair (U, φ) is a **chart** of M.

(ii) Two charts $(U_1, \varphi_1), (U_2, \varphi_2)$ are **C^∞-compatible** if $U_1 \cap U_2 = \varnothing$ or

$$\varphi_2 \circ \varphi_1^{-1} : \underbrace{\varphi_1(U_1 \cap U_2)}_{\subset\mathbb{R}^n_-} \to \underbrace{\varphi_2(U_1 \cap U_2)}_{\subset\mathbb{R}^n_-}$$

is a diffeomorphism.

(iii) A **smooth atlas** on M is a set $\mathscr{A} = \{(U_i, \varphi_i) : i \in I\}$ of pairwise compatible charts such that $M = \bigcup_{i\in I} U_i$.

We have an equivalence relation $\mathscr{A} \sim \mathscr{A}' \iff \mathscr{A} \cup \mathscr{A}'$ is a C^∞-atlas.

A **C^∞-manifold with boundary** is a pair $(M, [\mathscr{A}])$ of a topological manifold with boundary M and an equivalence class $[\mathscr{A}]$ of smooth atlases on M.

(iv) $p \in M$ is called a **boundary point** of M if there exists a chart $(U, \varphi = (x^1, \dots, x^n))$ such that $x^1(p) = 0$. We denote by ∂M the set of all boundary points on M.

Lemma 7.4.3

If $p \in M$ is such that $x^1(p) = 0$ for a chart $(U, \varphi = (x^1, \dots, x^n))$, then we have $y^1(p) = 0$ for any other chart $(V, \psi = (y^1, \dots, y^n))$.

Proof. Suppose $y^1(p) < 0$. Let $v \in \mathbb{R}^n$ be such that $D(\varphi \circ \psi^{-1})|_{\psi(p)}(v) = e^1 = (1, 0, \dots, 0)$. For small t, we have $c : t \mapsto \psi(p) + tv$ is a curve in $\psi(U \cap V)$, hence $\widetilde{c} = \varphi \circ \psi^{-1} \circ c$ is a curve in $\varphi(U \cap V)$. However, $\widetilde{c}(0) = \varphi(p) \in \partial\mathbb{R}^n_-$

and

$$\tilde{c}'(0) = D\left(\varphi \circ \psi^{-1}\right)\Big|_{\psi(p)} \Big(\underbrace{c'(0)}_{=v}\Big) = e^1,$$

so $\tilde{c}(t) \notin \mathbb{R}^n_-$ for $t > 0$ small. This is a contradiction. □

Remark 7.4.4. If M is a C^∞-manifold of dimension n with boundary, then ∂M is a C^∞-manifold of dimension $n-1$ in its own right.

(i) M is Hausdorff and second countable, so ∂M is Hausdorff and second countable (by Problem 1).

(ii) The proof of Lemma 7.4.3 shows that if $U, V \subset \mathbb{R}^n_-$ are open, $\chi : U \to V$ is a diffeomorphism, then $\chi\left(U \cap \partial\mathbb{R}^n_-\right) \subset \partial\mathbb{R}^n_-$ and $\chi : U \cap \partial\mathbb{R}^n_- \to V \cap \partial\mathbb{R}^n_-$ is also a diffeomorphism. This implies that if $\mathscr{A} = \{(U_i, \varphi_i) : i \in I\}$ is a C^∞-atlas of M, then $\mathscr{A}_{\partial M} = \left\{\left(U_i \cap \partial M, \varphi_i|_{U_i\cap\partial M}\right) : i \in I\right\}$ is a C^∞-atlas on ∂M.

(iii) If $U, V \subset \mathbb{R}^n_-$ are open and $\chi : U \to V$ is an orientation-preserving diffeomorphism, then $\chi_\partial := \chi|_{U\cap\partial\mathbb{R}^n_-}$ is also orientation preserving: One can show that for $(0, x') = x \in U \cap \partial\mathbb{R}^n_-$, we have

$$D\chi|_x = \begin{pmatrix} \partial_1\chi^1 & 0 & \cdots & 0 \\ * & & & \\ \vdots & & D\chi_\partial|_{x'} & \\ * & & & \end{pmatrix}$$

with $\partial_1\chi^1 > 0$ so $\det\left(D\chi|_x\right) > 0 \iff \det\left(D\chi_\partial|_{x'}\right) > 0$.

Here, the orientation on $\partial\mathbb{R}^n_- \subset \mathbb{R}^n_-$ is defined as follows: A basis $v_1, \ldots, v_{n-1}$ of $\mathbb{R}^{n-1} \cong \partial\mathbb{R}^n_-$ is positively oriented if and only if $e_1, v_1, \ldots, v_{n-1}$ is a positively oriented basis of $\mathbb{R}^n$. Now if $\mathscr{A}$ is an oriented atlas on M, we see from the above that $\mathscr{A}_{\partial M}$ is an oriented atlas on ∂M. Thus, ∂M is orientable (c.f. Lemma 7.3.2). We equip ∂M with the orientation such that $\mathscr{A}_{\partial M}$ is a positively oriented atlas whenever $\mathscr{A}$ is positively oriented.

(iv) Many concepts we developed so far (C^∞-maps, tangent space, tensor fields) carry over to this setting without problems. We have that the inclusion map $i_\partial : \partial M \hookrightarrow M$ is C^∞ and $T_p\partial M \subset T_pM$ is a subspace.

Theorem 7.4.5: Stokes' theorem

Let M^n be an oriented manifold with boundary and let $\omega \in \Omega^{n-1}(M)$ be of compact support. THen

$$\int_M d\omega = \int_{\partial M} i_\delta^* \omega,$$

where $i_\delta : \partial M \to M$ is the inclusion map.

Proof. Let $\mathscr{A} = \{(U_i, \varphi_i) : i \in I\}$ be a positively oriented atlas of M and $\mathscr{A}_{\partial M} = \left\{(V_i, \psi_i) := (U_i \cap \partial M, \varphi_i|_{U_i\cap\partial M}) : i \in I\right\}$ be the induced (positively oriented) atlas on ∂M. Let $\{\chi_i\}_{i\in I}$ be a partition of unity subordinate to $\{U_i\}_{i\in I}$. Then

$$\begin{aligned} \int_M d\omega = \sum_{i\in I}\int_M d(\chi_i\omega) &= \sum_{i\in I}\int_{\varphi_i(U_i)} \left(\varphi_i^{-1}\right)^* \left(d\left(\chi_i\omega\right)\right), && \text{by defn of integral} \\ &= \sum_{i\in I}\int_{\varphi_i(U_i)} d\left(\left(\varphi_i^{-1}\right)^* \left(\chi_i\omega_i\right)\right), && \text{by Lemma 7.2.9(i).} \end{aligned}$$

Since $\varphi_i \circ i_\partial = i_{\partial\mathbb{R}^n_-} \circ \psi_i$ for the inclusion $i_{\partial\mathbb{R}^n_-} : \partial\mathbb{R}^n_- \hookrightarrow \mathbb{R}^n_-$, we have

$$\int_{\partial M} i_\partial^*\omega = \sum_{i\in I}\int_{\psi_i(V_i)} \left(\psi_i^{-1}\right)^* \left(i_\partial^*\left(\chi_i\omega_i\right)\right) = \sum_{i\in I}\int_{\varphi_i(U_i)\cap\partial\mathbb{R}^n_-} i_{\partial\mathbb{R}^n_-}\left(\varphi_i^{-1}\right)^* \left(\chi_i\omega_i\right).$$

This means that it suffices to prove Stokes' theorem for open subsets in $\mathbb{R}^n_-$. Let $\omega = \sum_{j=1}^n \omega_j dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n \in \Omega^{n-1}(\mathbb{R}^n_-)$ be compactly supported. Then

$$\begin{aligned} d\omega &= \sum_{j=1}^n \left(\sum_{k=1}^n \frac{\partial \omega^j}{\partial x^k} dx^k \right) \wedge dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n \\ &= \sum_{j=1}^n \frac{\partial \omega^j}{\partial x^j} dx^j \wedge dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n \\ &= \left(\sum_{j=1}^n (-1)^{j-1} \frac{\partial \omega^j}{\partial x^j} \right) dx^1 \wedge \cdots \wedge dx^n . \end{aligned}$$

Since $i^*_{\partial \mathbb{R}^n_-} dx^1 = d\left(i^*_{\partial \mathbb{R}^n_-} x^1 \right) = d\left(x^1 \circ i_{\partial \mathbb{R}^n_-} \right) = 0$, we have

$$i^*_{\partial \mathbb{R}^n_-} \omega = \left(\omega_1 \circ i_{\partial \mathbb{R}^n_-} \right) dx^2 \wedge \cdots \wedge dx^n .$$

So,

$$\begin{aligned} \int d\omega &= \int_{\mathbb{R}^n_-} \left(\sum_{j=1}^n (-1)^{j-1} \frac{\partial \omega_j}{\partial x^j} \right) dx^1 \cdots dx^n \\ &= \sum_{j=1}^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (-1)^{j-1} \frac{\partial \omega_j}{\partial x^j} dx^1 dx^2 \cdots dx^n \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{0} \frac{\partial \omega_1}{\partial x^1} dx^1 \right]}_{=\omega_1(0,x^2,\ldots,x^n)} dx^2 \cdots dx^n - \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{0} \underbrace{\left[\int_{-\infty}^{\infty} \frac{\partial \omega_2}{\partial x^2} dx^2 \right]}_{=0 \text{ (cpt. support)}} dx^1 \cdots dx^n + \text{ similar terms} \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \omega_1 \circ i_{\partial \mathbb{R}^n_-} dx^2 \cdots dx^n \\ &= \int i^*_{\partial \mathbb{R}^n_-} \omega , \end{aligned}$$

where we have used the fundamental theorem of calculus in the third equality. □

7.5 Integration on Riemannian Manifolds

Goal: We want to integrate functions! For doing this, we need a (semi-)Riemannian manifold. For simplicity, we restrict to the Riemannian case here. Let M^n be an oriented Riemannian manifold (with or without boundary ∂M).

Definition 7.5.1: volume form (dvol)

The **volume form** is the form $\mathrm{dvol} \in \Omega^n(M)$ defined by

$$\mathrm{dvol}(e_1, \ldots, e_n) = 1 \tag{7.5.1}$$

for each positively oriented orthogonormal basis $\{e_1, \ldots, e_n\}$ of T_pM (for all $p \in M$).

Remark 7.5.2. For any basis $\{v_1, \ldots, v_n\}$ of T_pM and the matrix $A = \left\{ a_i^j \right\}_{1 \le i,j \le n}$ with $v_i = a_i^j e_j$, we get

$$\mathrm{dvol}(v_1, \ldots, v_n) = \det A \cdot \mathrm{dvol}(e_1, \ldots, e_n) = \det A .$$

(See Prop. 7.1.7.)

(i) In particular, if $\{v_1, \ldots, v_n\}$ is another positively oriented orthonormal basis, then $\det A = 1$ so dvol is well-defined (i.e. if (7.5.1) holds for one positively oriented orthonormal basis it holds for all of them).

(ii) If (U, φ) is a chart and $v_i = \partial_i$ in (i), then

$$g_{ij} = \langle \partial_i, \partial_j \rangle = \langle a_i^k e_k, a_j^\ell e_\ell \rangle = \sum_{k=1}^n a_i^k a_j^k = (A \circ A^T)_{ij}.$$

So,

$$\det\left((g_{ij})_{1 \le i,j \le n} \right) = \det\left(A \circ A^T \right) = (\det A)^2 .$$

This implies $\mathrm{dvol}\,(\partial_1, \dots, \partial_n) = \det A = \sqrt{\det\left(g_{ij}\right)_{1 \le i,j \le n}}$ and thus

$$\mathrm{dvol} = \sqrt{\det\left(g_{ij}\right)_{1 \le i,j \le n}}\, dx^1 \wedge \cdots \wedge dx^n .$$

Definition 7.5.3: Integral of a compactly supported function $f \in C^\infty(M)$

The **integral** of a function $f \in C^\infty(M)$ with compact support is

$$\int_M f\, \mathrm{dvol} .$$

In other words, we define the integral of f to be the integral of the n-form $f\,\mathrm{dvol} \in \Omega^n(M)$. (In the physics literature, you often see the notation $\int f \sqrt{\det g}\, dx$.)

Theorem 7.5.4: Gauß divergence theorem

Let (M, g) be an oriented Riemannian manifold with boundary ∂M and let $h = g|_{\partial M}$. Then if $X \in \mathfrak{X}(M)$ has compact support, we have

$$\int_M \mathrm{div}\, X \cdot \mathrm{dvol}_g = \int_{\partial M} \langle X, \nu \rangle\, \mathrm{dvol}_h ,$$

where ν is the outward pointing unit normal.

(In the above, $\mathrm{div} = \mathrm{tr} \circ \nabla : \mathfrak{X}(M) \to C^\infty(M)$, which locally looks like $\partial_i X^i + \Gamma^i_{ij} X^j$.)

Idea of proof. Let $\omega = \mathrm{dvol}_g\,(X, -, \dots, -) \in \Omega^{n-1}(M)$, show

$$d\omega = \mathrm{div}\, X \cdot \mathrm{dvol}_g$$
$$i_\partial^* \omega = \langle X, \nu \rangle\, \mathrm{dvol}_h$$

and then apply Stokes' theorem. □

Remark 7.5.5. Theorem 7.5.4 has a physical interpretation: $\int_{\partial M} \langle X, \nu \rangle\, \mathrm{dvol}_h$ is the total flux of X through ∂M and $\mathrm{div}\, X$ is the infinitesimal change of dvol along X.

Chapter 8

Outline: Important questions in Riemannian geometry

Note. *Material from this chapter will not be on the exam.*

For a compact orientable surface M, there exists a topological invariant, called the **genus** g, counting the number of holes. One can show $\chi(M) = 2(1-g)$.

Theorem 8.0.1: Gauß-Bonnet theorem
For each metric g on a compact oriented surface, we have $$\int_M K_g \,\mathrm{dvol}_g = 2\pi\chi(M) = 4\pi(1-g).$$

This allows you to deduce from local assumptions ("curvature") information on the global structure, e.g.

- $K > 0$ everywhere implies $\int K \,\mathrm{dvol} > 0$ so M is diffeomorphic to $\mathbb{S}^2$ (no holes).
- $K < 0$ everywhere implies $\int K \,\mathrm{dvol} < 0$ so M is a surface of genus $g \geq 2$ and hence not diffeomorphic to a sphere or torus.

Many statements are also known in higher dimensions, e.g.

Theorem 8.0.2: Sphere-Theorem (Brendle-Schoen, 2009)
If M^n is a simply-connected Riemannian manifold with $\frac{1}{4} < K(\sigma) \leq 1$ for all 2-planes $\sigma \subset TM$, then M^n is diffeomorphic to $\mathbb{S}^n$.

But many questions are still open, e.g.

Conjecture 8.0.3 (Hopf conjecture)**.** $\mathbb{S}^2 \times \mathbb{S}^2$ does not admit a Riemannian metric such that $K(\sigma) > 0$ for all 2-planes in TM. (The product metric satisfies $K(\sigma) \geq 0$.)

Definitions

Terms & Notation