SF1690
KTH

Basic course in mathematics
HT 2022

Exam

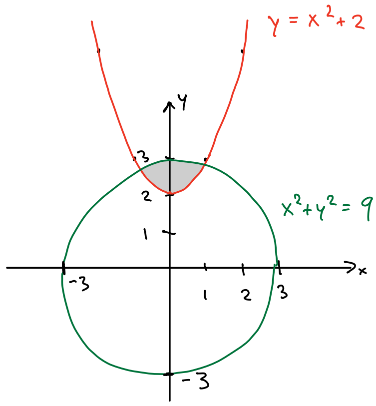
24th of October 2022
14:00-17:00

Properly motivate your answers where suitable!

1. (2 points)

   (a) Sketch the graphs of $y = x^2 + 2$ and $x^2 + y^2 = 9$. Label the points where the graphs cross the x and y axis.

   (b) Shade the region for which $y \geq x^2 + 2$ and (simultaneously) $x^2 + y^2 \leq 9$.
   Solution to (a) and (b): The first graph is the graph of $f(x) = x^2$ moved two steps upward. The second graph is the circle centered at the origin with radius 3. Here is a picture:



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2. (2 points)

   (a) Give an example of a surjective function which is not bijective.
   Solution: An example is

$$f : \mathbb{R} \to [0, \infty)$$
$$f(x) = x^2.$$

   It is surjective, because for any $x \in [0, \infty)$, we have $f(\sqrt{x}) = x$ so the range is $[0, \infty)$. It is not injective, because $f(-1) = f(1)$. Thus it is not bijective.

   (b) Give an example of an injective function which is not bijective.
   Solution: An example is

$$f : \mathbb{R} \to \mathbb{R},$$
$$f(x) = e^x.$$

   It is injective because if $x \neq y$, it follows that $f(x) \neq f(y)$ It is not surjective, since $f(x) > 0$ for all $x \in \mathbb{R}$ so the range of f is not equal to $\mathbb{R}$.

3. (2 points)

   (a) Let $\vec{v} = (1,2)$ and $\vec{w} = (2,-1)$. Are the vectors parallell? Are they orthogonal?
   Lösning: The vectors $\vec{v}$ and $\vec{w}$ are orthogonal since

$$\langle \vec{v}, \vec{w} \rangle = 1 \cdot 2 + 2 \cdot (-1) = 0.$$

   Namely, the angle between the vectors is $\frac{\pi}{2}$ and it follows that they are not parallell.

   (b) Write an equation for the line which is perpendicular to $\vec{v} = (1,2)$ and which passes through the point $(3,4)$.
   Lösning: The point normal equation gives us

$$(x-3) + 2(y-4) = 0.$$

   Alternatively, in parametric form, if it is perpendicular to $(1,2)$ then it is parallell to $(-2,1)$ and so the line is given by

$$t(-2,1) + (3,4).$$

4. (3 points)

   Find all real numbers x satisfying

$$\frac{1}{x} \leq x + 1. \tag{1}$$

Solution: The inequality is undefined for $x = 0$.

If $x > 0$, then multiplying by x on both sides of the inequality we obtain

$$1 \leq x^2 + x. \tag{2}$$

Solving $1 = x^2 + x$ we obtain the roots

$$x = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}.$$

Since we are considering $x > 0$, we are left with the root $-\frac{1}{2} + \frac{\sqrt{5}}{2}$, which solves (2). Also if x is bigger than this root then it will also solve (2) since $x^2 + x$ is monotone increasing for $x > 0$. Thus if

$$x \in \left[-\frac{1}{2} + \frac{\sqrt{5}}{2}, \infty\right)$$

then solves the inequality.

Now consider (1) when $x < 0$. When multiplying by x on both sides of the inequality, we also need to change the inequality around so we obtain

$$1 \geq x^2 + x. \tag{3}$$

Since we are considering $x < 0$, we take the negative root $x = -\frac{1}{2} - \frac{\sqrt{5}}{2}$. If

$$-\frac{1}{2} - \frac{\sqrt{5}}{2} \leq x < 0$$

then x satisfies the inequality (4) because it certainly holds for e.g. $x = -1/2$ and equality is only achieved at the endpoint of the interval $x = -\frac{1}{2} - \frac{\sqrt{5}}{2}$. So the solution to the inequality (1) is

$$\left[-\frac{1}{2} - \frac{\sqrt{5}}{2}, 0\right) \bigcup \left[-\frac{1}{2} + \frac{\sqrt{5}}{2}, \infty\right)$$

5. (3 points)

   (a) What is the domain of $\cos(\pi + \operatorname{Arcsin}(x))$?

   Solution: The domain of Arcsin is $[-1, 1]$. Since the domain of cos is $\mathbb{R}$, it follows that the domain of $\cos(\pi + \operatorname{Arcsin}(x))$ is $[-1, 1]$.

   (b) Simplify $\cos(\pi + \operatorname{Arcsin}(x))$.

   Solution: We have

$$\begin{aligned}\cos(\pi + \operatorname{Arcsin}(x)) &= -\cos(\operatorname{Arcsin}(x)) \\ &= \pm\sqrt{\cos^2(\operatorname{Arcsin}(x))} \\ &= \pm\sqrt{1 - \sin^2(\operatorname{Arcsin}(x))} \\ &= \pm\sqrt{1 - x^2}.\end{aligned}$$

   We proceed to determine whether $\pm$ means $+$ or $-$. The range of Arcsin is $[-\pi/2, \pi/2]$. Since cos is non-negative on $[-\pi/2, \pi/2]$, it follows that $-\cos(\operatorname{Arcsin}(x))$ is non-negative, and so

$$\cos(\pi + \operatorname{Arcsin}(x)) = -\sqrt{1 - x^2}.$$

6. (4 points)

   Consider the vectors

$$\begin{aligned}\vec{u} &= (-1, 1, -2)\,, \\ \vec{v} &= (2, 0, -2)\,, \\ \vec{w} &= (1, 0, -1)\,.\end{aligned}$$

   Find all vectors which are orthogonal to *all* three of the vectors $\vec{u}, \vec{v}$, and $\vec{w}$.

   Solution: We begin by observing that $\vec{v} = 2\vec{w}$, which means that $\vec{v}$ and $\vec{w}$ are parallell. Therefore, all vectors orthogonal to $\vec{v}$ are also orthogonal to $\vec{w}$, and vice versa. Thus we have reduced the question to finding all vectors orthogonal to $\vec{u}$ and $\vec{w}$. We see that $\vec{u}$ and $\vec{w}$ can't be parallell, since if there were a real number k satisfying

$$k(-1, 1, -2) = (-k, k, -2k) = (1, 0, -1),$$

   then it follows from the equation for the second component that $k = 0$. So we seek all vectors orthogonal to $\vec{u}$ and $\vec{w}$. Such a vector is given by the cross product

$$\begin{aligned}\vec{u} \times \vec{w} &= ((1 \cdot (-1) - (-2) \cdot 0, -((-1) \cdot (-1) - (-2) \cdot 1)\,, -1 \cdot 0 - 1 \cdot 1) \\ &= (-1, -3, -1).\end{aligned}$$

   (It is a good idea to verify that the solution obtained indeed is orthogonal to $\vec{u}$ and $\vec{w}$, such a verification is done by checking that the dot product with $\vec{u}$ and $\vec{v} = 0$.) We conclude that $(-1, -3, -1)$ and all vectors parallell to it are orthgonal to $\vec{u}$ and $\vec{w}$. Thus the solution is

$$k(-1, -3, -1),$$

   for $k \in \mathbb{R}$.

7. (4 points) Consider the polynomial $P(x) = 2x^3 + 3x^2 + 2x + 3$. Find the roots of P.

   Hint: $P(i) = 0$.

   Solution: Since i is a root, and P is a real polynomial, it follows that $-i$ is also a root. Thus P has the form

$$P(x) = (ax + b)(x - i)(x + i), \tag{4}$$

for some constants a and b. It follows that

$$\begin{aligned} P(x) &= (ax+b)(x^2+1) \\ &= ax^3+bx^2+ax+b \end{aligned}$$

Matching a and b with the original form for P yields $a = 2$ and $b = 3$. Thus

$$P(x) = 2(x+3/2)(x^2+1),$$

and so the roots of P are i, $-i$, and $-3/2$.

8. (4 points) Let

$$g(x) = 2\log(\sin x) - \log(1+\cos x) - \log(1-\cos x),$$

where log is the natural logarithm.

(a) For which $x \in [0, 2\pi)$ is g well-defined?
Solution: Since the domain of log is $(0, \infty)$, it follows that g is well-defined when all of the following are positive:

$$\sin x, \quad 1+\cos x, \quad 1-\cos x.$$

Now $\sin x > 0$ for $x \in (0, \pi)$. If $x \in (0, \pi)$ then $1 + \cos x > 0$ and also $1 - \cos x > 0$. So g is well-defined if $x \in (0, \pi)$.

(b) Take the domain of g to be the set of all $x \in [0, 2\pi)$ for which g is well-defined (this set is the answer to part (a)). What is the range of g?
Solution: Observe that if $\sin x > 0$ then $2\log(\sin x) = \log(\sin^2(x))$, and similarly $-\log(1+\cos x) - \log(1-\cos x) = -\log(1-\cos^2 x)$. So

$$g(x) = \log(\sin^2 x) - \log(1-\cos^2 x)$$

Writing $1 - \cos^2 x = \sin^2 x$, we obtain

$$g(x) = \log(\sin^2 x) - \log(\sin^2 x) = 0.$$

The only value g can take is 0, and so the range of g is $\{0\}$.

9. (4 points)

(a) Find a root to the equation

$$z^{20} = 1 - i.$$

Solution: First write $1 - i = \sqrt{2}(\cos(-\pi/4) + \sin(-\pi/4))$. Let $\zeta = \frac{z}{2^{1/40}}$. Then we need to solve

$$\zeta^{20} = (\cos(-\pi/4) + \sin(-\pi/4)).$$

By de Moivre's theorem, a solution is

$$\zeta_1 = (\cos(-\pi/80) + \sin(-\pi/80)).$$

Thus a solution to the original equation is

$$z_1 = 2^{1/40}(\cos(-\pi/80) + \sin(-\pi/80)).$$

(b) Find all remaining roots to the same equation

$$z^{20} = 1 - i.$$

Solution: If w_k is a twentieth root of unity solving

$$w_k^{20} = 1,$$

and z_1 is the solution from part (a), then $z_1 w_k$ solves

$$(z_1 w_k)^{20} = 1 - i$$

as desired. Since

$$w_k = \cos(\pi k/10) + i \sin(\pi k/10)$$

for $k = 0, \dots, 19$, we obtain that

$$z_1 w_k = 2^{1/40} \left(\cos(-\pi/80 + \pi k/10) + i \sin(-\pi/80 + \pi k/10)\right) \tag{5}$$

solves the equation for $k = 0, \dots, 19$.

These are 20 distinct solutions (since the roots of unity are distinct). Since there are at most 20 roots to an equation of degree 20, we have found all the solutions.

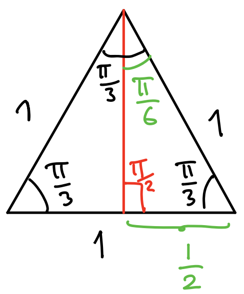
10. (2 points)

Prove that $\sin(\pi/6) = 1/2$.

Solution: Consider an equilateral triangle with lengths 1. Since the sum of angles of the triangle is π radians, every angle must be $\frac{\pi}{3}$ radians. Now split the triangle in two pieces by pulling a straight line from a vertex to the middle of the opposite edge. We are left with two new triangles, which are reflections of each other through the straight line, each with angles

$$\frac{\pi}{6}, \quad \frac{\pi}{3}, \quad \frac{\pi}{2}$$

radians. Consider one of these new triangles, which looks like this:



It has a right angle, since $\frac{\pi}{2}$ radians corresponds to 90 degrees. The length of the edge opposite the angle $\frac{\pi}{6}$ is $\frac{1}{2}$, since this is an edge of length 1 divided in two. The hypothenuse of the new triangle is of length one. We conclude that

$$\sin\left(\frac{\pi}{6}\right) = \frac{\frac{1}{2}}{1} = \frac{1}{2}.$$