SF1690 Basic course in mathematics
KTH HT 2022

Coursework 3

Upload your solutions to Canvas at the latest 23:59 the 11th of October 2022.

Properly motivate your answers where suitable!

1. (3 points)

Assume that a function $f : (0, \infty) \to \mathbb{R}$ satisfies

$$f(xy) = f(x) + f(y), \tag{1}$$

and that

$$f(x) > 0, \tag{2}$$

for all $x > 1$, and that

$$f(x) < 0, \tag{3}$$

for all $x < 1$. Prove that f is a (strictly) monotone increasing function.

Comment: Since the function

$$f(x) = \log(x)$$

satisfies these criteria, we have proven that $\log$ *is a (strictly) monotone increasing function!*

Solution: Assume that $b > a$. Then $f(b) = f\left(a\frac{b}{a}\right)$. Setting $x = a$ and $y = \frac{b}{a}$ in (1), it follows that

$$f(b) = f(a) + f\left(\frac{b}{a}\right). \tag{4}$$

Since $b > a > 0$, it follows that $\frac{b}{a} > 1$. Thus by (2), $f\left(\frac{b}{a}\right) > 0$. Thus, by (4),

$$f(b) > f(a)$$

which proves that f is monotone increasing.

2. (3 points)

Find all complex numbers z satisfying

$$z^4 - 2z^2 + 2 = 0.$$

Hint: First write

$$x = z^2$$

and solve for x, subsequently solve for z.

Solution: We first solve for $x = z^2$: The equation

$$x^2 - 2x + 2 = 0$$

has roots

$$x = 1 \pm i.$$

Thus

$$z^2 = 1 \pm i,$$

so

$$z^2 = \sqrt{2}(\cos(\pm\pi/4) + i\sin(\pm\pi/4)).$$

Considering first the + sign gives a solution

$$z_1 = 2^{1/4}\left(\cos(\pi/8) + i\sin(\pi/8)\right),$$

and a second solution

$$z_2 = -z_1.$$

Now considering the − sign gives the third solution

$$z_3 = 2^{1/4}\left(\cos(-\pi/8) + i\sin(-\pi/8)\right),$$

and a fourth solution

$$z_4 = -z_3.$$

3. (3 points) Let P be a polynomial with solely real or solely imaginary coefficients. Show that if

$$P(z) = 0$$

for a complex number z, then it follows that

$$P(\overline{z}) = 0.$$

Solution: First, assume that the polynomial has solely real coefficients. Then, as covered in class, $P\left(\overline{z}\right) = \overline{P(z)}$ (it is an important condition that P is a real polynomial, it does not hold otherwise). Thus, if $P(z) = 0$, then $\overline{P(z)} = 0$ and so $P(\overline{z}) = 0$.

Now assume that P is a polynomial with solely imaginary coefficients. Then

$$\begin{aligned} P(z) &= ib_n z^n + ib_{n-1}z^{n-1} + \cdots + ib_0 \\ &= i(b_n z^n + b_{n-1}z^{n-1} + \cdots + b_0). \end{aligned}$$

Thus, if $Q(z) = -iP(z)$. Then $Q(z) = b_n z^n + b_{n-1}z^{n-1} + \cdots + b_0$ is a real polynomial. We just showed that if Q is a real polynomial and $Q(z) = 0$, then $Q\left(\overline{z}\right) = 0$. So, if $P(z) = 0$ then $Q(z) = 0$ and so $Q\left(\overline{z}\right) = 0$ and thus again $P\left(\overline{z}\right) = 0$, solving the problem.

4. (3 points)

Consider the vectors

$$\begin{aligned} \vec{u} &= (2, 2, 1), \\ \vec{v} &= \left(-\pi, -\pi, -\frac{\pi}{2}\right), \\ \vec{w} &= (1, 0, 1). \end{aligned}$$

(a) Are any of the vectors parallell? If so, which?

Solution: Observe that $\vec{v} = \frac{\pi}{2}\vec{u}$. Thus $\vec{u}$ and $\vec{v}$ are parallell. However $\vec{u}$ and $\vec{w}$ are not parallell, because if

$$\vec{u} = t\vec{w}$$

for some $t \in \mathbb{R}$, then considering the second entry of the vector, we would have

$$2 = t0 = 0.$$

This is false for all t, so we conclude that $\vec{u}$ and $\vec{v}$ are not parallell. Since $\vec{v}$ and $\vec{u}$ are parallell, it also follows that $\vec{v}$ and $\vec{w}$ are not parallell.

(b) Are any of the vectors orthogonal? If so, which?

Solution: Observe that

$$\begin{aligned} \vec{u} \cdot \vec{v} &= -9\pi/2, \\ \vec{u} \cdot \vec{w} &= 3, \\ \vec{v} \cdot \vec{w} &= -3\pi/2. \end{aligned}$$

All of the above are unequal to zero, so none of the vectors are orthogonal.

(c) What is the angle between each pair of vectors?

Solution: Although $\vec{u}$ and $\vec{v}$ are parallell, they point in the opposite directions and the angle between them is π. Now $\|\vec{u}\| = 3$ and $\|\vec{w}\| = \sqrt{2}$ so since $\vec{u} \cdot \vec{w} = 3$ we obtain that

$$\frac{\vec{u} \cdot \vec{w}}{\|\vec{u}\|\|\vec{w}\|} = 1/\sqrt{2}$$

and thus, if θ is the angle between $\vec{u}$ and $\vec{w}$ then $\cos\theta = 1/\sqrt{2}$. Now since $\cos(\pi/4) = 1/\sqrt{2}$, it follows that the angle between $\vec{u}$ and $\vec{w}$ is $\pi/4$.

Now since $\vec{v}$ and $\vec{w}$ point in the opposite directions, that the angle between $\vec{v}$ and $\vec{w}$ is $\pi - \pi/4 = 3\pi/4$.

5. (3 points) Let L be the line in $\mathbb{R}^2$ which passes through the points $(0, 1)$ and $(2, 2)$.

(a) Write the line L in parameterform.

Solution: Since $(2, 2) - (0, 1) = (2, 1)$ is parallell to the line and since $(0, 1)$ is a point on the line it follows that L is given by

$$(2t, 1 + t), \qquad t \in \mathbb{R}.$$

Any alternative parameter description will do, e.g. L is the set of points $(x, y) \in \mathbb{R}^2$ satisfying

$$(x, y) = t(2, 1) + (0, 1),$$

for $t \in \mathbb{R}$.

(b) What is the distance between the line L and the point $(3, 0)$?

(Here, we mean the distance between $(3, 0)$ and the closest point on the line L.)

Solution: If $P = (P_1, P_2)$ is the point in L which is closest to $(3, 0)$ then $(3, 0) - P$ will be perpendicular to the line L, and thus orthogonal to $(2, 1)$. Observe that $(1, -2)$ is orthogonal to $(2, 1)$ and so P is the (unique) intersection of L and the line

$$(3, 0) + s(1, -2), \qquad s \in \mathbb{R}.$$

Observe that the point $(2, 2)$ is in both lines (with $t = 1$ and $s = -1$), and so $P = (2, 2)$. The distance from $(3, 0)$ to $(2, 2)$ is $\sqrt{5}$ and so $\sqrt{5}$ is the solution.