SF1690 | Basic course in mathematics
KTH | HT 2022

Solutions - Coursework 2

Upload your solutions to Canvas at the latest 23:59 the 26th of September 2022.

Properly motivate your answers where suitable!

1. (3 points)

   Observe that $x = 1$ is a root of the equation

$$x^3 + ax^2 - ax - 1 = 0, \tag{1}$$

for all real numbers a. The Fundamental Theorem of Algebra combined with the factor theorem guarantees that the equation has exactly 3 roots (when counting multiplicities). For which real numbers a are the other two roots real?

Solution: Since $x = 1$ is a root, it follows by the factor theorem that the polynomial on the left-hand side must have the form

$$(x-1)(x^2 + bx + c) \tag{2}$$

for some real constants b and c. Now (2) equals

$$x^3 + x^2(b-1) + x(c-b) - c. \tag{3}$$

Matching the coefficient of the x^2 term in (3) with the x^2 term of (1), it follows that

$$a = (b-1). \tag{4}$$

Similarly matching coefficients of the constant term yields

$$c = 1 \tag{5}$$

Thus (1) becomes

$$(x-1)\left(x^2 + (a+1)x + 1\right). \tag{6}$$

The roots of $x^2 + (a+1)x + 1$ are given by

$$\frac{-a-1 \pm \sqrt{(a+1)^2 - 4}}{2}. \tag{7}$$

Thus these are the other roots of (1), and they are real if and only if

$$(a+1)^2 - 4 \geq 0.$$

Since

$$(a+1)^2 - 4 = (a+3)(a-1),$$

we need to solve

$$(a+3)(a-1) \geq 0. \tag{8}$$

This is solved by

$$a \in (-\infty, -3] \cup [1, \infty), \tag{9}$$

which is also the region for which the remaining roots of (1) are real.

2. (3 points)

   Compute
   $$\sin\left(\frac{\pi}{12}\right)?$$
   Present your entire solution!

   Hint: You may rely on the values we obtained for
   $$\sin\left(\frac{\pi}{3}\right), \quad \cos\left(\frac{\pi}{3}\right), \quad \sin\left(\frac{\pi}{4}\right), \quad \text{and} \quad \cos\left(\frac{\pi}{4}\right)$$
   in class.

   Solution: Observe that
   $$\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}.$$
   Thus
   $$\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right).$$
   Since
   $$\sin(x+y) = \sin x \cos y + \cos x \sin y,$$
   we obtain
   $$\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{3}\right)\cos\left(-\frac{\pi}{4}\right) + \sin\left(-\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right). \tag{10}$$
   We have $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$, and also
   $$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}, \quad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \quad \text{and} \quad \sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$
   So we obtain
   $$\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

3. (3 points)

   Let
   $$f(x) = \sin^4\left(\frac{1}{1+x}\right).$$
   What is the domain and range of f?

   Solution: First consider the domain. The domain $\frac{1}{1+x}$ is $\mathbb{R} \setminus \{-1\}$. Since $\sin^4$ has domain $\mathbb{R}$, it follows that the domain of f is the same as the domain of $\frac{1}{1+x}$, namely $\mathbb{R} \setminus \{-1\}$.

   Now consider the range. The range of $\sin^4$ is $[0,1]$, and since the range of $\frac{1}{1+x}$ is $\mathbb{R} \setminus \{0\}$, we know that $(0,1]$ is in the range of f, the only question is whether 0 is in the range as well. Note that π is in the range of $\frac{1}{1+x}$ and $\sin \pi = 0$, so 0 is also in the range of f.

4. (3 points)

   Consider the polynomial
   $$p(x) = x^2 - 3x + 2$$
   with domain
   $$D = \left[\frac{3}{2}, \infty\right).$$

(a) Prove that p is injective.

Hint: note that

$$x^2 - 3x + 2 = \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}.$$

Solution: We will show that p is (strictly) monotone increasing, from which it follows that p is injective. If $b > a \geq 3/2$, then $b = a + c$ for some $c > 0$, and

$$\begin{aligned} p(b) - p(a) &= \left(a + c - \frac{3}{2}\right)^2 - \left(a - \frac{3}{2}\right)^2 \\ &= c^2 + 2ac - 3c. \end{aligned}$$

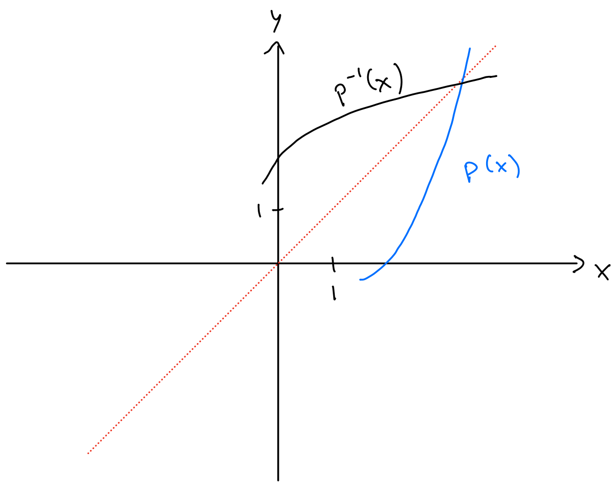
Since $a \geq 3/2$ it follows that $2ac - 3c \geq 0$, and since $c > 0$ we also have $c^2 > 0$. Thus $p(b) - p(a) > 0$ on $[3/2, \infty)$.

(b) What is the range of p?

Solution: The range of $\left(x - \frac{3}{2}\right)^2$ is $[0, \infty)$. If we subtract $1/4$, then the resulting range is $[-1/4, \infty)$, which is the range of p.

(c) Denote the range of p by R. Sketch the graph of the inverse function

$$p^{-1} : R \to D.$$



Solution:

5. (3 points)

Let

$$g : \left[0, \frac{\pi}{2}\right] \to [0, 1]$$

be given by

$$g(x) = \sin^2(x).$$

(a) Prove that g is invertible.

Hint: You may rely on the fact that the sine function is injective on the interval $\left[0, \frac{\pi}{2}\right]$.

Solution: If $a \neq b$, then since sin is injective on $\left[0, \frac{\pi}{2}\right]$, it follows that $\sin a \neq \sin b$. Since sin is positive on this interval, it follows that $(\sin a)^2 \neq (\sin b)^2$. Thus g is injective.

Since $\sin^2(x)$ is a continuous increasing function between 0 and 1 it follows that it takes all the values between 0 and 1, and thus the range is $[0, 1]$. Since the range equals the codomain, it is surjective.

Since g is both injective and surjective, it follows that it is bijective and thus invertible.

(b) Find a simple formula for the function

$$g^{-1}\left(\sin^2(x)\right),$$

defined for all

$$x \in \left[-\pi, -\frac{\pi}{2}\right].$$

Solution: Let $t \in [0, \pi/2]$. Then by definition of the inverse,

$$g^{-1}(\sin^2(t)) = t. \tag{11}$$

Now we aim to find $t \in [0, \pi/2]$ such that $\sin^2 x = \sin^2 t$. Then it will follow by (11) that $g^{-1}(\sin^2 x) = t$.

Observe that

$$\sin^2(x + \pi) = (-\sin(x))^2 = \sin^2(x). \tag{12}$$

Since $x + \pi \in [0, \pi/2]$ we obtain (with $t = x + \pi$ in (11))

$$g^{-1}\left(\sin^2(x + \pi)\right) = x + \pi. \tag{13}$$

Thus, by (12) and (13), we obtain

$$g^{-1}(\sin^2(x)) = x + \pi. \tag{14}$$