SF1690 Basic Course in Mathematics
KTH HT 2022

Coursework 1

Upload solutions to Canvas by 23:59 the 12th of September 2022.

Properly motivate your answers where suitable!

1. (3 points)

   (a) Sketch the graph of the function

   $$f(x) = |x + 1| \,.$$

   (b) Sketch the graph of the function

   $$g(x) = |2x - 2| \,.$$

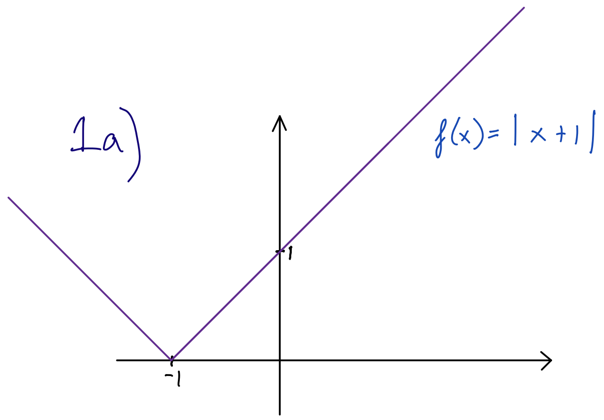
   (c) For which real numbers x is
   $$f(x) = g(x)?$$

   (d) Rely on (a), (b) and (c) to determine which real numbers x satisfy

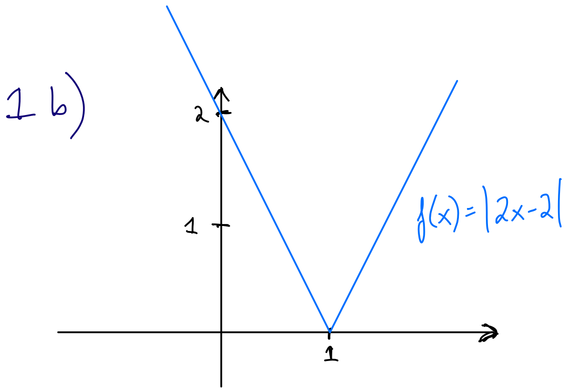
   $$|2x - 2| < |x + 1| \,.$$

Solution:

(a)



(b)



(c) We see from the graphs in exercise (a) and (b) that there must be two intersection points between the graphs. In the left intersection point we see that $x + 1 \geq 0$ and $2x - 2 \leq 0$, and thus at this intersection point x solves

$$x + 1 = -(2x - 2),$$

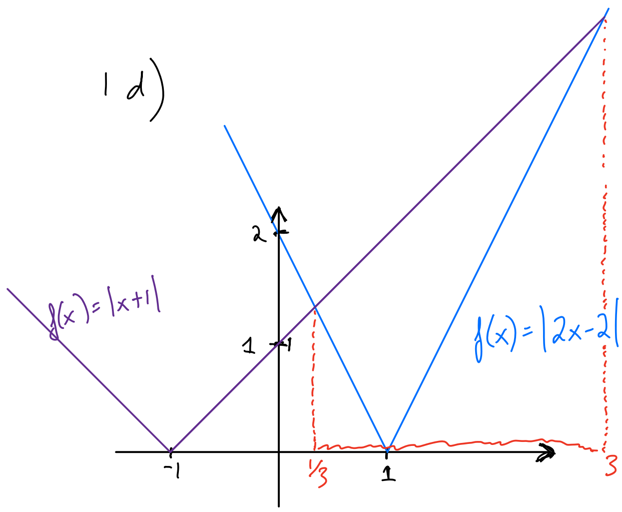
and thus we obtain $x = 1/3$ upon solving the equation. In the right intersection point we see that $x + 1 \geq 0$ och $2x - 2 \geq 0$, which provides the following equation for x:

$$x + 1 = 2x - 2,$$

and thus we obtain $x = 3$. In conclusion, the real values of x where $f(x) = g(x)$ are given by

$$x = \frac{1}{3} \quad \text{and} \quad x = 3.$$

(d) Now sketching the results from (a), (b) and (c) in a single picture we obtain:



From the picture we see that

$$|2x - 2| = g(x) < f(x) = |x + 1|$$

for x values between the intersection points of $f(x) = g(x)$. We conclude that

$$|2x - 2| < |x + 1|$$

for all

$$x \in \left(\frac{1}{3}, 3\right).$$

(Observe that the endpoints of this interval are not included in the solution which explains why we have round brackets instead of square brackets in the solution.)

2. (3 points)

Sketch the region in the xy-plane which is described by the following set of inequalities:

$$\begin{cases} y \geq x^2, \\ 4x + 2y \geq 1, \\ x^2 + \frac{y^2}{4} \leq 1. \end{cases}$$

Solution:

The first inequality describes the area above the parabola

$$y = x^2,$$

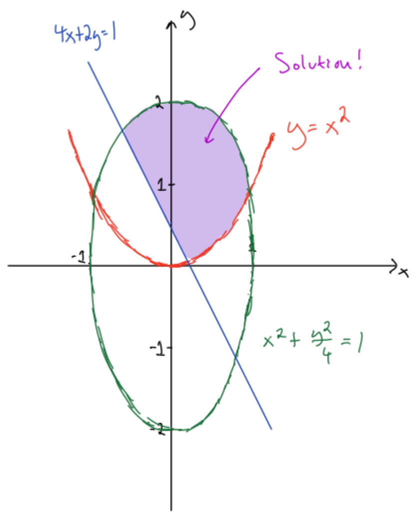
which includes the points $(-1, 1)$, $(0, 0)$ and $(1, 1)$. The second inequality describes the area above

$$y = -2x + \frac{1}{2},$$

which includes the points $\left(0, \frac{1}{2}\right)$ och $\left(\frac{1}{4}, 0\right)$. The third inequality describes the interior of the ellipse

$$x^2 + \frac{y^2}{2^2} = 1,$$

which contains the points $(1, 0)$, $(-1, 0)$, $(0, 2)$ and $(0, -2)$. Here is the sketch: (NB blue line $4x + 2y = 1$ is incorrect fix before posting online)



3. (3 points)

We define two functions

$$f(x) = \frac{1}{1-x} \quad \text{and} \quad g(x) = 1 - \frac{1}{x}.$$

(a) What is $f \circ g$, and what is the domain and range?

(b) What is $g \circ f$, and what is the domain and range?

(c) What is $f \circ f \circ f$, and what is the domain and range? Simplify the expression as much as possible.

Solution:

(a) We calculate $f \circ g$ as follows:

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= \frac{1}{1 - g(x)} \\ &= \frac{1}{1 - \left(1 - \frac{1}{x}\right)} \\ &= \frac{1}{\frac{1}{x}} \\ &= x. \end{aligned}$$

The function f is not defined for $x = 1$, but is defined for all other real values meaning that the domain of f, which we denote $D(f)$, is given by

$$D(f) = (-\infty, 1) \cup (1, \infty). \tag{1}$$

The function g is not defined for $x = 0$, but is defined for all other real values, so

$$D(g) = (-\infty, 0) \cup (0, \infty).$$

The domain of $f \circ g$ consists of all $x \in D(g)$ such that $g(x) \in D(f)$. We therefore need to find $x \in D(g)$ satisfying $g(x) = 1$ and these will not be in the domain of $f \circ g$. Observe that $g(x) = 1$ if and only if $\frac{1}{x} = 0$, which doesn't hold for any x! So the domain of $f \circ g$ is the same as the domain of g in this case. We conclude that

$$D(f \circ g) = (-\infty, 0) \cup (0, \infty).$$

The range consists of numbers y where there exists $x \in D(f \circ g)$ satisfying

$$y = f \circ g(x).$$

Since we know that $f \circ g(x) = x$, we see that the range of $f \circ g$ is the same as the domain of $f \circ g$. We conclude that

$$\text{Range}(f \circ g) = (-\infty, 0) \cup (0, \infty).$$

(b) We start by calculating $g \circ f$ as follows:

$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= 1 - \frac{1}{f(x)} \\ &= 1 - \frac{1}{\frac{1}{1-x}} \\ &= 1 - (1 - x) \\ &= x. \end{aligned}$$

The domain of $g \circ f$, is the set of all $x \in D(f)$ such that $f(x) \in D(g)$. Since $D(g) = (-\infty, 0) \cup (0, \infty)$, we need to decide for which x we obtain $f(x) = 0$. Observe that $f(x)$ can never be 0, so we conclude that the domain of $g \circ f$ is given by

$$D(g \circ f) = D(f) = (-\infty, 1) \cup (1, \infty).$$

Since $g \circ f(x) = x$, we conclude precisely as in (a) that

$$\text{Range}(g \circ f) = D(g \circ f) = (-\infty, 1) \cup (1, \infty).$$

(c) We calculate $f \circ f$ as follows:

$$\begin{aligned} f \circ f(x) &= f(f(x)) \\ &= \frac{1}{1 - \frac{1}{1-x}} \\ &= \frac{1}{\frac{1-x-1}{1-x}} \\ &= \frac{1-x}{-x} \\ &= 1 - \frac{1}{x}, \end{aligned}$$

which is the same as $g(x)$, but potentially with a different domain! The domain of $f \circ f$ consists of all $x \in D(f)$ satisfying $f(x) \in D(f)$. Since $D(f) = (-\infty, 1) \cup (1, \infty)$, we exclude any x satisfying $f(x) = 1$ (such x will not be in the domain of $f \circ f$). Since $f(0) = 1$, it follows that $x = 0$ is not in $D(f \circ f)$, and so

$$D(f \circ f) = (-\infty, 0) \cup (0, 1) \cup (1, \infty). \tag{2}$$

We now calculate $f \circ f \circ f$:

$$\begin{aligned} f \circ f \circ f(x) &= f(f \circ f(x)) \\ &= f\left(1 - \frac{1}{x}\right) \\ &= \frac{1}{1 - \left(1 - \frac{1}{x}\right)} \\ & x, \end{aligned}$$

for x in the domain of $f \circ f \circ f$. Now the domain of $f \circ f \circ f$ consists of all $x \in D(f \circ f)$ such that $f \circ f(x) \in D(f)$. Now $D(f \circ f)$ is given by (2), meaning that $x \neq 0, 1$. The statement $f \circ f(x) \in D(f)$ is in fact not a restriction in this case, because by (1) it means that $f \circ f(x) \neq 1$, but we knew this already because $f \circ f(x)$ cannot be equal to 1. Thus the domain of $f \circ f \circ f$ consists of all $x \in D(f \circ f)$, so by (2) it follows that

$$D(f \circ f \circ f) = (-\infty, 0) \cup (0, 1) \cup (1, \infty).$$

Since $f \circ f \circ f(x) = x$, it follows that the range is equal to the domain in this case, and thus

$$\text{Range}(f \circ f \circ f) = (-\infty, 0) \cup (0, 1) \cup (1, \infty).$$

4. (3 points) The graph of the function

$$f(x) = x^2 + 4x + 4$$

is shifted two steps the right and one step downwards.

(a) Compute the function which yields the new graph.

(b) Is the new function even?

(c) Is the new function odd?

Solution:

(a) The new function $g(x)$ is given by

$$g(x) = f(x-2) - 1$$

Since $f(x) = (x+2)^2$, we obtain

$$g(x) = (x+2-2)^2 - 1 = x^2 - 1$$

(b) Yes it is even, because

$$\begin{aligned} g(x) &= x^2 - 1 \\ &= (-x)^2 - 1 \\ &= g(-x) \end{aligned}$$

for all x in the domain of g, which is given by $(-\infty, \infty)$.

(c) No, the function is not odd, because

$$g(1) = 0 \neq -2 = -g(-1).$$

(This is compatibel with (b), because the only function which is both even and odd is $f(x) = 0$.)

5. (3 points) Consider the equation

$$\sqrt{y^2 + (x-p)^2} + \sqrt{y^2 + (x+p)^2} = c, \tag{3}$$

with $c > 2p$. Observe that $\sqrt{y^2 + (x-p)^2}$ is the distance from (x, y) to $(p, 0)$ in $\mathbb{R}^2$. Similarly, $\sqrt{y^2 + (x+p)^2}$ is the distance from (x, y) to $(-p, 0)$ in $\mathbb{R}^2$.

(a) Sketch the graph of the equation (3), identifying the main features.

(b) Square the left-hand side and the right hand side of (3), and conclude that

$$2y^2 + 2x^2 + 2p^2 - c^2 = -2\sqrt{(y^2 + (x-p)^2)\,(y^2 + (x+p)^2)}. \tag{4}$$

(c) Square the left and the right hand side of (4). Conclude that if (x, y) satisfies (3), then

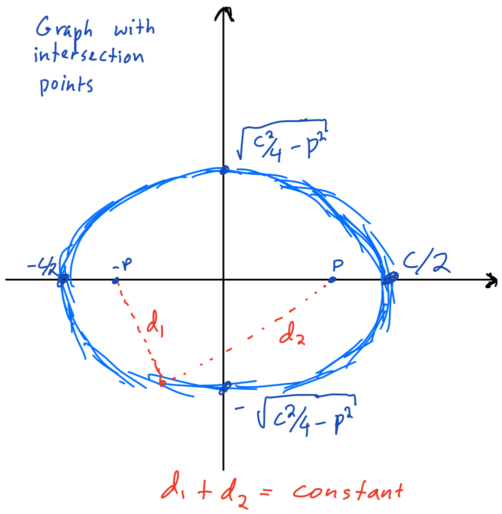
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{5}$$

for suitable constants a and b. Provide a formula for a and b in terms of p and c.

Comment: Recall that (5) *is the equation of an ellipse. We have thus shown that the set of all points* (x, y) *such that the sum of the distance to two points* $(p, 0)$ *and* $(-p, 0)$ *is constant forms an ellipse.*

Lösning:

(a)



OBSERVE THAT GRAPH IS WIDER THAN IT IS TALL BECAUSE $c/2 > \sqrt{c^2/4 - p^2}$.

(b) We square the left-hand side of (3) and obtain

$$\begin{aligned}
&\left(\sqrt{y^2+(x-p)^2}+\sqrt{y^2+(x+p)^2}\right)^2\\
&=2y^2+(x-p)^2+(x+p)^2+2\sqrt{(y^2+(x-p)^2)(y^2+(x+p)^2)}\\
&=2x^2+2y^2+2p^2+2\sqrt{(y^2+(x-p)^2)(y^2+(x+p)^2)}.
\end{aligned}$$

By setting the bottom line equal to c^2 (which is the square of the right hand side of (3)), and subtracting $c^2+2\sqrt{(y^2+(x-p)^2)(y^2+(x+p)^2)}$ from both sides of the equation we obtain (4).

(c) We square both sides of (4) to obtain

$$\left(2y^2+2x^2+2p^2-c^2\right)^2=4\left(y^2+(x-p)^2\right)\left(y^2+(x+p)^2\right)$$

Thus we obtain

$$\begin{aligned}
&4y^4+8y^2x^2+8y^2p^2-4y^2c^2+4x^4+8x^2p^2-4x^2c^2+4p^4-4p^2c^2+c^4\\
&=4y^4+4y^2(x+p)^2+4(x-p)^2y^2+4(x-p)^2(x+p)^2
\end{aligned}\tag{6}$$

At this stage, the student who wants to make their life easy will try to match it with the end goal, and observe that there are no terms x^4, y^4, or x^2y^2 in (5). So we identify these terms on the left hand side of (6), and subtract them: namely we subtract $(4y^4+4x^4+8y^2x^2)$ from both the left and right hand side of (6), to obtain:

$$\begin{aligned}
&8y^2p^2-4y^2c^2+8x^2p^2-4x^2c^2+4p^4-4p^2c^2+c^4\\
&=8y^2p^2-8x^2p^2+4p^4
\end{aligned}$$

Now subtract the right hand side from both sides to obtain

$$-4y^2c^2+16x^2p^2-4x^2c^2-4p^2c^2+c^4=0.$$

Moving the constant terms to the right hand side we obtain

$$y^2\left(-4c^2\right)+4x^2\left(4p^2-c^2\right)=c^2\left(4p^2-c^2\right).$$

We divide by $c^2\left(4p^2 - c^2\right)$ on both sides to obtain

$$y^2 \frac{4}{c^2 - 4p^2} + x^2 \frac{4}{c^2} = 1.$$

Thus we have obtained the form (5), and we thus have

$$a = \frac{c}{2}, \qquad b = \sqrt{c^2/4 - p^2}. \tag{7}$$

For brownie points we state that our result for b makes sense because we assumed $c > 2p$ which implies that $c^2 - 4p^2 > 0$ and thus we can take the square root in b. Also we note that the values for a and b precisely match what we calculated when we initially made the graph, so we know we have the right answer.