Problem sheet 7

We recall the terminology regarding functions.

- A function $f : D \to S$ maps elements from the domain D to a codomain S.
- The subset of S given by $\{f(x) : x \in D\}$ is called the range of f.
- If the range of f is equal to S then f is surjective (onto). This means that for each element $y \in S$ there is an element $x \in D$ such that $f(x) = y$.
- If, for each $x, y \in D$ such that $x \neq y$, we have $f(x) \neq f(y)$, then f is injective.
- If a function is both injective and surjective, then it is bijective. Bijective functions are invertible.
- The inverse satisfies $f^{-1}(f(x)) = x$, and also $f(f^{-1}(x)) = x$.
- If a function is invertible, and we draw a graph $y = f(x)$, then the inverse is found by changing the role of x and y so that $f^{-1}(y) = x$.

(1) Every resident in Sweden has a personal number (personnummer). Let D be the set of all personal numbers in Sweden, and let S be the set made up of all the names of all the people in the world. Let $f : D \to S$ be a function mapping each personal number to that person's name. Determine whether f is surjective, injective, bijective, and whether it is invertible.

(2) Same question as above, but now let S be the set made up of all the names of all Swedish residents instead of all the names of all the people in the world.

(3) In the classroom, there are students and chairs. The teacher assigns students to chairs. The set of all students is the domain, and the set of all chairs is the codomain. Under which circumstance is the function injective? Surjective? Give conditions for a bijection.

For the remaining problems, we recall the assumptions we usually make in this course:

- We assume that the domain of a function is the largest subset of $\mathbb{R}$ such that the function is well defined and real.
- We assume that the function is surjective, in other words the range is equal to the codomain.

Under the above assumptions, a function is invertible if it is injective (one-to-one).

(4) Show that the following functions are one-to-one, and calculate the inverse functions f^{-1}:

- $f(x) = -\sqrt{x-1}$.
- $f(x) = x^3$.
- $f(x) = 1 + \sqrt[3]{x}$.
- $f(x) = x^2, \; x \leq 0$.
- $f(x) = \frac{x}{1+x}$.
- $f(x) = \frac{x}{\sqrt{x^2+1}}$.
- $f(x) = x|x| + 1$.

(5) Assume that f and h are bijective. Show that $f \circ h$ is bijective.

(6) Show that $f(x) = \frac{1+x}{1-x}$ is bijective.

(7) Rely on (5) and (6) to show that if f is a bijective function with range $\mathbb{R} \setminus \{1\}$, then

$$s(x) = \frac{1 + f(x)}{1 - f(x)} \qquad (1)\ \{?\}$$

is bijective. Compute the inverse of s in terms of f^{-1}.