

VEKTORANALYS

HT 2022

CELTE / CENMI

ED1110

**SOME SPECIAL VECTOR FIELDS
AND
LAPLACE AND POISSON EQUATIONS
ÖVNINGAR**

Kursvecka 6

Kapitel 16

Avsnitt: målproblem, 17.1, 17.2, 17.3

Avsnitt 17.5 (inte bevisen)



PROBLEM 1

A dipole is formed by two point sources with charge $+c$ and $-c$
 Calculate the flux of the dipole field on a closed surface S that

- (a) encloses both poles
- (b) encloses only the plus pole
- (c) does not enclose any pole

SOLUTION

Vector field from
 a point source
 located in $\vec{r}_0$:

$$\vec{A}(\vec{r}) = c \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|^3}$$

Vector field from
 two point sources with
 opposite charge
 located in $\vec{r}_1$ and $\vec{r}_2$

$$\vec{A}(\vec{r}) = c \frac{\vec{r} - \vec{r}_1}{|\vec{r} - \vec{r}_1|^3} - c \frac{\vec{r} - \vec{r}_2}{|\vec{r} - \vec{r}_2|^3}$$

Flux from
 a point source:

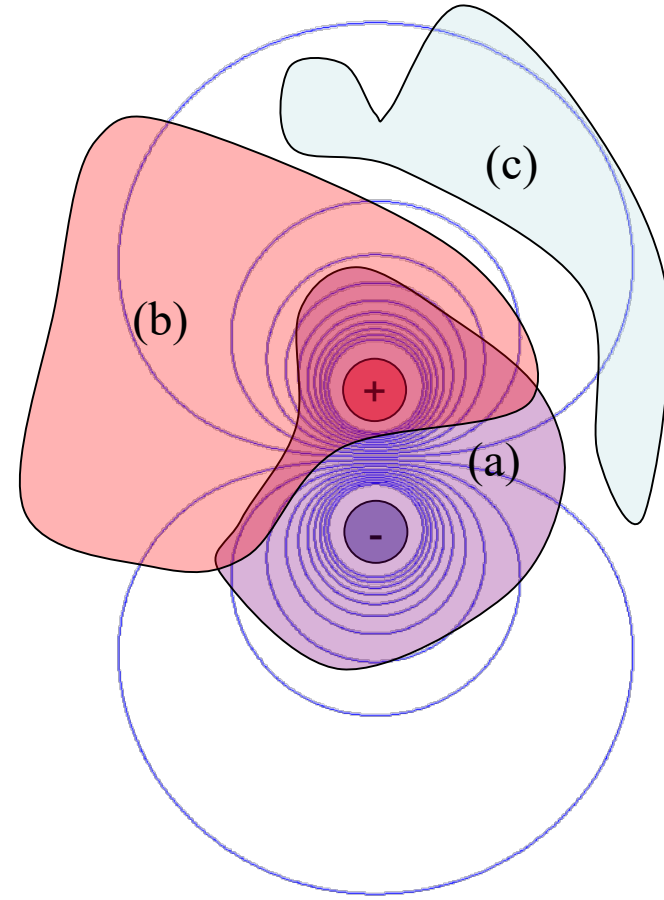
$$= \begin{cases} 0 & \text{If the origin is outside } V \\ 4\pi c & \text{If the origin is inside } V \end{cases}$$

$$\oiint_S \left(c \frac{\vec{r} - \vec{r}_1}{|\vec{r} - \vec{r}_1|^3} - c \frac{\vec{r} - \vec{r}_2}{|\vec{r} - \vec{r}_2|^3} \right) \cdot d\vec{S} = \oiint_S c \frac{\vec{r} - \vec{r}_1}{|\vec{r} - \vec{r}_1|^3} \cdot d\vec{S} - \oiint_S c \frac{\vec{r} - \vec{r}_2}{|\vec{r} - \vec{r}_2|^3} \cdot d\vec{S}$$

$$(a) \quad \oiint_S () \cdot d\vec{S} = 4\pi c \quad -4\pi c \quad = 0$$

$$(b) \quad \oiint_S () \cdot d\vec{S} = 4\pi c \quad 0 \quad = 4\pi c$$

$$(c) \quad \oiint_S () \cdot d\vec{S} = 0 \quad 0 \quad = 0$$

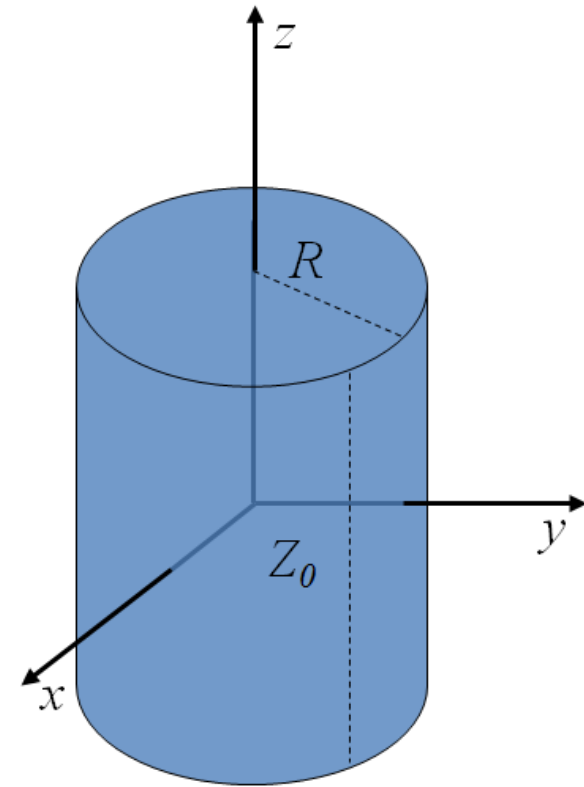


PROBLEM 2

Betrakta följande vektorfält (i sfäriska koordinater):

$$\vec{A} = \frac{3 + r^4 \sin \theta}{r^2} \hat{e}_r + r^2 \hat{e}_\phi$$

Beräkna flödet av vektorfältet ut genom en cylinderyta S med radien R , axel längs $\hat{e}_z$, längd z_0 och centrum i origo. Flödet genom cylinderns två cirkulära lock skall medräknas (S är en sluten yta).



SOLUTION to problem 2

$$\int_s \bar{A} \cdot d\bar{S} = \int_s \left(\frac{3+r^4 \sin \theta}{r^2} \hat{e}_r + r^2 \hat{e}_\varphi \right) \cdot d\bar{S} = \int_s \left(\frac{3}{r^2} \hat{e}_r + r^2 \sin \theta \hat{e}_r + r^2 \hat{e}_\varphi \right) \cdot d\bar{S} =$$

$$= \underbrace{\int_s \left(\frac{3}{r^2} \hat{e}_r \right) \cdot d\bar{S}}_{\text{Theorem 16.1}} + \underbrace{\int_s \left(r^2 \sin \theta \hat{e}_r + r^2 \hat{e}_\varphi \right) \cdot d\bar{S}}_{\text{Gauss' theorem}} = 12\pi + \int_V \nabla \cdot (r^2 \sin \theta \hat{e}_r + r^2 \hat{e}_\varphi) dV =$$

$$\left\{ \begin{array}{l} \text{in a spherical coordinate system: } \nabla \cdot \bar{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} (v_\varphi) \\ \text{so: } \nabla \cdot (r^2 \sin \theta \hat{e}_r + r^2 \hat{e}_\varphi) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^4 \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} (r^2) = \frac{4r^3 \sin \theta}{r^2} = 4r \sin \theta \end{array} \right\}$$

$$= 12\pi + \int_V 4r \sin \theta dV =$$

{Note that the volume is a cylinder. And that $r \sin \theta = \rho$ }

$$= 12\pi + \int_V 4\rho dV = 12\pi + \int_V 4\rho^2 d\varphi d\rho dz = 12\pi + 8\pi Z_0 \int_V \rho^2 d\rho = 12\pi + 8\pi Z_0 \left[\frac{1}{3} \rho^3 \right]_0^R = 12\pi + \frac{8}{3} \pi Z_0 R^3$$

SCALAR LAPLACIAN: USEFUL EXPRESSIONS

The Laplace equation is $\nabla^2 \phi = 0$

The **Laplace operator** ∇^2 is the divergence of the gradient:

$$\nabla^2 \phi \equiv \nabla \cdot \nabla \phi$$

CARTESIAN COORDINATE SYSTEM

$$\left. \begin{array}{l} \bullet \text{GRADIENT} \quad \nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) \\ \bullet \text{DIVERGENCE} \quad \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \end{array} \right\} \Rightarrow$$

CARTESIAN

$$\nabla^2 \phi = \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right)$$

CYLINDRICAL COORDINATE SYSTEM

$$\left. \begin{array}{l} \bullet \text{GRADIENT} \quad \nabla \phi = \left(\frac{\partial \phi}{\partial \rho}, \frac{1}{\rho} \frac{\partial \phi}{\partial \varphi}, \frac{\partial \phi}{\partial z} \right) \\ \bullet \text{DIVERGENCE} \quad \nabla \cdot \vec{A} = \left(\frac{1}{\rho} \frac{\partial (\rho A_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z} \right) \end{array} \right\} \Rightarrow$$

CYLINDRICAL

$$\nabla^2 \phi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \phi}{\partial \varphi^2} + \frac{\partial^2 \phi}{\partial z^2}$$

SPHERICAL COORDINATE SYSTEM

$$\left. \begin{array}{l} \bullet \text{GRADIENT} \quad \nabla \phi = \left(\frac{\partial \phi}{\partial r}, \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \right) \\ \bullet \text{DIVERGENCE} \quad \nabla \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} (A_\varphi) \end{array} \right\} \Rightarrow$$

SPHERICAL

$$\nabla^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \varphi^2}$$

PROBLEM 3

An infinitely long cylinder with radius R has a charge density ρ_c .
Calculate the potential and the electric field:

- (a) Inside the cylinder
- (b) Outside the cylinder

Assume the potential on the surface is V_0 .

SOLUTION

Due to the symmetry of the problem,
the solution will depend on the radius only: $\phi = \phi(\rho)$

Inside the cylinder

$$\nabla^2 \phi = -\frac{\rho_c}{\epsilon_0}$$

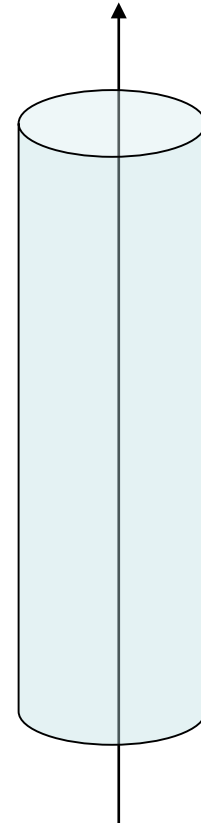
cylindrical coord.

$$\nabla^2 \phi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \phi}{\partial \rho} \right) + \underbrace{\frac{1}{\rho^2} \frac{\partial^2 \phi}{\partial \varphi^2}}_{=0} + \underbrace{\frac{\partial^2 \phi}{\partial z^2}}_{=0} = -\frac{\rho_c}{\epsilon_0}$$

$\leftarrow$ Because the solution depends only on ρ

The equation becomes:

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \phi}{\partial \rho} \right) = -\frac{\rho_c}{\epsilon_0}$$



$$\begin{aligned}
 \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \phi}{\partial \rho} \right) &= -\frac{\rho_c}{\epsilon_0} \xRightarrow{\text{Multiplying by } \rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \phi}{\partial \rho} \right) = -\frac{\rho_c}{\epsilon_0} \rho \xRightarrow{\text{Integrating in } \rho} \rho \frac{\partial \phi}{\partial \rho} = -\frac{\rho_c}{2\epsilon_0} \rho^2 + a \xRightarrow{\text{Dividing by } \rho} \frac{\partial \phi}{\partial \rho} = -\frac{\rho_c}{2\epsilon_0} \rho + \frac{a}{\rho} \Rightarrow \\
 \left[\bar{E} = -\nabla \phi &= -\left(\frac{d\phi(\rho)}{d\rho}, \underbrace{\frac{1}{\rho} \frac{d\phi(\rho)}{d\rho}}_{=0}, \underbrace{\frac{d\phi(\rho)}{dz}}_{=0} \right) \Rightarrow E_r = -\frac{d\phi(\rho)}{d\rho} = +\frac{\rho_c}{2\epsilon_0} \rho - \frac{a}{\rho} \right] \begin{array}{l} \text{Divergent at } \rho=0 \\ \text{A solution with } a \neq 0 \text{ would} \\ \text{not have a physical meaning} \\ \Rightarrow a=0 \end{array} \leftarrow \text{Because the solution depends only on } \rho
 \end{aligned}$$

$$\frac{\partial \phi}{\partial \rho} = -\frac{\rho_c}{2\epsilon_0} \rho \Rightarrow \boxed{\phi(\rho) = -\frac{\rho_c}{4\epsilon_0} \rho^2 + b}$$

Integrating in ρ

Outside the cylinder

$$\nabla^2 \phi = 0 \quad \leftarrow \text{There is no charge}$$

And the equation becomes:

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \phi}{\partial \rho} \right) = 0 \xRightarrow{\text{Multiplying by } \rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \phi}{\partial \rho} \right) = 0 \xRightarrow{\text{Integrating in } \rho} \rho \frac{\partial \phi}{\partial \rho} = c \xRightarrow{\text{Dividing by } \rho} \frac{\partial \phi}{\partial \rho} = \frac{c}{\rho} \xRightarrow{\text{Integrating in } \rho} \boxed{\phi(\rho) = c \ln \rho + d}$$

$$E_r = -\frac{d\phi(\rho)}{d\rho} = \boxed{-\frac{c}{\rho}}$$

$$E_r^{in}(\rho) = +\frac{\rho_c}{2\epsilon_0}\rho \quad \phi^{in}(\rho) = -\frac{\rho_c}{4\epsilon_0}\rho^2 + b$$

$$E_r^{out} = -\frac{c}{\rho} \quad \phi^{out}(\rho) = c \ln \rho + d$$

Now we must determine the three constants b , c and d .

We have three conditions:

- (1) Continuity of the electric field at $\rho = R$
- (2) The potential at $\rho = R$
- (3) Continuity of the potential at $\rho = R$

$$(1) \quad E_r^{in}(R) = E_r^{out}(R) \Rightarrow +\frac{\rho_c}{2\epsilon_0}R = -\frac{c}{R} \Rightarrow c = -\frac{\rho_c}{2\epsilon_0}R^2 \Rightarrow \phi^{out}(\rho) = -\frac{\rho_c}{2\epsilon_0}R^2 \ln \rho + d$$

$$(2) \quad \phi^{out}(R) = V_0 \Rightarrow -\frac{\rho_c}{2\epsilon_0}R^2 \ln R + d = V_0 \Rightarrow d = V_0 + \frac{\rho_c}{2\epsilon_0}R^2 \ln R$$

$$\Rightarrow \phi^{out}(\rho) = -\frac{\rho_c}{2\epsilon_0}R^2 \ln \rho + d = -\frac{\rho_c}{2\epsilon_0}R^2 \ln \rho + V_0 + \frac{\rho_c}{2\epsilon_0}R^2 \ln R = V_0 - \frac{\rho_c}{2\epsilon_0}R^2 \ln \left(\frac{\rho}{R} \right)$$

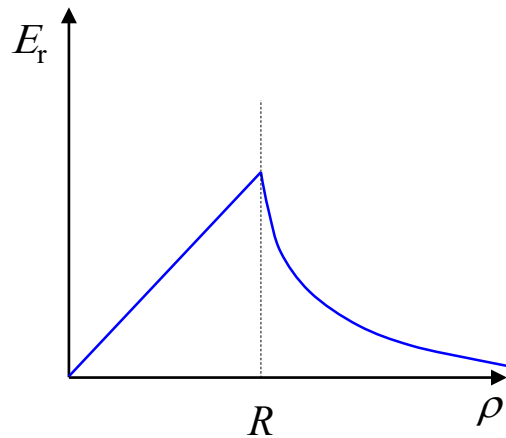
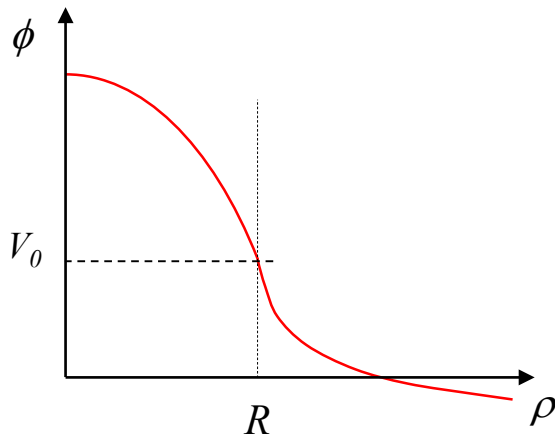
$$(3) \quad \phi^{out}(R) = \phi^{in}(R) \Rightarrow V_0 - \underbrace{\frac{\rho_c}{2\epsilon_0}R^2 \ln \left(\frac{R}{R} \right)}_{=0} = -\frac{\rho_c}{4\epsilon_0}R^2 + b \Rightarrow b = V_0 + \frac{\rho_c}{4\epsilon_0}R^2$$

$$\phi^{in}(\rho) = V_0 + \frac{\rho_c}{4\epsilon_0} R^2 \left(1 - \frac{\rho^2}{R^2} \right)$$

$$E_r^{in}(\rho) = + \frac{\rho_c}{2\epsilon_0} \rho$$

$$\phi^{out}(\rho) = V_0 - \frac{\rho_c}{2\epsilon_0} R^2 \ln\left(\frac{\rho}{R}\right)$$

$$E_r^{out} = \frac{\rho_c}{2\epsilon_0} \frac{R^2}{\rho}$$



Note that: $\lim_{\rho \rightarrow \infty} \phi(\rho) = -\infty$

So, this solution is unphysical.

How is this possible? This is because infinitely long cylinders do not exist.

Nonetheless, the solution can be considered as a reasonable approximation for cylinders with $R \ll L$ (where L is the length of the cylinder).

PROBLEM 4

Use the Gauss theorem to calculate the flux of the vector field: $\vec{A}(\rho, \varphi, z) = z \frac{\rho^2 - 1}{\rho} \hat{e}_\rho$

on the surface S: $x^2 + y^2 + (z - 2)^2 = 4$

SOLUTION

The field is singular at $\rho=0$ (the z-axis)
The Gauss theorem cannot be applied on S.

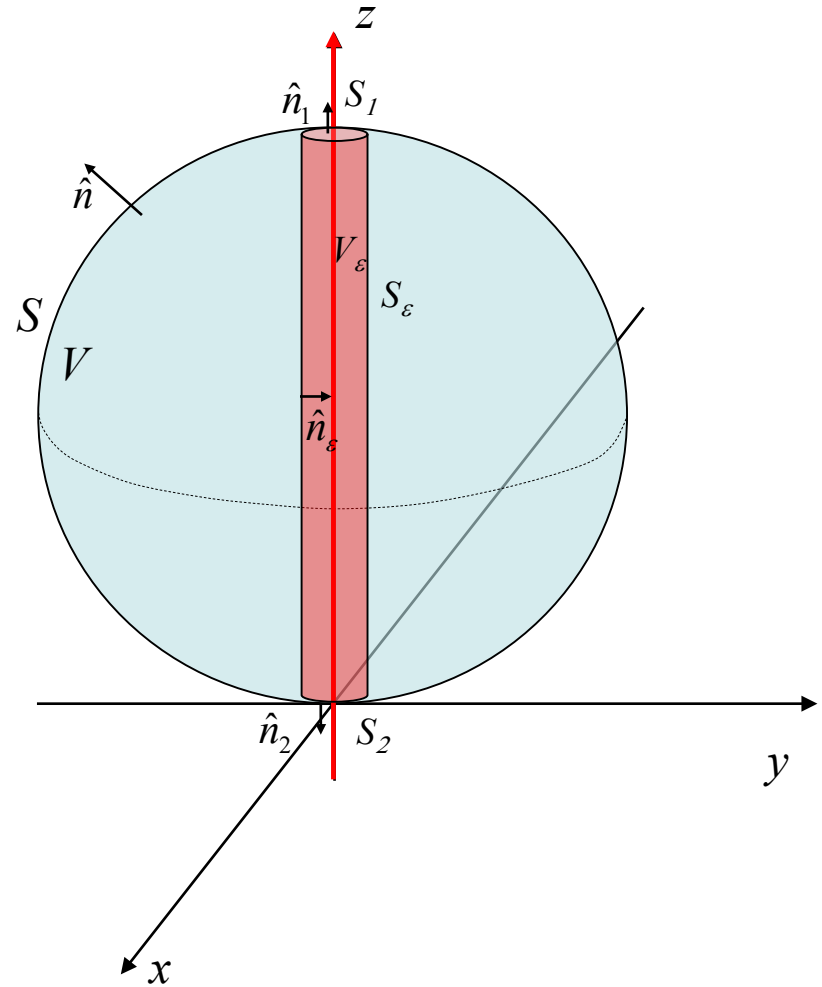
We divide V into two volumes:

$$V = V_0 + V_\varepsilon$$

Thin cylinder with radius ε along the z-axis

The surface, boundary to V_0 , is: $S + S_\varepsilon - S_1 - S_2$

V_0 does not contain the z-axis
 $\Rightarrow$ we can apply Gauss



$$\iint_S \vec{A}(\vec{r}) \cdot d\vec{S} = \iint_{\substack{S+S_\varepsilon-S_1-S_2 \\ -S_1+S_2-S_2}} \vec{A}(\vec{r}) \cdot d\vec{S} =$$

$$= \iint_{S+S_\varepsilon-S_1-S_2} \vec{A}(\vec{r}) \cdot d\vec{S} + \iint_{-S_\varepsilon+S_1+S_2} \vec{A}(\vec{r}) \cdot d\vec{S} = \iiint_{V_0} \text{div} \vec{A} dV + \iint_{-S_\varepsilon+S_1+S_2} \vec{A}(\vec{r}) \cdot d\vec{S}$$

$$= \iiint_{V_0} \text{div} \bar{A} dV + \iint_{-S_\varepsilon + S_1 + S_2} \bar{A}(\bar{r}) \cdot d\bar{S} = \iiint_{V_0} \text{div} \bar{A} dV - \iint_{S_\varepsilon} \bar{A}(\bar{r}) \cdot d\bar{S} + \iint_{S_1} \bar{A}(\bar{r}) \cdot d\bar{S} + \iint_{S_2} \bar{A}(\bar{r}) \cdot d\bar{S}$$

I
II
III
IV

Integrals *III* and *IV* are zero:

$$\iint_{S_1} \bar{A}(\bar{r}) \cdot d\bar{S} = \iint_{S_1} z \frac{\rho^2 - 1}{\rho} \underbrace{\hat{e}_\rho \cdot \hat{e}_z}_{=0} dS = \boxed{0}$$

$\lim_{\varepsilon \rightarrow 0} d\bar{S} = \hat{e}_z dS$

$$\iint_{S_2} \bar{A}(\bar{r}) \cdot d\bar{S} = - \iint_{S_2} z \frac{\rho^2 - 1}{\rho} \underbrace{\hat{e}_\rho \cdot \hat{e}_z}_{=0} dS = \boxed{0}$$

$\lim_{\varepsilon \rightarrow 0} d\bar{S} = -\hat{e}_z dS$

Integrals *II* is:

$$\begin{aligned}
 - \iint_{S_\varepsilon} \bar{A}(\bar{r}) \cdot d\bar{S} &= - \iint_{S_\varepsilon} z \frac{\rho^2 - 1}{\rho} \hat{e}_\rho \cdot (-\hat{e}_\rho) dS = \iint_{S_\varepsilon} z \frac{\rho^2 - 1}{\rho} dS = \iint_{S_\varepsilon} z \frac{\varepsilon^2 - 1}{\varepsilon} \varepsilon d\varphi dz = \\
 &= \int_0^{2\pi} \int_{2-\sqrt{4-\varepsilon^2}}^{2+\sqrt{4-\varepsilon^2}} z (\varepsilon^2 - 1) d\varphi dz = 2\pi (\varepsilon^2 - 1) \int_{2-\sqrt{4-\varepsilon^2}}^{2+\sqrt{4-\varepsilon^2}} z dz = 8\pi (\varepsilon^2 - 1) \sqrt{4 - \varepsilon^2}
 \end{aligned}$$

$d\bar{S} = -\hat{e}_\rho dS$ (pointing to the first equation)
 $dS = \varepsilon d\varphi dz$ because on S_ε : $\rho = \varepsilon$ (pointing to the second equation)

But the cylinder is very “thin”, which means that we need to calculate the limit for : $\varepsilon \rightarrow 0$

$$\lim_{\varepsilon \rightarrow 0} 8\pi(\varepsilon^2 - 1)\sqrt{4 - \varepsilon^2} = -16\pi$$

Integrals I is:

$$\iiint_{V_0} \operatorname{div} \bar{A} dV = \iiint_{V_0} \operatorname{div} \left(z \frac{\rho^2 - 1}{\rho} \hat{e}_\rho \right) dV = \iiint_{V_0} \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho z \frac{\rho^2 - 1}{\rho} \right) dV = \iiint_{V_0} 2z dV$$

The volume is a sphere with centre in the point (0,0,2)

Therefore, to use spherical coord. a coordinate transformation is necessary: $z' = z - 2$

$$\iiint_{V_0} 2z dV = \iiint_{V_0} 2(z' + 2) dV = \iiint_{V_0} 4 dV + \iiint_{V_0} 2z' dV = 4 \frac{4\pi 2^3}{3} + \iiint_{V_0} 2r \cos \theta r^2 \sin \theta dr d\theta d\varphi$$

$$= \frac{128\pi}{3} + 2 \left[\frac{r^4}{4} \right]_0^2 \underbrace{2\pi \left[-\frac{\cos^2 \theta}{2} \right]_0^\pi}_0 = \frac{128\pi}{3}$$

$$I + II + III + IV = \frac{128\pi}{3} - 16\pi + 0 + 0 = \frac{80\pi}{3}$$