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ED1110 VEKTORANALYS
SUMMARY
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Warning: I have tried to be extramely concise. So this summary might not contain all the topics that you need to learn. It might contain minor imprecisions and typos. If you find any of them, please contact me.

1. GRADIENT

A **scalar field** associates a real number ϕ(x,y,z) to each point (x,y,z) of the space.

A **vector field** associates a vector $\bar{\mathbf{A}}$(x,y,z) to each point (x,y,z) of the space.

A level surface is a surface on which the scalar field is constant: ϕ(x,y,z)=c

The **gradient** of a <u>scalar</u> ϕ is the <u>vector</u> defined by: $\boxed{grad\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)}$ *(in cartesian)*

The increase of a scalar field in direction $\hat{e}$ is the component of the gradient in that direction:

Directional derivative

$$\boxed{\frac{d\phi}{ds} = grad\phi \cdot \hat{e}}$$

THEOREM: The direction of $grad\phi$ is the direction of the maximum increase of ϕ
The maximum increase of ϕ per unit length is $|grad\phi|$

THEOREM: The gradient in the point P is zero if ϕ has a maximum or a minimum in P

THEOREM: $grad\phi$ is orthogonal to the level surfaces of ϕ.

2. LINE INTEGRAL

The line integral of the vector field $\bar{F}$ on the path L parameterized by the equation $\underset{u:a\to b}{\bar{r}=\bar{r}(u)}$ is:

$$\boxed{\int_L \bar{F}(\bar{r})\cdot d\bar{r} = \int_a^b \bar{F}(\bar{r}(u))\cdot \frac{d\bar{r}}{du}du}$$

THEOREM: The line integral along $-L$ is the opposite of the line integral along L

$$\int_{-L} \bar{F}(\bar{r})\cdot d\bar{r} = -\int_L \bar{F}(\bar{r})\cdot d\bar{r}$$

The line of integral along a closed path is called circulation (*cirkulationen*)

$$\oint_C \bar{A}(\bar{r})\cdot d\bar{r}$$

THEOREM: The circulation is zero if and only if for all points P and Q the line integral is independent from the integration path between P and Q.
Such a field is called conservative.

THEOREM: If $\bar{A}=grad\phi$ then : $\int_P^Q \bar{A}(\bar{r})\cdot d\bar{r} = \phi(Q)-\phi(P)$

3. FLUX INTEGRAL

The flux integral of the vector field $\bar{A}$ on the surface S parameterized by $\underset{\substack{u:a_1\to a_2\\ v:b_1\to b_2}}{\bar{r}=\bar{r}(u,v)}$ is:

$$\boxed{\iint_S \bar{A}\cdot d\bar{S} = \int_{a_1}^{a_2}\int_{b_1}^{b_2} \bar{A}(\bar{r}(u,v))\cdot\left(\frac{\partial\bar{r}}{\partial u}\times\frac{\partial\bar{r}}{\partial v}\right)dudv}$$

The infinitesimal surface element $d\bar{S}$ is:

$$d\bar{S} = \hat{n}dS = \left(\frac{\partial\bar{r}}{\partial u}\times\frac{\partial\bar{r}}{\partial v}\right)dudv$$

4. DIVERGENCE AND CURL

The **divergence** of a vector $\bar{A}$ is a scalar defined by:
It is a measure of how much the field diverges (*or converges*) from (*to*) a point.

$$\boxed{div\bar{A} \equiv \frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}}$$

The **curl** of a vector $\bar{A}$ is a vector defined by:
The curl is a measure of how much the direction of a vector field changes in space, i.e. how much the field "rotates".
The direction is the axis of rotation. The amplitude is the amplitude of rotation.

$$\boxed{rot\bar{A} \equiv \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}}$$

THEOREM: A vector field $\bar{A}$ has a scalar potential ϕ if and only if $rot\bar{A}=0$ (virvälfria fält)

THEOREM: $\bar{B}$ has a vector potential $\bar{A}$, $\bar{B}=rot\bar{A} \Leftrightarrow div\bar{B}=0$ (solenoidal fält)

5. GAUSS' THEOREM and STOKES' THEOREM

GAUSS' THEOREM

$$\oiint_S \bar{A}\cdot d\bar{S} = \iiint_V div\bar{A}dV$$

where S is a closed surface that forms the boundary of the volume V and $\bar{A}$ is a continuously differentiable vector field on V.

STOKES' THEOREM

$$\oint_L \bar{A}\cdot d\bar{r} = \iint_S rot\bar{A}\cdot d\bar{S}$$

where $\bar{A}$ is a continuously differentiable vector field on S, L is a closed curve and S is a surface whit boundary on L.

6. CURVILINEAR COORDINATES

The basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ of an orthogonal curvilinear coordinate system (u_1, u_2, u_3) are defined as:

$$\boxed{\hat{e}_i = \frac{1}{h_i}\frac{\partial \bar{r}}{\partial u_i} \quad with \quad scale\ factor \quad h_i = \left|\frac{\partial \bar{r}}{\partial u_i}\right|}$$

In a curvilinear coordinate system, gradient, curl and divergence are:

$$\boxed{grad\phi = \sum_i \frac{1}{h_i}\frac{\partial \phi}{\partial u_i}\hat{e}_i}$$

$$\boxed{rot\bar{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}}$$

$$\boxed{div\bar{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_3 h_1) + \frac{\partial}{\partial u_3}(A_3 h_1 h_2)\right]}$$

See the formula sheet for applications to cylindrical and spherical coordinate systems

7. NABLA OPERATOR AND NABLARÄKNING

An **operator** *T* is a law that to each function *f* in the function class D_t associates a function *T(f)*.

An operator *T* is **linear** if ***T(af+bg)=aT(f)+bT(g)***, where *f* and g are two functions

The **nabla** operator is: $\boxed{\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)}$

$$\boxed{\begin{aligned} grad\phi &= \nabla\phi \\ div\bar{A} &= \nabla\cdot\bar{A} \\ rot\bar{A} &= \nabla\times\bar{A} \end{aligned}}$$

To calculate an expression that contains nabla:

STEP 1 Rewrite the expression as a sum with N terms, where N is the number of fields in the expression. Every term will be identical to the original expression, but the *i-th* field in the *i-th* term will have a dot.

$$\nabla\cdot\cdot(\phi,\bar{A},\psi,\bar{B},\ldots) = \nabla\cdot\cdot(\dot{\phi},\bar{A},\psi,\bar{B},\ldots) + \nabla\cdot\cdot(\phi,\dot{\bar{A}},\psi,\bar{B},\ldots) + \nabla\cdot\cdot(\phi,\bar{A},\dot{\psi},\bar{B},\ldots) + \nabla\cdot\cdot(\phi,\bar{A},\psi,\dot{\bar{B}},\ldots) + \ldots$$

STEP 2 The nabla can now be formally considered as a vector. Each term will be rewritten using vector algebra rules.

STEP 3 In each rewritten term, **ONLY the field with the "dot" must appear after the nabla.** Finally, you can remove the "dot".

8. INDEXRÄKNING

Notations: substitute *x,y,z* with suffices *1,2,3*. Examples: $A_x = A_1 \quad \hat{e}_y = \hat{e}_2 \quad \frac{\partial\phi}{\partial y} = \partial_2\phi \quad \frac{\partial A_x}{\partial y} = \partial_2 A_1$

Summation convention: whenever a suffix is repeated in a single term, summation from 1 to 3 is understood. Example: $\bar{a}\cdot\bar{b} = a_x b_x + a_y b_y + a_z b_z = a_i b_i$

Kronecker delta $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & otherwise \end{cases}$ some properties: $\boxed{l_{jm}\delta_{km} = l_{jk}}$ $\boxed{\delta_{ii} = 3}$

Alternating tensor $\varepsilon_{ijk} = \hat{e}_i\cdot(\hat{e}_j\times\hat{e}_k) = \begin{cases} 0 & if\ any\ of\ i,j,k\ are\ equal \\ +1 & if\ (i,j,k) = (1,2,3)\ or\ (2,3,1)\ or\ (3,1,2) \quad (even\ permutation) \\ -1 & if\ (i,j,k) = (1,3,2)\ or\ (2,1,3)\ or\ (3,2,1) \quad (odd\ permutation) \end{cases}$

The alternating tensor is useful to write the cross product in tensor notation: $\boxed{(\bar{a}\times\bar{b})_i = \varepsilon_{ijk} a_j b_k}$

The nabla calculation can be performed using the tensor notation, the Kronecker delta and the alternating tensor. This procedure is called Index calculation (indexräkning).

Gradient, divergence and curl in index notation: $\boxed{(\nabla\phi)_i = \phi_{,i}}$ $\boxed{\nabla\cdot\bar{A} = A_{i,i}}$ $\boxed{(\nabla\times\bar{A})_i = \varepsilon_{ijk} A_{k,j}}$

9. INTEGRAL THEOREMS

Integral theorems are a generalization of Gauss and Stokes' theorem

Generalized Gauss theorem: $\boxed{\oint\!\!\oint_S d\bar{S}(\ldots) = \iiint_V dV\nabla(\ldots)}$

Generalized Stokes theorem: $\boxed{\oint_L d\bar{r}(\ldots) = \iint_S \left(d\bar{S}\times\nabla\right)(\ldots)}$

where (...) can be substituted with:

$\phi\ ,\ \cdot\bar{A}\ ,\ \times\bar{A}$

10. IMPORTANT VECTOR FIELDS

POINT SOURCE

The **vector field generated by a point source** is: $\boxed{\bar{A}(\bar{r}) = \frac{s}{r^2}\hat{e}_r}$

The flux of the field generated by a point source is: $\boxed{\oint\!\!\oint_S \frac{s}{r^2}\hat{e}_r\cdot d\bar{S} = \begin{cases} 0 & \text{If the source is outside } V \\ 4\pi s & \text{If the source is inside } V \end{cases}}$

The potential from a point source is: $\boxed{\phi = -\frac{s}{r}}$

DIPOLE

The potential and the electric field generated by a dipole are: $\boxed{\phi(\bar{r}) = \frac{\bar{p}\cdot\bar{r}}{r^3}}$ $\boxed{\bar{E}(\bar{r}) = -\frac{\bar{p}}{r^3} + \frac{3(\bar{p}\cdot\bar{r})\bar{r}}{r^5}}$

where $\bar{p}$ is the dipole moment

VIRVELTRÅDEN $\boxed{\oint_L \frac{k}{\rho}\hat{e}_\varphi\cdot d\bar{r} = 2\pi kN}$ *Where N is number of turns of L around the z-axis*

11. LAPLACE AND POISSON EQUATIONS

Laplace's equation $\boxed{\nabla^2\phi = 0}$

Poisson's equation $\boxed{\nabla^2\phi = \rho}$

THEOREM: If ϕ has continuous second derivatives in the volume *V* and if $\phi = 0$ on the surface S that encloses *V* then the solution to the Laplace equation $\nabla^2\phi = 0$ is $\phi(x,y,z) = 0$ everywhere in *V*

Dirichlet's boundary condition

$$\nabla^2\phi = \rho \qquad \phi = \sigma \quad \text{on } S$$

Neumann's Boundary condition

$$\nabla^2\phi = \rho \qquad \frac{\partial\phi}{\partial n} = \hat{n}\cdot\nabla\phi = \gamma \quad \text{on } S$$

THEOREM: The Poisson's equation $\nabla^2\phi = \rho$ in the volume *V* with boundary condition $\phi = \sigma$ on the surface S *(boundary of V)* has only one solution.

THEOREM: If ϕ_s is a solution to the Poisson's equation $\nabla^2\phi = \rho$ in *V* with boundary condition $\hat{n}\cdot\nabla\phi = \gamma$, then also ϕ_s+c is a solution, with *c* an arbitrary constant.

Solutions of the Laplace's equation $\nabla^2\phi = 0$ in simple cases:

PLANAR SYMMETRY $\phi = \phi(x) \Rightarrow \frac{d^2\phi(x)}{dx^2} = 0 \Rightarrow \boxed{\phi(x) = ax + b}$

CYLINDRICAL SYMMETRY $\phi = \phi(\rho) \Rightarrow \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\phi(\rho)}{d\rho}\right) = 0 \Rightarrow \boxed{\phi(\rho) = a\ln\rho + b}$

SPHERICAL SYMMETRY $\phi = \phi(r) \Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi(r)}{dr}\right) = 0 \Rightarrow \boxed{\phi(r) = -\frac{a}{r} + b}$