

VEKTORANALYS

HT 2022

CELTE / CENMI

ED1110

**NABLAOPERATOR och NABLARÄKNING,
INTEGRALSATSER,
TENSORER och INDEXRÄKNING
ÖVNINGAR**

Kapitel 11, 12, 14

Kapitel 15



PROBLEM 1

Use “nablaräkning” to verify: $rot(\phi \bar{A}) = grad \phi \times \bar{A} + \phi rot \bar{A}$

ID3

SOLUTION

Add the dots

$$rot(\phi \bar{A}) = \nabla \times (\phi \bar{A}) \overset{\swarrow}{=} \nabla \times (\phi \dot{\bar{A}}) + \nabla \times (\phi \bar{\dot{A}}) =$$

Now nabla can be considered as a vector: $\bar{n} \times (c\bar{a}) + \bar{n} \times (c\bar{a}) = (\bar{n}c) \times \bar{a} + c(\bar{n} \times \bar{a})$ (because c is a scalar)

$$= (\nabla \phi) \times \bar{A} + \phi (\nabla \times \bar{A}) =$$

$$= (\nabla \phi) \times \bar{A} + \phi (\nabla \times \bar{A})$$

PROBLEM 2

Use “nablaräkning” to verify: $div rot \bar{A} = 0$

ID8

SOLUTION

Now nabla can be considered as a vector.

$$div rot \bar{A} = \nabla \cdot (\nabla \times \bar{A}) \overset{\swarrow}{=} \text{Because: } \bar{n} \cdot (\bar{n} \times \bar{a}) = \bar{a} \cdot (\underbrace{\bar{n} \times \bar{n}}_{=0})$$

$$= \bar{\dot{A}} \cdot (\nabla \times \nabla) = (\nabla \times \nabla) \cdot \bar{\dot{A}} = 0$$

PROBLEM 3

Use “nablaräkning” to verify: $(\vec{A} \times \nabla) \times \vec{A} = \frac{1}{2} \nabla A^2 - \vec{A}(\nabla \cdot \vec{A})$

SOLUTION

$$\begin{aligned}
 (\vec{A} \times \nabla) \times \vec{A} &= (\vec{A} \times \nabla) \times \vec{A} = \leftarrow \dots \dots \dots (\vec{a} \times \vec{n}) \times \vec{b} = (\vec{a} \cdot \vec{b}) \vec{n} - (\vec{n} \cdot \vec{b}) \vec{a} \\
 &= \nabla(\vec{A} \cdot \vec{A}) - \vec{A}(\nabla \cdot \vec{A}) = \\
 &= \frac{1}{2} \nabla A^2 - \vec{A}(\nabla \cdot \vec{A}) \quad \begin{array}{l} \nearrow \dots \dots \dots \nabla A^2 = \nabla(\vec{A} \cdot \vec{A}) = \nabla(\vec{A} \cdot \vec{A}) + \nabla(\vec{A} \cdot \vec{A}) = 2\nabla(\vec{A} \cdot \vec{A}) \\ \Rightarrow \nabla(\vec{A} \cdot \vec{A}) = \frac{1}{2} \nabla A^2 \end{array}
 \end{aligned}$$

PROBLEM 4

The divergence of the magnetic field is zero: $\nabla \cdot \vec{B} = 0$

Use “nablaräkning” to prove: $\vec{B} \times (\nabla \times \vec{B}) - (\vec{B} \times \nabla) \times \vec{B} = -(\vec{B} \cdot \nabla) \vec{B}$

SOLUTION

$$\begin{aligned}
 \vec{B} \times (\nabla \times \vec{B}) - (\vec{B} \times \nabla) \times \vec{B} &= \vec{B} \times (\nabla \times \vec{B}) - (\vec{B} \times \nabla) \times \vec{B} = \vec{B} \times (\nabla \times \vec{B}) + \vec{B} \times (\vec{B} \times \nabla) = \\
 &= \nabla(\vec{B} \cdot \vec{B}) - \vec{B}(\vec{B} \cdot \nabla) - \nabla(\vec{B} \cdot \vec{B}) + \vec{B}(\vec{B} \cdot \nabla) = \quad \text{using } \vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}) \\
 &= \nabla(\vec{B} \cdot \vec{B}) - (\vec{B} \cdot \nabla) \vec{B} - \nabla(\vec{B} \cdot \vec{B}) + \vec{B}(\nabla \cdot \vec{B}) = \\
 &= -(\vec{B} \cdot \nabla) \vec{B} + \vec{B}(\nabla \cdot \vec{B}) = \underbrace{\vec{B}(\nabla \cdot \vec{B})}_{=0} - (\vec{B} \cdot \nabla) \vec{B} = -(\vec{B} \cdot \nabla) \vec{B}
 \end{aligned}$$

PROBLEM 5

Calculate $\epsilon_{ijk} \epsilon_{ljk}$

SOLUTION

We know that $\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$

Therefore:

We re-arrange the suffixes to have an expression similar to

the same expression with $m=j$

$$\epsilon_{ijk} \epsilon_{ljk} = \epsilon_{ijk} \epsilon_{klj} = \delta_{il} \delta_{jj} - \delta_{ij} \delta_{jl} = \delta_{il} 3 - \delta_{il} = 2\delta_{il}$$

even permutations does NOT change the sign

Remember that $\delta_{ii} = 3$

Remember that $\delta_{km} p_m = p_k$

PROBLEM 6

Prove $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}$

using the suffix notation.

SOLUTION

We know that the i -component of the cross product can be written as: $(\bar{a} \times \bar{b})_i = \varepsilon_{ijk} a_j b_k$
Therefore:

$$\begin{aligned}
 (\bar{a} \times (\bar{b} \times \bar{c}))_i &= \varepsilon_{ijk} a_j (\bar{b} \times \bar{c})_k = \varepsilon_{ijk} a_j \varepsilon_{klm} b_l c_m \\
 &= \varepsilon_{ijk} a_j \varepsilon_{klm} b_l c_m = \varepsilon_{ijk} \varepsilon_{klm} a_j b_l c_m = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) a_j b_l c_m \\
 &= \delta_{il} \delta_{jm} a_j b_l c_m - \delta_{im} \delta_{jl} a_j b_l c_m = a_m b_i c_m - a_l b_l c_i = (\bar{a} \cdot \bar{c}) b_i - (\bar{a} \cdot \bar{b}) c_i
 \end{aligned}$$

$(\bar{b} \times \bar{c})_k = \varepsilon_{klm} b_l c_m$
 $\varepsilon_{ijk} \varepsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$
 $\delta_{ij} d_{kj} = d_{ki}$

PROBLEM 7

Use “indexräkning” to verify: $div(\bar{A} \times \bar{B}) = (rot \bar{A}) \cdot \bar{B} - (rot \bar{B}) \cdot \bar{A}$ **ID4**

SOLUTION

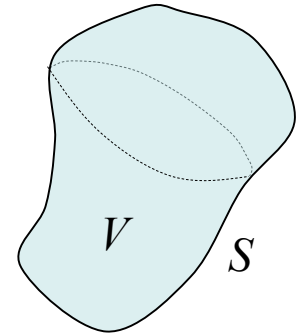
$$\begin{aligned} div(\bar{A} \times \bar{B}) &= (\bar{A} \times \bar{B})_{i,i} = (\varepsilon_{ijk} A_j B_k)_{,i} = \varepsilon_{ijk} (A_j B_k)_{,i} = \varepsilon_{ijk} (A_{j,i} B_k + A_j B_{k,i}) = \\ &= \varepsilon_{ijk} A_{j,i} B_k + \varepsilon_{ijk} A_j B_{k,i} = \varepsilon_{kij} A_{j,i} B_k - \varepsilon_{jik} B_{k,i} A_j = (rot \bar{A}) \cdot \bar{B} - (rot \bar{B}) \cdot \bar{A} \end{aligned}$$


remember that: $(\nabla \times \bar{A})_i = \varepsilon_{ijk} A_{k,j}$

PROBLEM 8

Show that:
$$\iiint_V \bar{r} \times \text{rot} \bar{A} dV = 2 \iiint_V \bar{A} dV$$

if on the boundary surface S of V the vector field is $\bar{A} = 0$



SOLUTION

Let's consider only the i -th component of the left hand side:

$$\hat{e}_i \cdot \iiint_V \bar{r} \times \text{rot} \bar{A} dV = \iiint_V \hat{e}_i \cdot (\bar{r} \times \text{rot} \bar{A}) dV = \iiint_V (\text{rot} \bar{A}) \cdot (\hat{e}_i \times \bar{r}) dV =$$

$\nearrow$
 $\bar{a} \cdot (\bar{b} \times \bar{c}) = \bar{c} \cdot (\bar{a} \times \bar{b})$

To continue, we must remember that: $\nabla \cdot (\bar{a} \times \bar{b}) = \bar{b} \cdot (\nabla \times \bar{a}) - \bar{a} \cdot (\nabla \times \bar{b})$ **(ID4)**

therefore,

$$\nabla \cdot (\bar{A} \times (\hat{e}_i \times \bar{r})) = (\hat{e}_i \times \bar{r}) \cdot (\nabla \times \bar{A}) - \bar{A} \cdot \nabla \times (\hat{e}_i \times \bar{r})$$

re-arranging the terms:
$$(\text{rot} \bar{A}) \cdot (\hat{e}_i \times \bar{r}) = \text{div}(\bar{A} \times (\hat{e}_i \times \bar{r})) + \bar{A} \cdot \text{rot}(\hat{e}_i \times \bar{r})$$

and we substitute

$$= \iiint_V \left[\operatorname{div}(\bar{A} \times (\hat{e}_i \times \bar{r})) + \bar{A} \cdot \operatorname{rot}(\hat{e}_i \times \bar{r}) \right] dV =$$

$$= \iiint_V \operatorname{div}(\bar{A} \times (\hat{e}_i \times \bar{r})) dV + \iiint_V \bar{A} \cdot \operatorname{rot}(\hat{e}_i \times \bar{r}) dV =$$

Generalized Gauss theorem

ID5

$$\begin{aligned} \operatorname{rot}(\hat{e}_i \times \bar{r}) &= (\bar{r} \cdot \nabla) \hat{e}_i - (\hat{e}_i \cdot \nabla) \bar{r} + \hat{e}_i (\nabla \cdot \bar{r}) - \bar{r} (\nabla \cdot \hat{e}_i) \\ &= 0 - \frac{\partial \bar{r}}{\partial x_i} + 3\hat{e}_i - 0 = 2\hat{e}_i \end{aligned}$$

$$= \underbrace{\iint_S (\bar{A} \times (\hat{e}_i \times \bar{r})) \cdot d\bar{S}}_{=0} + \iiint_V \bar{A} \cdot 2\hat{e}_i dV = 2 \iiint_V A_i dV$$

Because on S , $\bar{A}=0$

So, we have:

$$\hat{e}_i \cdot \iiint_V \bar{r} \times \operatorname{rot} \bar{A} dV = 2 \iiint_V A_i dV$$