VEKTORANALYS

HT 2022

CELTE / CENMI

ED1110

GAUSS' THEOREM and STOKES' THEOREM

Kursvecka 3

Kapitel 8-9 (*Vektoranalys,* 1:e uppl, Frassinetti/Scheffel)

version: 11-sept-2022

This week

Gauss' theorem:

- Divergence
  - definition
  - physical meaning
- The Gauss' theorem

Stokes' theorem:

- Curl
  - definition
  - physical meaning
- Stokes' theorem
- The Green's formula in the plane
- Culf-free fields and scalar potentials
- Solenoidal fields and vector potentials

Connections with previous and next topics

Gauss' theorem:

- vector fields
- It can be used to calculate the flux (in some specific cases)
- Applications: in "Electromagnetic Theory" to calculate the flux of electric field (i.e. with the Gauss' law).

Stokes' theorem:

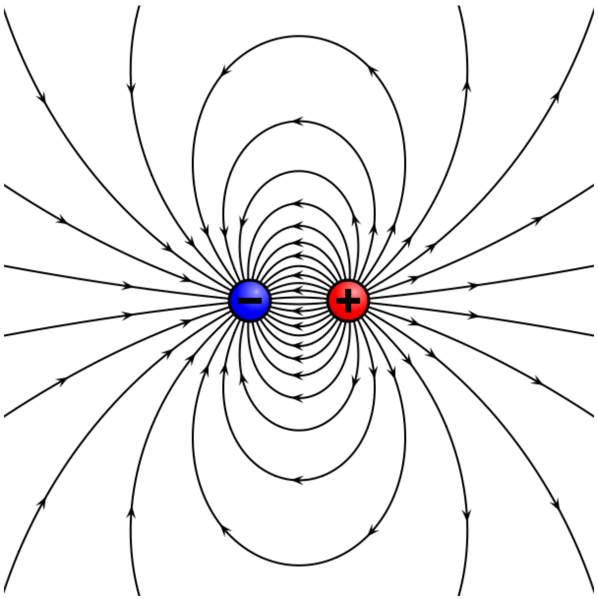
- Vector fields
- It can be used to calculate line integrals (in some specific cases).
- Important implication for the conservative fields and the potential
- Applications in "Electromagnetic Theory" to calculate the magnetic field (Ampere's law).

TARGET PROBLEM : the 1st and 2nd equations of Maxwell

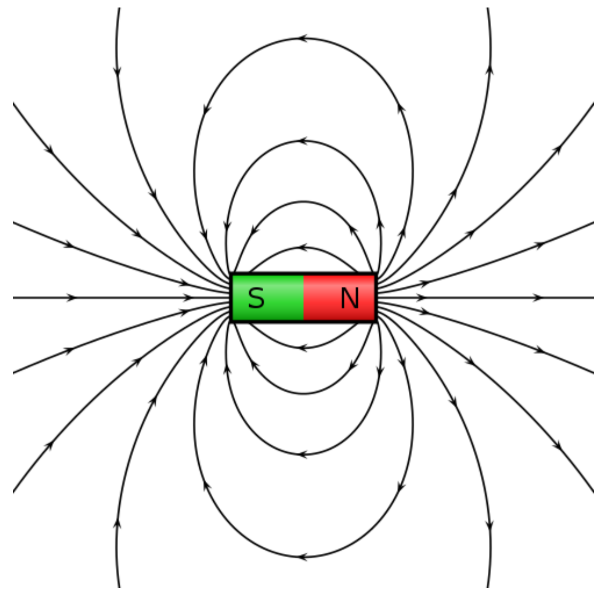
ELECTRIC FIELD $\overline{E}$

MAGNETIC FIELD $\overline{B}$

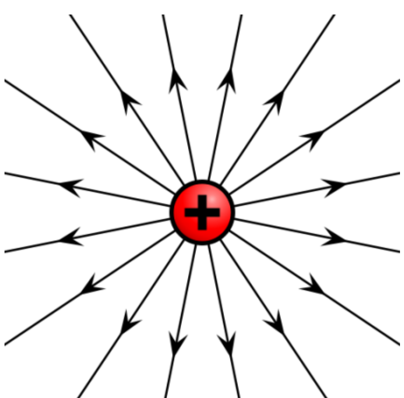
$\overline{E}$



$\overline{B}$



$\overline{E}$

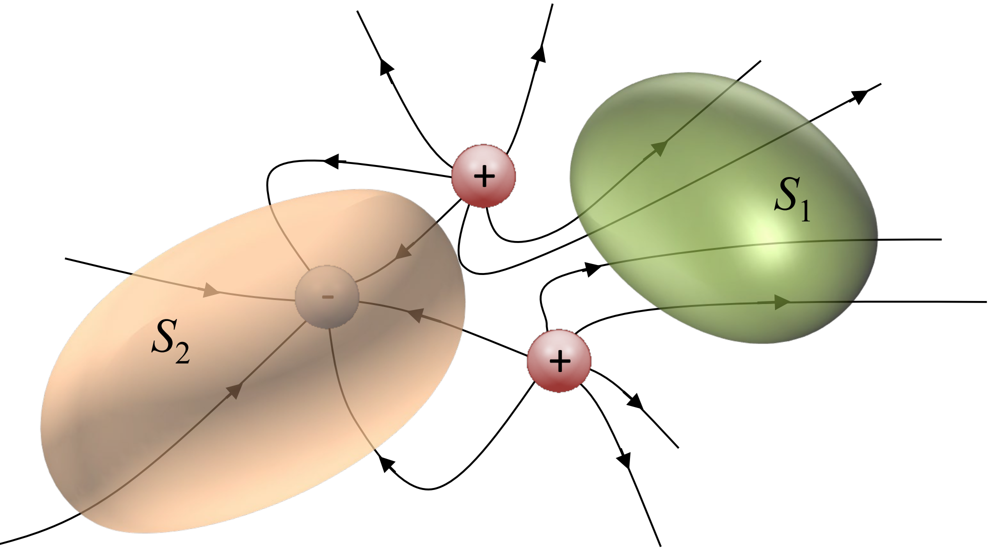


Magnetic monopoles do not exist in nature.

- **How can we express this information for $\overline{E}$ and $\overline{B}$ using the mathematical formalism?**

TARGET PROBLEM: the 1st equation of Maxwell

Let's consider some ELECTRIC CHARGES and two closed surfaces, S_1 and S_2



S_1 does not contain any charge.
It has no sources and no sinks:
no field lines destroyed and
no field lines created inside S1

$$\iint_{S_1} \overline{E} \cdot d\overline{S} = 0$$

S_2 contains a negative charge (a sink).
The field lines are destroyed inside S2

$$\iint_{S_2} \overline{E} \cdot d\overline{S} < 0$$

$$\boxed{\int_S \overline{E} \cdot d\overline{S} = \frac{Q}{\varepsilon_0} \quad \textbf{Gauss' law}}$$

(see the 6th week of this course for details or "Teoretisk elektroteknik")

We want to find: (1) the differential form of the Gauss' law.
(i.e. to express the Guass's law without using integrals)
(2) the corresponding expressions for the magnetic field

- the divergence of a vector field $\overline{A}$, $div\,\overline{A}$
- the Gauss's theorem $\iint_S \overline{A} \cdot d\overline{S} = \iiint_V div\overline{A}\,dV$

THE DIVERGENCE (DIVERGENSEN)

In a Cartesian coordinate system , the divergence of a vector field $\overline{A}$ is:

DEFINITION

$$\boxed{div\overline{A} \equiv \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}} \qquad (1)$$

It is a measure of how much the field diverges (*or converges*) from (*to*) a point.

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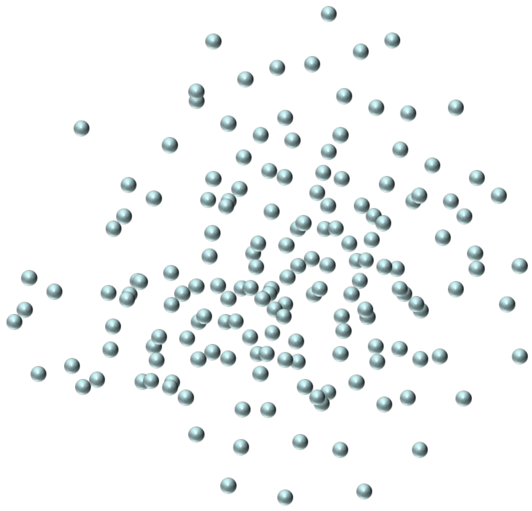
EXAMPLE:

- Assume that $\overline{A}$ is the velocity field of a gas.



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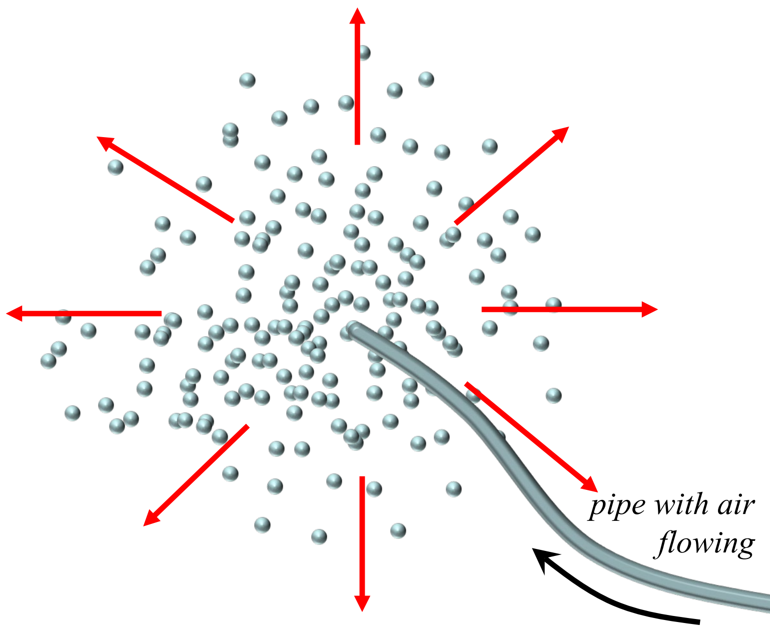
EXAMPLE:

- Assume that $\overline{A}$ is the velocity field of a gas.
- If there is gas flowing out of the pipe, the vector field will diverge
  - The position where the gas flows out of the pipe is a **source** of the vector field
  - In this case, we will see that the divergence is positive



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EXAMPLE:

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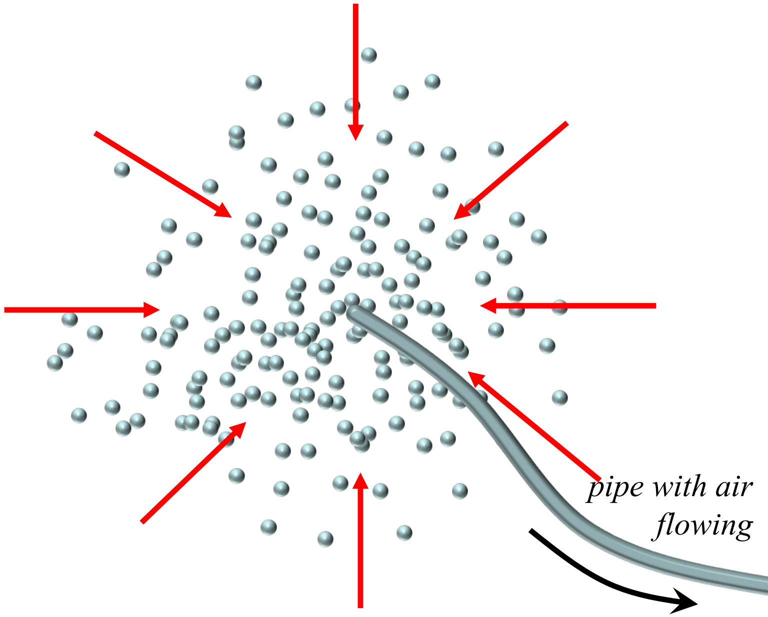
- If there is gas flowing out of the pipe, the vector field will diverge
  - The position where the gas flows out of the pipe is a **source** of the vector field
  - In this case, we will see that the divergence is positive
- If the pipe sucks gas, we will have gas flowing in the pipe and the vector field will converge
  - The position where the gas flows in the pipe is a **sink** of the vector field
  - In this case, we will see that the divergence is negative



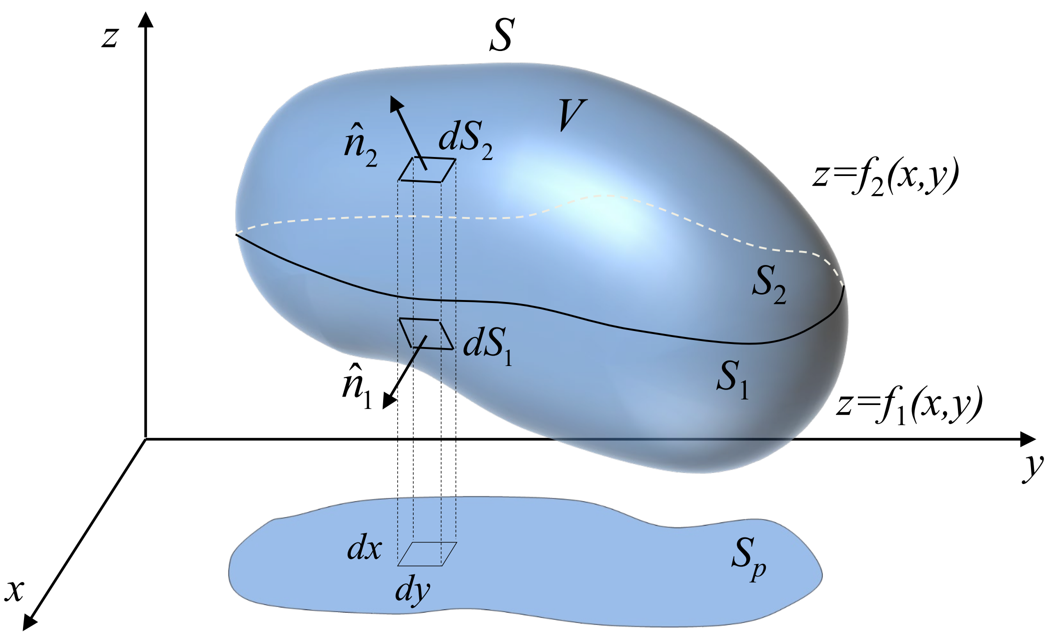
The divergence is a measure of the strength of sources and sinks.

(This is only "intuitive". From a formal point of view, this statement will be clear using the Gauss' theorem)

THE GAUSS' THEOREM

$$\boxed{\iint_S \bar{A} \cdot d\bar{S} = \iiint_V div\bar{A}dV} \qquad (2)$$

where **S is a closed surface** that forms the boundary of the volume V and $\bar{A}$ is a continuously differentiable vector field defined on V.



$$dxdy = dS_2\hat{n}_2 \cdot \hat{e}_z = \quad d\bar{S}_2 \cdot \hat{e}_z$$

$$dxdy = -dS_1\hat{n}_1 \cdot \hat{e}_z = -d\bar{S}_1 \cdot \hat{e}_z$$

THE GAUSS' THEOREM

PROOF

$$\iiint_V div\bar{A}dV = \iiint_V \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) dxdydz =$$

$$\iiint_V \frac{\partial A_x}{\partial x} dxdydz + \iiint_V \frac{\partial A_y}{\partial y} dxdydz + \iiint_V \frac{\partial A_z}{\partial z} dxdydz$$

Let's calculate the last term:

$$\iiint_V \frac{\partial A_z}{\partial z} dxdydz = \iint_{S_p} dxdy \int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial A_z}{\partial z} dz = \iint_{S_p} \left[A_z(x,y,f_2(x,y)) - A_z(x,y,f_1(x,y)) \right] dxdy =$$

$dxdy$ is the projection on S_p of the small element surfaces on dS_1 and dS_2.

Therefore: $dxdy = -\hat{e}_z \cdot \hat{n}_1 dS_1 = \hat{e}_z \cdot \hat{n}_2 dS_2$

$$= \iint_{S_2} A_z(x,y,f_2(x,y))\hat{e}_z \cdot \hat{n}_2 dS_2 + \iint_{S_1} A_z(x,y,f_1(x,y))\hat{e}_z \cdot \hat{n}_1 dS_1 = \iint_S A_z \hat{e}_z \cdot \hat{n} dS$$

Which means:

$$\iiint_V \frac{\partial A_z}{\partial z} dV = \iint_S A_z \hat{e}_z \cdot \hat{n} dS \qquad (3)$$

THE GAUSS' THEOREM

PROOF

In the same way we get:

$$\iiint\limits_V \frac{\partial A_x}{\partial x} dV = \iint\limits_S A_x \hat{e}_x \cdot \hat{n} dS \tag{4}$$

$$\iiint\limits_V \frac{\partial A_y}{\partial y} dV = \iint\limits_S A_y \hat{e}_y \cdot \hat{n} dS \tag{5}$$

Adding together equations (3), (4) and (5) we finally obtain:

$$\iiint\limits_V div\bar{A} dV = \iiint\limits_V \frac{\partial A_x}{\partial x} dxdydz + \iiint\limits_V \frac{\partial A_y}{\partial y} dxdydz + \iiint\limits_V \frac{\partial A_z}{\partial z} dxdydz =$$

$$\iint\limits_S A_x \hat{e}_x \cdot \hat{n} dS + \iint\limits_S A_y \hat{e}_y \cdot \hat{n} dS + \iint\limits_S A_z \hat{e}_z \cdot \hat{n} dS = \iint\limits_S \bar{A} \cdot d\bar{S}$$

Rearrange in logic order the steps to prove the Gauss' theorem

- **Add all the three terms together in order to obtain the flux of $\bar{A}$.**
- **Write down the volume integral of $\mathrm{div}\bar{A}$**
- **Consider the projection of the surface element on the xy plane, it will be $dxdy$. The projection will identify a infinitesimal surface element (dS_2) on the lower surface.**
- **Consider a closed surface.**
- **Split the volume integral into three terms. Then:**
  - (a) consider only the term which depends on the z-derivative of A_z,
  - (b) remove the z-derivative by solving the integral in dz, (what will remain is just the integral in $dxdy$)
  - (c) express $dxdy$ in order to obtain dS_1 and dS_2,
  - (d) re-arrange the integrals in dS_1 and dS_2 in order to have obtain a flux integral of $(0,0,A_z)$.
- **Repeat the same for the terms which depend on the x-derivative of A_x and on the y-derivative of A_y.**
- **Divide the surface in two parts, an upper surface and a lower surface and consider an infinitesimal surface element dS_1 on the upper surface.**
- **Write the expression that relates $dxdy$ to dS_1 and dS_2.**

Rearrange in logic order the steps to prove the Gauss' theorem

8 - **Add all the three terms together in order to obtain the flux of $\bar{A}$.**

5 - **Write down the volume integral of $\mathbf{div}\bar{A}$**

3 - **Consider the projection of the surface element on the xy plane, it will be $dxdy$. The projection will identify a infinitesimal surface element (dS_2) on the lower surface.**

1 - **Consider a closed surface.**

6 - **Split the volume integral into three terms. Then:**

6(a) consider only the term which depends on the z-derivative of A_z,

6(b) remove the z-derivative by solving the integral in dz, (what will remain is just the integral in $dxdy$)

6(c) express $dxdy$ in order to obtain dS_1 and dS_2,

6(d) re-arrange the integrals in dS_1 and dS_2 in order to have obtain a flux integral of $(0,0,A_z)$.

7 - **Repeat the same for the terms which depend on the x-derivative of A_x and on the y-derivative of A_y.**

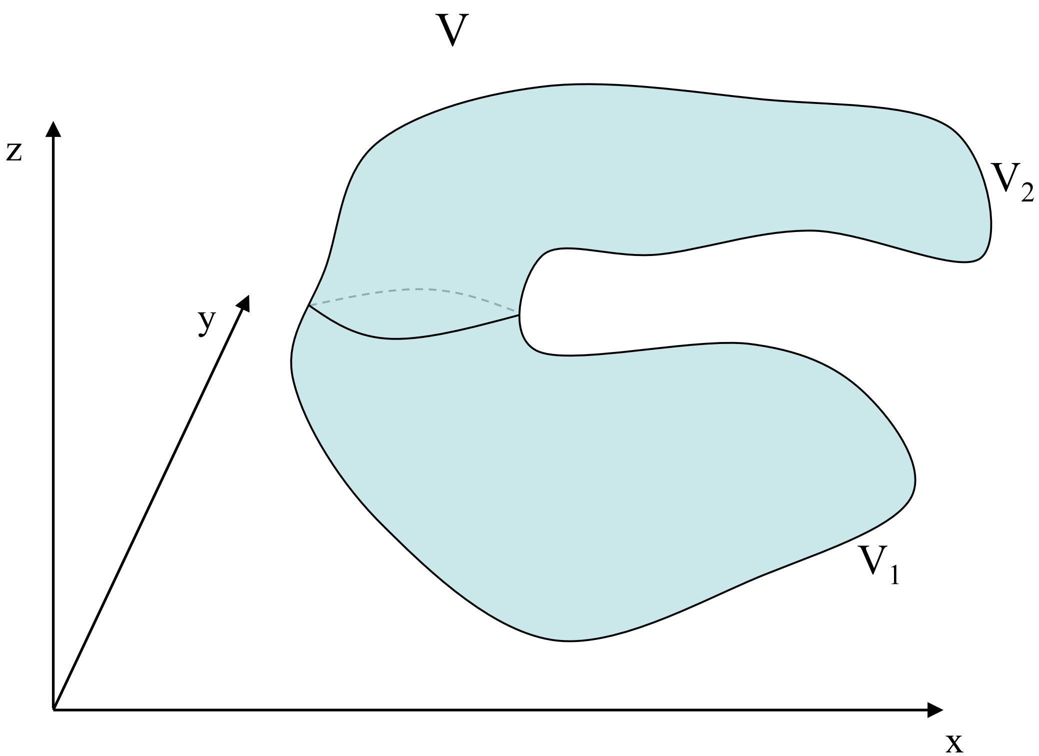
2 - **Divide the surface in two parts, an upper surface and a lower surface and consider an infinitesimal surface element dS_1 on the upper surface.**

4 - **Write the expression that relates $dxdy$ to dS_1 and dS_2.**

THE GAUSS' THEOREM

PROOF

What if we consider a more complicated volume?



We divide the volume V in smaller and "simpler" volumes

$$V = V_1 + V_2 + \ldots = \sum_i V_i$$

$$\iiint_V divA dV = \sum_i \iiint_{V_i} divA dV =$$

$$\sum_i \iint_{S_i} A \cdot dS = \iint_S A \cdot dS$$

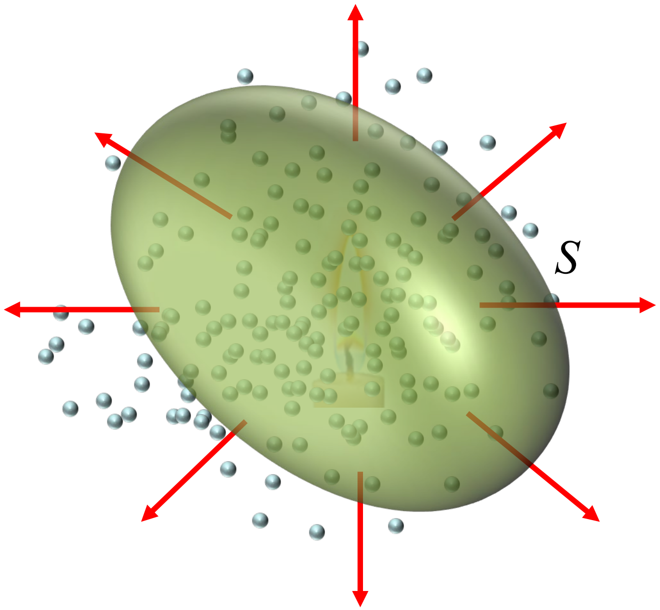
PHYSICAL MEANING

Suppose that $\overline{v}(\overline{r})$ is the velocity field of a gas

Let's apply the Gauss' theorem to a volume V of the gas

$$\iint_S \overline{v} \cdot d\overline{S} = \iiint_V div(\overline{v}) dV$$

This term is the gas volume per second [m^3/s]
that flows outwards *(or inwards)* through a closed surface S



If there are no sinks and no sources:
the amount of gas that flows inwards through
a closed surface S is equal to the amount of gas
that flows outwards.
This implies that the flow $\iint_S \overline{v} \cdot d\overline{S}$ is zero.
Therefore, $div(\overline{v}) = 0$

TARGET PROBLEM

Magnetic monopoles do not exist in nature.

How can this statement be mathematically expressed?

Magnetic monopoles do not exists $\Rightarrow$ the flux of **B** is zero

Let's apply the Gauss' theorem to the magnetic field:

Gauss

$$\iint_S \overline{B} \cdot d\overline{S} = \iiint_V div\overline{B} dV$$

$$\iint_S \overline{B} \cdot d\overline{S} = 0$$

$$div\overline{B} = 0$$

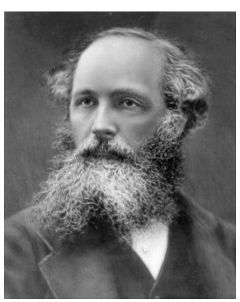
Exercise: apply the Gauss' theorem

to the Gauss' law: $\iint_S \overline{E} \cdot d\overline{S} = \frac{Q}{\varepsilon_0}$

where S is a closed surface and Q the total charge inside S.

Tip: Q is related to the charge density ρ_c via $Q = \int_V \rho_c dV$

One of the four Maxwell's equations



WHICH STATEMENT IS WRONG?

1- The divergence of a vector field is a scalar

2- The divergence is related to the flux

3- The Gauss' theorem translates a surface integral into a volume integral

4- The Gauss' theorem can be applied also to an open surface

VEKTORANALYS

CURL (ROTATIONEN)

and

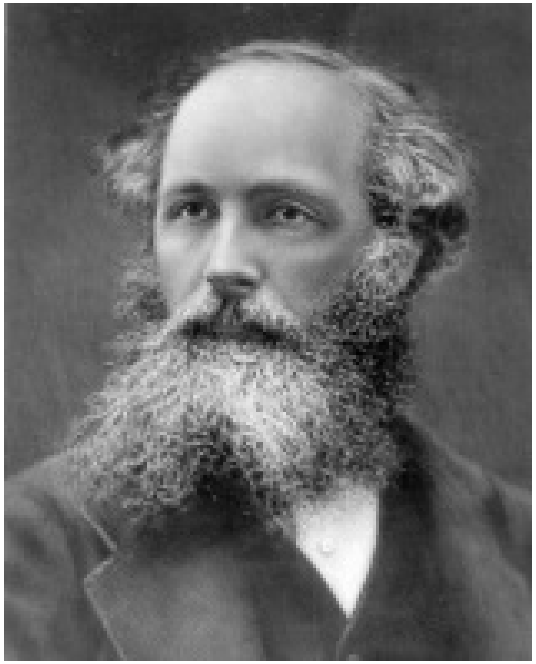
STOKES' THEOREM

THE CURRENT DENSITY

One of the main properties of electromagnetism is that a current density $\overline{j}$ produces a magnetic field $\overline{B}$. The current density and the magnetic field are related via the 4th Maxwell's equation:

$$rot\overline{B} = \mu_0 \overline{j}$$

(in stationary condition)
See the "Teoretisk elektroteknik" course.



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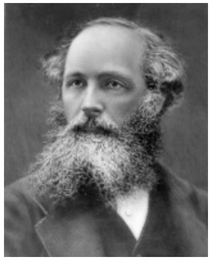
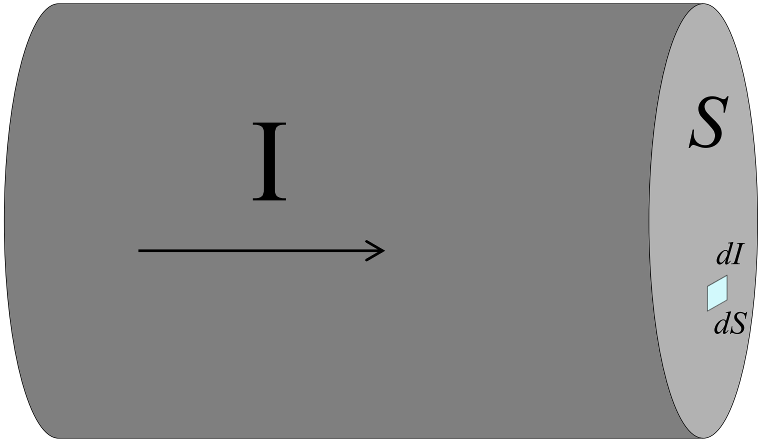


$$rot\overline{B} = \mu_0 \overline{j}$$

(in stationary condition)
See the "Teoretisk elektroteknik" course.

Consider a conductor with an electric current I.
Assume that the section of the conductor perpendicular to I has area S.

- If the electric current is uniform, then the current density $\overline{j}$ is: $\left|\overline{j}\right| = \frac{I}{S}$

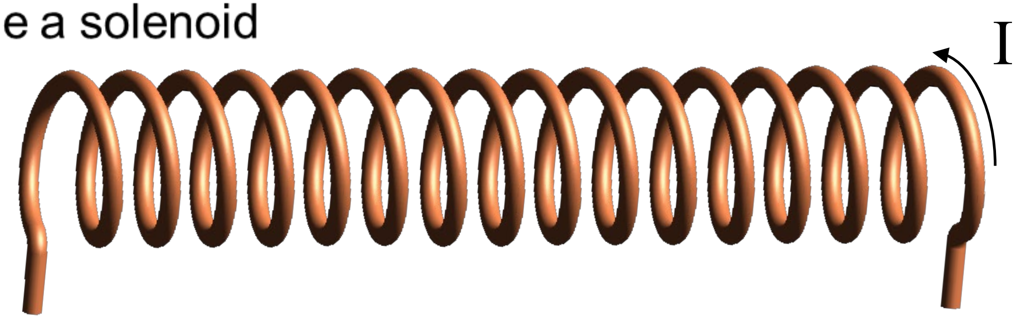


TARGET PROBLEM

$$\begin{cases} rot\overline{B} = \mu_0 \overline{j} \\ I = \iint\limits_S \overline{j} \cdot d\overline{S} \end{cases}$$

(4th Maxwell's equation in stationary conditions)

- Calculate the magnetic field generated by the current I
- Calculate the magnetic field inside a solenoid



We need:

(1) the definition of "curl" (or rotor) of a vector field: $rot\,\overline{A}$

(2) the Stokes' theorem $\oint\limits_L \overline{A} \cdot d\overline{r} = \iint\limits_S rot\overline{A} \cdot d\overline{S}$

THE CURL (ROTATIONEN) $rot\,\overline{A}$

DEFINITION (in a Cartesian coordinate system)

$$rot\overline{A} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \quad \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \quad \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

rot stands for “rotation”
In fact, the curl is a measure of how much the direction of a vector field changes in space, i.e. how much the field “rotates”.

In every point of the space, $rot\,\overline{A}$ is a vector whose length and direction describe the rotation of the field $\overline{A}$.

The direction is the axis of rotation of $\overline{A}$

The magnitude is the magnitude of rotation of $\overline{A}$

THE CURL $rot\,\overline{A}$

PHYSICAL MEANING

Consider the rotation of a rigid body around the z-axis.
The position vector of a point P on located at the distance ρ from the origin is:

$$\overline{r} = (x, y, 0) \quad \text{with} \quad \begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases}$$

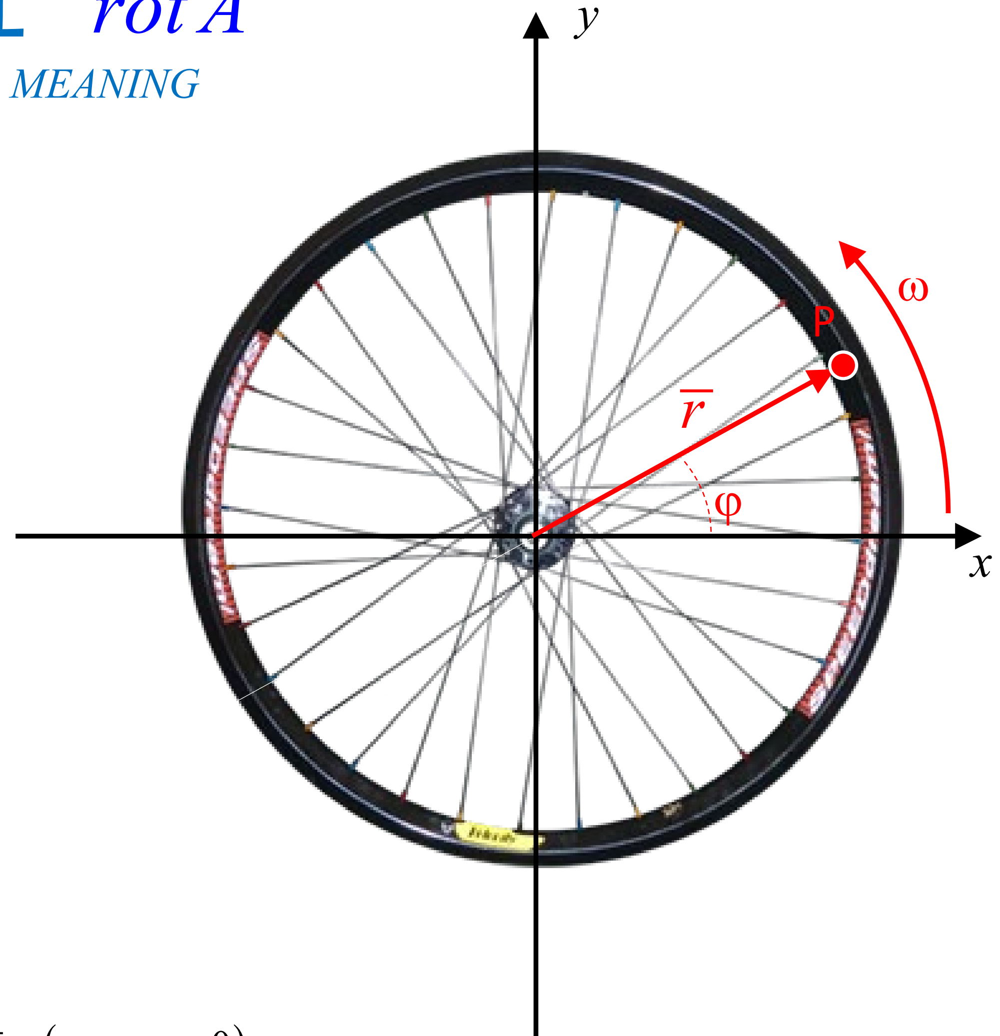
If P rotates with constant angular velocity ω, the angle φ is : $\varphi(t)=\omega t$.

$$\begin{cases} x(t) = \rho\cos(\omega t) \\ y(t) = \rho\sin(\omega t) \end{cases}$$

The velocity of the point P is:

$$\left.\begin{aligned} v_x(t) &= \frac{dx(t)}{dt} = -\rho\omega\sin\omega t = -\omega y(t) \\ v_y(t) &= \frac{dy(t)}{dt} = \rho\omega\cos\omega t = \omega x(t) \end{aligned}\right\} \Rightarrow \overline{v} = \left(-\omega y, \omega x, 0\right)$$

$$\overline{\omega} = \omega\hat{e}_z$$

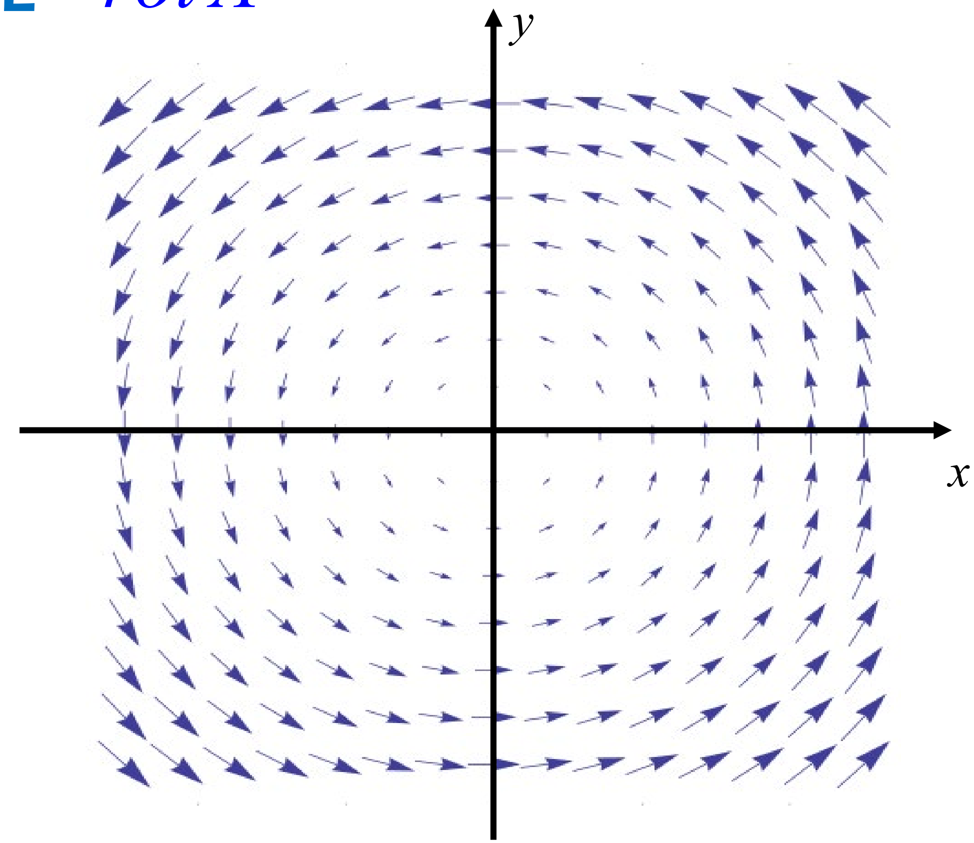


THE CURL $rot\,\overline{A}$

EXAMPLE

$$\overline{v}(x,y,z)=(-\omega y,\omega x,0)$$

Exercise: calculate the curl of $\vec{v}$



Direction: the direction is the axis of rotation, i.e. perpendicular to the plane of the figure
The sign (negative, in this case) is determined by the right-hand rule

Magnitude: the amount of rotation
In this example, it is constant and independent of the position, i.e. the amount of rotation is the same at any point.

THE CURL $rot\,\overline{A}$

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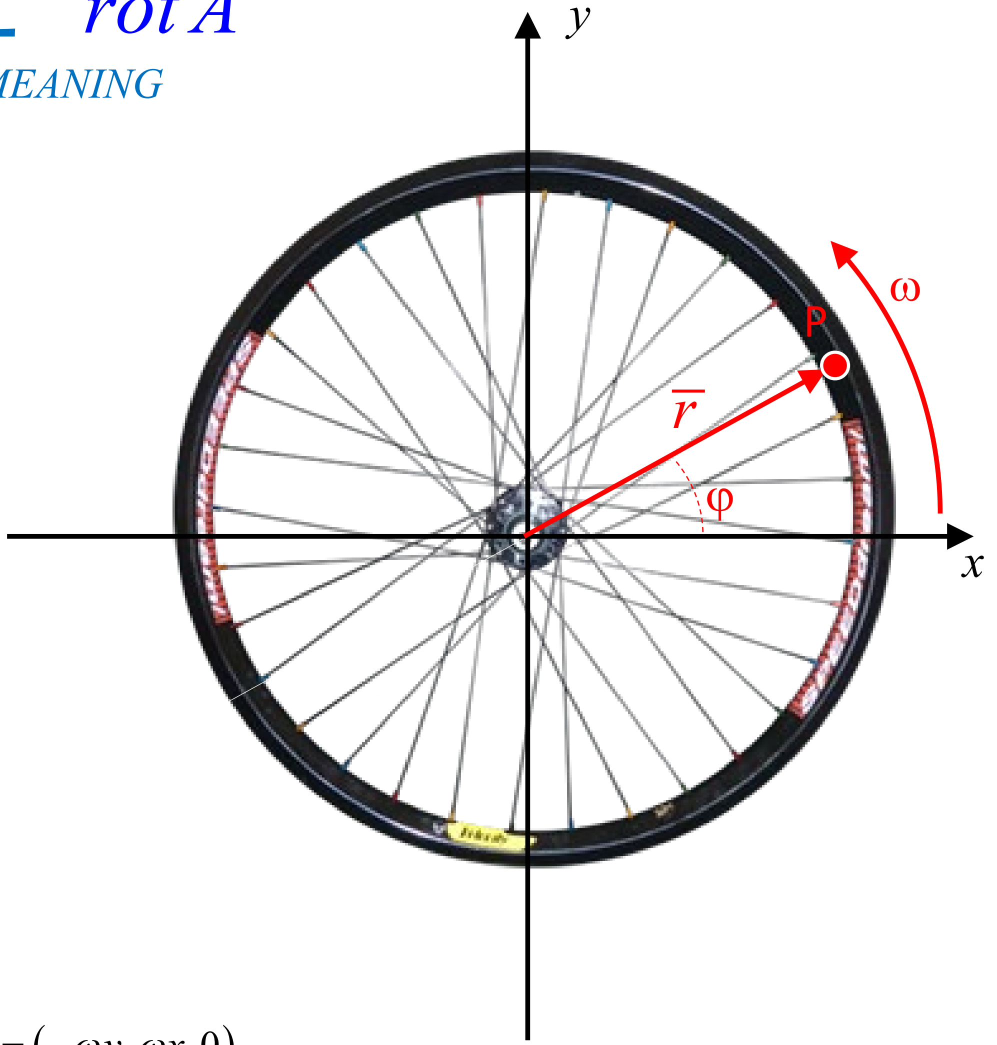
$$\overline{\omega} = \omega\hat{e}_z$$



Therefore $\quad rot\,\overline{v} = (0, 0, 2\omega) \quad \Rightarrow \quad \overline{\omega} = \frac{1}{2} rot\,\overline{v}$

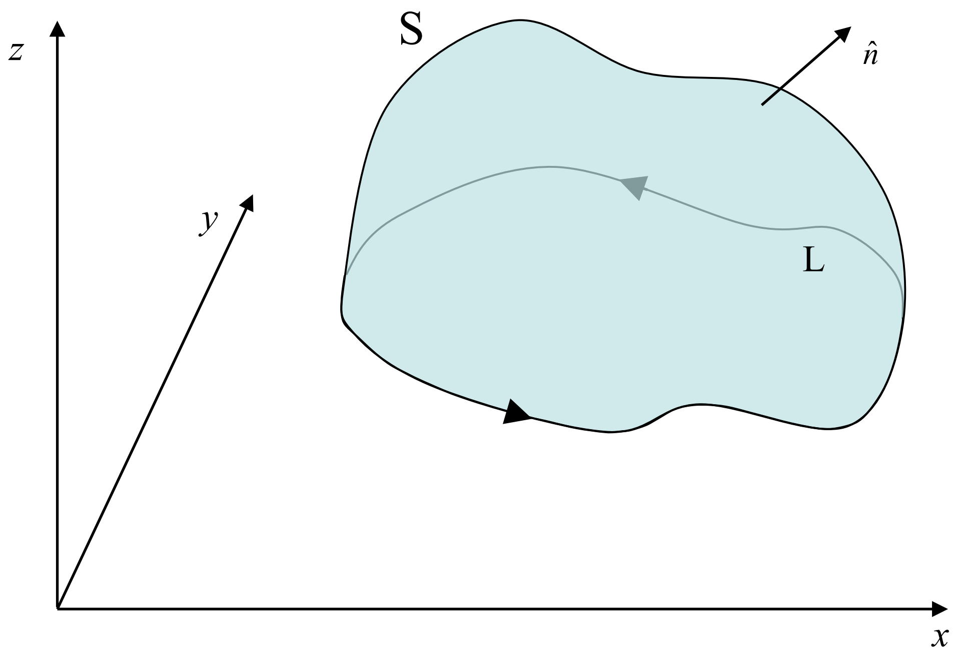
THE STOKES' THEOREM

$$\oint_L \overline{A} \cdot d\overline{r} = \iint_S rot\overline{A} \cdot d\overline{S}$$

where $\bar{A}$ is a vector field, ***L* is a closed curve** and S is a surface whose boundary is defined by L.

L must be positively oriented relatively to S. Both L and S must be "stykvis glatta".

$\bar{A}$ must be continuously differentiable on S.



THE STOKES' THEOREM

PROOF

Five steps:

1. We divide S in "many" "smaller" *(infinitesimal)* surfaces:

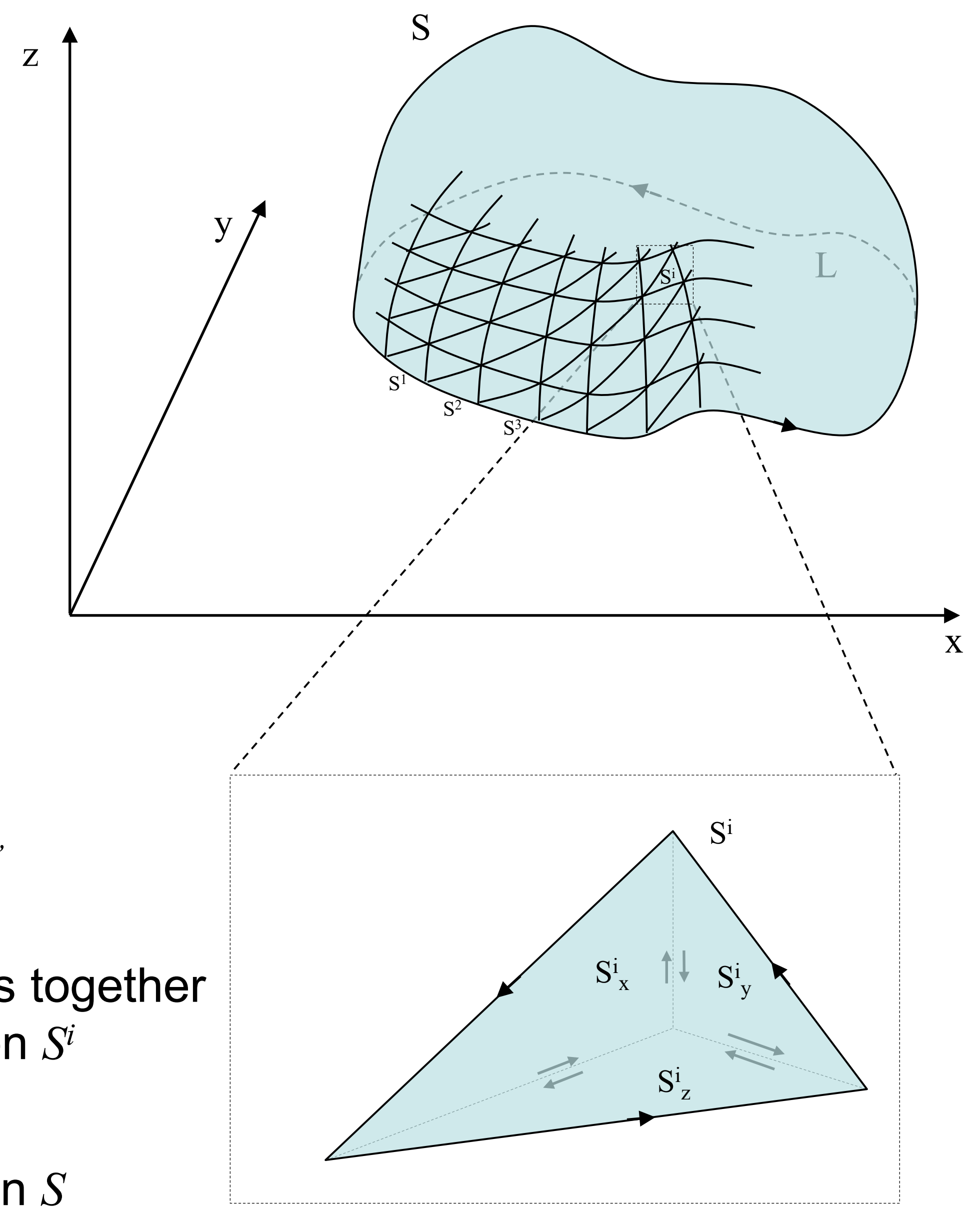
$$S = \sum_i S^i$$

2. We project S^i on:
   - the xy-plane S^i_z
   - the yz-plane S^i_x
   - the xz-plane S^i_y

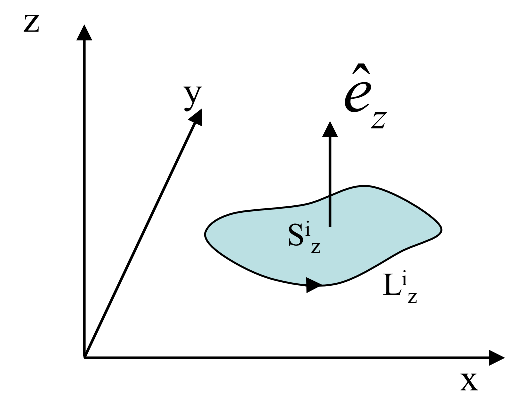
3. We prove the Stokes' theorem on S^i_z, *(the only difficult part)*

4. We add the results for the projections together and we obtain the Stokes' theorem on S^i

5. We add the results for S^i together and we obtain the Stokes' theorem on S



THE STOKES' THEOREM



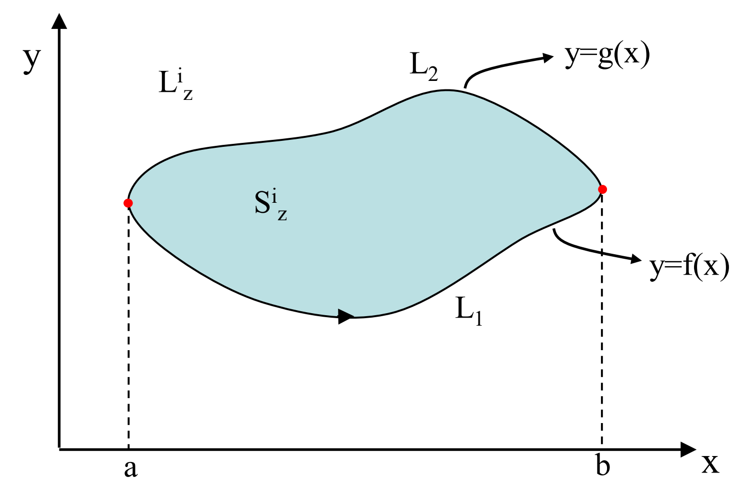
PROOF
Let's consider the plane surface S^i_z
located in the xy-plane (i.e. z=constant=z_0)
with boundary defined by the curve L^i_z

Let's calculate $\oint_{L^i_z} \overline{A} \cdot d\overline{r}$

$$\oint_{L^i_z} \overline{A} \cdot d\overline{r} = \oint_{L^i_z} \underbrace{A_x(x,y,z_0)dx}_{\textbf{Term 1}} + \underbrace{A_y(x,y,z_0)dy}_{\textbf{Term 2}} + \underbrace{A_z(x,y,z_0)dz}_{\textbf{Term 3}}$$

Term 3 =0 *(z=constant!* $\Rightarrow$ *dz=0)*

Term 1



$$\oint_{L^i_z} A_x(x,y,z_0)dx = \int_{L_1+L_2} A_x(x,y,z_0)dx =$$

$$\int_{L_1} A_x(x,y,z_0)dx + \int_{L_2} A_x(x,y,z_0)dx =$$

$$\int_a^b A_x(x,f(x),z_0)dx + \int_b^a A_x(x,g(x),z_0)dx =$$

THE STOKES' THEOREM

PROOF

$$= \int_a^b A_x(x, f(x), z_0)dx - \int_a^b A_x(x, g(x), z_0)dx = \int_a^b \left[A_x(x, f(x), z_0) - A_x(x, g(x), z_0) \right] dx =$$

$$\int_a^b \int_{g(x)}^{f(x)} \frac{\partial A_x(x, y, z_0)}{\partial y} dxdy = -\int_a^b \int_{f(x)}^{g(x)} \frac{\partial A_x}{\partial y} dxdy = -\iint_{S^i{}_z} \frac{\partial A_x}{\partial y} dxdy$$

Therefore we get:

Term 1 $$\oint_{L^i{}_z} A_x(x, y, z_0)dx = -\iint_{S^i{}_z} \frac{\partial A_x}{\partial y} dxdy$$

In a similar way:

Term 2 $$\oint_{L^i{}_z} A_y(x, y, z_0)dx = \iint_{S^i{}_z} \frac{\partial A_y}{\partial x} dxdy$$

It is the z-component of $rot\bar{A}$!!

Adding **Term 1**, **Term 2** and **Term 3**:

$$\oint_{L^i{}_z} \bar{A} \cdot d\bar{r} = \iint_{S^i{}_z} \left(\overbrace{\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}} \right) dxdy$$

THE STOKES' THEOREM

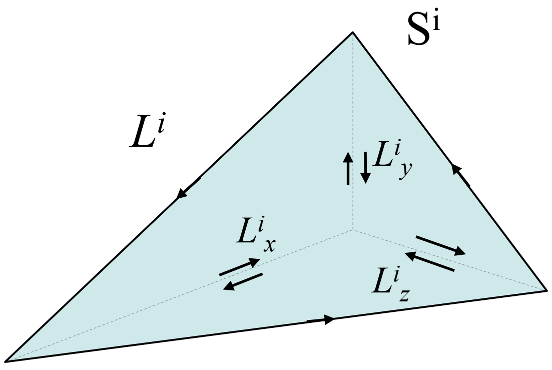
So can rewrite it as:

$$\oint_{L^i{}_z} \overline{A}\cdot d\overline{r} = \iint_{S^i{}_z} (rot\overline{A})_z\, dxdy \underset{\underbrace{dxdy = \hat{e}_z \cdot \hat{n} dS = \hat{e}_z \cdot d\overline{S}}}{=} \iint_{S^i} (rot\overline{A})_z \hat{e}_z \cdot d\overline{S}$$

In a similar way we have:

$$\oint_{L^i{}_y} \overline{A}\cdot d\overline{r} = \iint_{S^i} (rot\overline{A})_y \hat{e}_y \cdot d\overline{S}$$

$$\oint_{L^i{}_x} \overline{A}\cdot d\overline{r} = \iint_{S^i} (rot\overline{A})_x \hat{e}_x \cdot d\overline{S}$$



Now let's add everything together:

$$\oint_{L^i{}_x} \overline{A}\cdot d\overline{r} + \oint_{L^i{}_y} \overline{A}\cdot d\overline{r} + \oint_{L^i{}_z} \overline{A}\cdot d\overline{r} = \oint_{L^i} \overline{A}\cdot d\overline{r}$$

THE STOKES' THEOREM

So can rewrite it as:

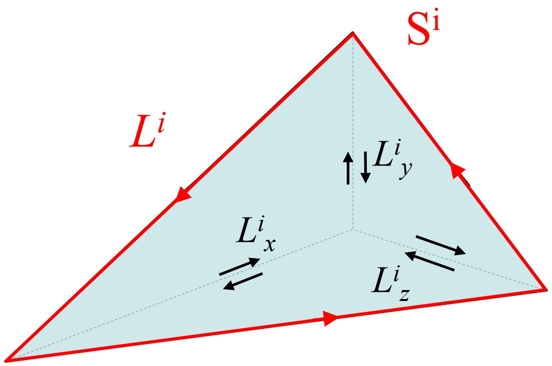
$$\oint_{L^i{}_z} \overline{A}\cdot d\overline{r} = \iint_{S^i{}_z} (rot\overline{A})_z\,dxdy = \iint_{S^i} (rot\overline{A})_z\,\hat{e}_z\cdot d\overline{S}$$

$$dxdy = \hat{e}_z\cdot\hat{n}dS = \hat{e}_z\cdot d\overline{S}$$

In a similar way we have:

$$\oint_{L^i{}_y} \overline{A}\cdot d\overline{r} = \iint_{S^i} (rot\overline{A})_y\,\hat{e}_y\cdot d\overline{S}$$

$$\oint_{L^i{}_x} \overline{A}\cdot d\overline{r} = \iint_{S^i} (rot\overline{A})_x\,\hat{e}_x\cdot d\overline{S}$$



Now let's add everything together:

$$\oint_{L^i{}_x} \overline{A}\cdot d\overline{r} + \oint_{L^i{}_y} \overline{A}\cdot d\overline{r} + \oint_{L^i{}_z} \overline{A}\cdot d\overline{r} = \oint_{L^i} \overline{A}\cdot d\overline{r}$$

$$\iint_{S^i} (rot\overline{A})_x\,\hat{e}_x\cdot d\overline{S} + \iint_{S^i} (rot\overline{A})_y\,\hat{e}_y\cdot d\overline{S} + \iint_{S^i} (rot\overline{A})_z\,\hat{e}_z\cdot d\overline{S} = \iint_{S^i} rot\overline{A}\cdot d\overline{S}$$

THE STOKES' THEOREM

PROOF

$$\oint_{L^i} \overline{A}\cdot d\overline{r} = \iint_{S^i} rot\overline{A}\cdot d\overline{S}$$

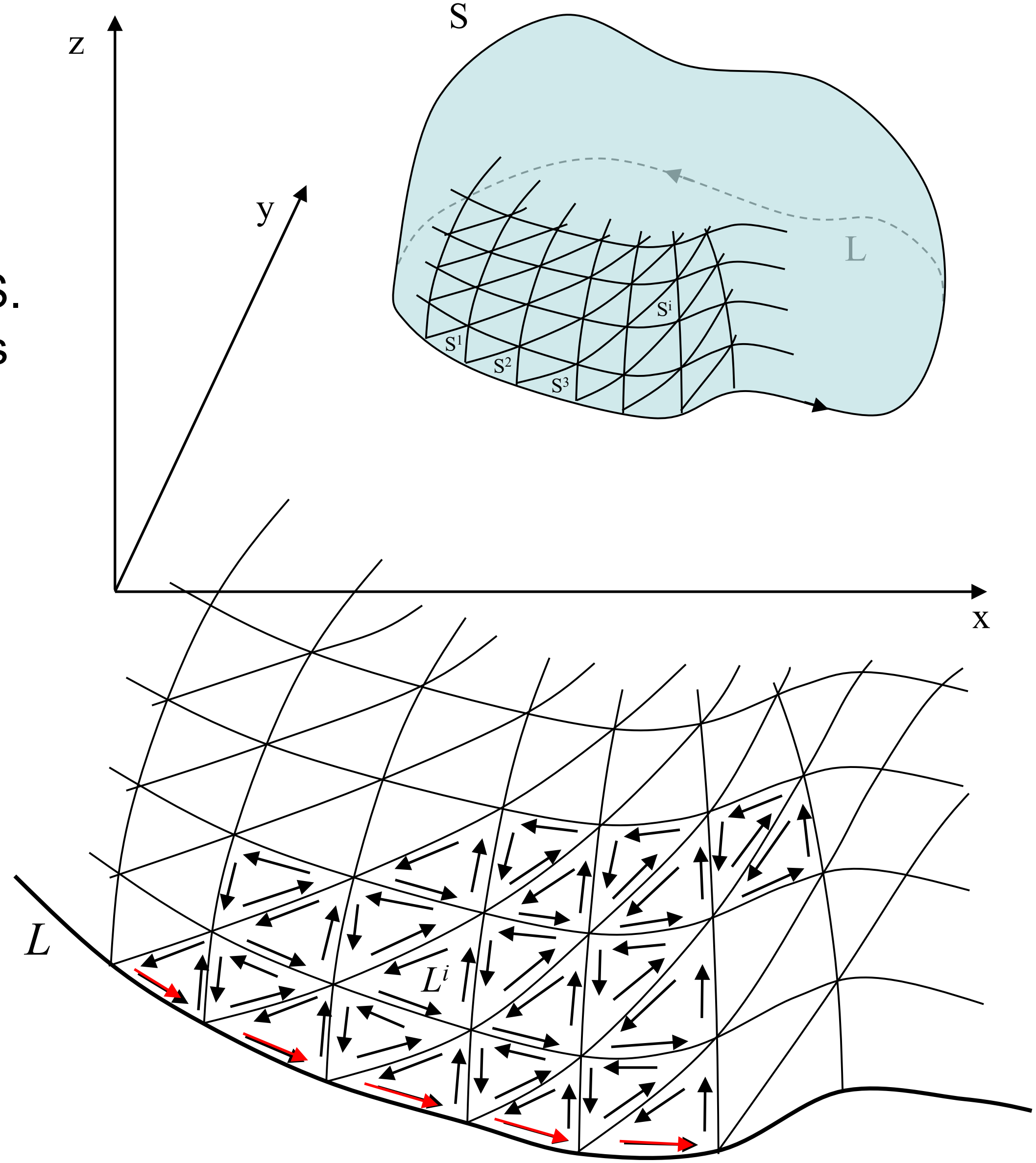
But we are interested in the whole S. So we add these small contributions altogether:

$$\underbrace{\sum_i \iint_{S^i} rot\overline{A}\cdot d\overline{S}}_{} = \iint_{S} rot\overline{A}\cdot d\overline{S}$$

$$=$$

$$\overbrace{\sum_i \oint_{L^i} \overline{A}\cdot d\overline{r}}^{} = \oint_{L} \overline{A}\cdot d\overline{r}$$

$$\boxed{\oint_{L} \overline{A}\cdot d\overline{r} = \iint_{S} rot\,\overline{A}\cdot d\,\overline{S}}$$



Rearrange in logic order the steps to prove the Stokes' theorem

- Prove the Stokes' theorem on $S^i{}_z$:
  - (a) - Write the line integral of the vector field along the boundary of $S^i{}_z$ and split the integral into three terms.
  - (b) - Consider only the integral in dx and prove that $\int_{L^i{}_z} A_x(x,y,z_0)dx = -\iint\limits_{S^i{}_z} \frac{\partial A_x}{\partial y} dxdy$
  - (c) -Repeat the same for the integral in dy and dz
  - (d) -Add the three integrals in dx, dy and dz to obtain $\int_{L^i{}_z} \bar{A} \cdot d\bar{r} = \iint\limits_{S^i{}_z} (rot\bar{A})_z dxdy$
  - (e) -Rewrite $dxdy$ to obtain $\int_{L^i{}_z} \bar{A} \cdot d\bar{r} = \iint\limits_{S^i} (rot\bar{A})_z \hat{e}_z \cdot d\bar{S}$

- Prove the Stokes' theorem on S: add together all the expressions obtained for S^i

- Consider a closed path and a surface whose boundary is defined by the closed path.

- Prove the Stokes' theorem on S^i:
  - (a) -Repeat the same procedure for $S^i{}_x$ and $S^i{}_y$
  - (b) - add together the expressions for the integrals in $S^i{}_x$ to $S^i{}_y$ and $S^i{}_z$ obtaining: $\int_{L^i} \bar{A} \cdot d\bar{r} = \iint\limits_{S^i} rot\bar{A} \cdot d\bar{S}$

- Divide the surface in small areas S^i and consider the projection of S^i on the xy, yz, xz planes

Rearrange in logic order the steps to prove the Stokes' theorem

3 - Prove the Stokes' theorem on ${S^i}_z$:

3.(a) - Write the line integral of the vector field along the boundary of ${S^i}_z$ and split the integral into three terms.

3.(b) - Consider only the integral in dx and prove that $\int_{{L^i}_z} A_x(x,y,z_0)dx = -\iint_{{S^i}_z} \frac{\partial A_x}{\partial y} dxdy$

3.(c) -Repeat the same for the integral in dy and dz

3.(d) -Add the three integrals in dx, dy and dz to obtain $\int_{{L^i}_z} \overline{A} \cdot d\overline{r} = \iint_{{S^i}_z} (rot\overline{A})_z dxdy$

3.(e) -Rewrite $dxdy$ to obtain $\int_{{L^i}_z} \overline{A} \cdot d\overline{r} = \iint_{S^i} (rot\overline{A})_z \hat{e}_z \cdot d\overline{S}$

5 - Prove the Stokes' theorem on S: add together all the expressions obtained for S^i

1 - Consider a closed path and a surface whose boundary is defined by the closed path.

4 - Prove the Stokes' theorem on S^i:

4.(a) -Repeat the same procedure for ${S^i}_x$ and ${S^i}_y$

4.(b) - add together the expressions for the integrals in ${S^i}_x$ to ${S^i}_y$ and ${S^i}_z$ obtaining: $\int_{L^i} \overline{A} \cdot d\overline{r} = \iint_{S^i} rot\overline{A} \cdot d\overline{S}$

2 - Divide the surface in small areas S^i and consider the projection of S^i on the xy, yz, xz planes

TARGET PROBLEM

$$\begin{cases} rot\overline{B} = \mu_0 \overline{j} \\ I = \iint\limits_S \overline{j} \cdot d\overline{S} \end{cases}$$

(4th Maxwell's equation in stationary conditions)

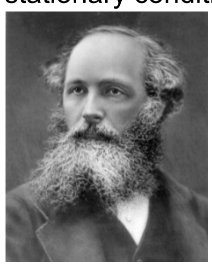
Using the Stokes' theorem we can find an expression that relates directly $\bar{B}$ with I:

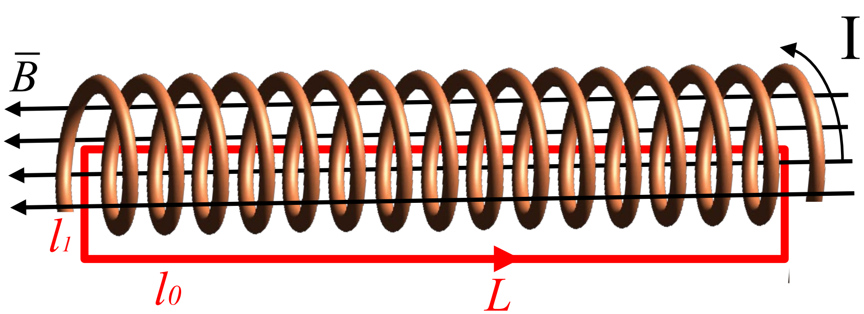
$$\oint_L \overline{B} \cdot d\overline{r} = \iint\limits_S rot\,\overline{B} \cdot d\overline{S} = \iint\limits_S \mu_0 \overline{j} \cdot d\overline{S} = \mu_0 \iint\limits_S \overline{j} \cdot d\overline{S} = \mu_0 I \quad \Rightarrow \quad \boxed{\oint_L \overline{B} \cdot d\overline{r} = \mu_0 I}$$

Stokes' theorem

4th Maxwell's equation (in stationary condition)

Ampere's law





Using the Ampere's law:

$$\left.\begin{aligned} &\int_L \overline{B} \cdot d\overline{r} = \mu_0 NI \\ &\int_L \overline{B} \cdot d\overline{r} = \left|\overline{B}\right| l_0 + 0 l_0 + 0 l_1 + 0 l_1 = \left|\overline{B}\right| l_0 \end{aligned}\right\} \Rightarrow \left|\overline{B}\right| = \frac{\mu_0 NI}{l_0}$$

THE GREEN FORMULA IN THE PLANE

THEOREM *(9.2 in the textbook)*

$$\boxed{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint_L \left(Pdx + Qdy \right)}$$

PROOF

We can start from Stokes' theorem $\oint_L \overline{A} \cdot d\overline{r} = \iint_S rot\,\overline{A} \cdot d\overline{S}$

$$\oint_L \overline{A} \cdot d\overline{r} = \oint \left(A_x dx + A_y dy + A_z dz \right) = \oint \left(A_x dx + A_y dy \right)$$

But we are in a plane,
so we can assume A=(A_x,A_y,0)

$$\iint_S rot\,\overline{A} \cdot d\overline{S} = \iint_S \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \underbrace{\hat{e}_z \cdot \hat{e}_z}_{=1} dxdy$$

$$rot\,\overline{A} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & 0 \end{vmatrix}$$

$$\iint_D \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dxdy = \oint_L \left(A_x dx + A_y dy \right)$$

which is the Green formula
for $P=A_x$ and $Q=A_y$

CURL FREE FIELD AND SCALAR POTENTIAL

(virvelfria fält och skalär potential)

DEFINITION: A vector field $\overline{A}$ is "curl free" if $rot\ \overline{A} = 0$ *Sometimes called "irrotational"*

If $\bar{A}$ is continuously derivable and defined in a simply connected domain, then:

THEOREM *(9.3 in the textbook)* $\boxed{rot\ \overline{A} = 0 \Leftrightarrow \overline{A} \text{ has a scalar potential } \phi,\ \overline{A} = grad\,\phi}$

PROOF

(1) $rot\ \overline{A} = 0$

$$\oint_L \overline{A} \cdot d\overline{r} = \iint_S rot\,\overline{A} \cdot d\,\overline{S} = 0$$

So, if the curl is zero, also the circulation is zero $\Rightarrow$ then the field is conservative and has a scalar potential. *See theorems 6.3 and 6.4 in the textbook or the slides of week 2.*

(2) $\overline{A} = grad\,\phi$

$$rot\ \overline{A} = rot\ grad\,\phi = rot\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} = \left(\frac{\partial}{\partial y}\frac{\partial\phi}{\partial z} - \frac{\partial}{\partial z}\frac{\partial\phi}{\partial y}, \ldots, \ldots\right) = (0,0,0)$$

SOLENOIDAL FIELD AND VECTOR POTENTIAL

DEFINITION: A vector field $\overline{B}$ is called **solenoidal** if $div\overline{B} = 0$

DEFINITION: $\overline{A}$ is a vector potential of the vector field $\overline{B}$ if $\overline{B} = rot\overline{A}$

THEOREM *(9.4 in the book)*

$\overline{B}$ has a vector potential $\overline{A}$ if and only if $\overline{B}$ has divergence zero:

$$\overline{B} = rot\overline{A} \Leftrightarrow div\overline{B} = 0$$

PROOF

(1) $\overline{B}$ has a vector potential $\Rightarrow \overline{B} = rot\overline{A} \Rightarrow div\overline{B} = div\left(rot\overline{A}\right) = 0$

(2) $div\overline{B} = 0$

Let's try to find a solution $\overline{A}$ to the equation $\overline{B} = rot\overline{A}$

We start looking for a particular solution A* of this kind:

$$\overline{A}^* = \left(A_x^*(x,y,z),\, A_y^*(x,y,z),\, 0\right)$$

PROOF

Assuming $\overline{B}=rot\overline{A}$ we obtain:

$$-\frac{\partial A_y^*}{\partial z}=B_x \quad \Rightarrow \quad A_y^*(x,y,z)=-\int_{z_0}^{z}B_x(x,y,z)dz+F(x,y)$$

$$\frac{\partial A_x^*}{\partial z}=B_y \quad \Rightarrow \quad A_x^*(x,y,z)=\int_{z_0}^{z}B_y(x,y,z)dz+G(x,y)$$

$$\frac{\partial A_y^*}{\partial x}-\frac{\partial A_x^*}{\partial y}=B_z \quad \Rightarrow \quad -\int_{z_0}^{z}\frac{\partial B_x}{\partial x}dz+\frac{\partial F}{\partial x}-\int_{z_0}^{z}\frac{\partial B_y}{\partial y}dz-\frac{\partial G}{\partial y}=B_z$$

$$\downarrow$$

But $div\overline{B}=0 \Rightarrow \dfrac{\partial B_x}{\partial x}+\dfrac{\partial B_y}{\partial y}=-\dfrac{\partial B_z}{\partial z} \longrightarrow \underbrace{\int_{z_0}^{z}\frac{\partial B_z}{\partial z}dz}_{=B_z(x,y,z)-B_z(x,y,z_0)}+\dfrac{\partial F}{\partial x}-\dfrac{\partial G}{\partial y}=B_z \Rightarrow \dfrac{\partial F}{\partial x}-\dfrac{\partial G}{\partial y}=B_z(x,y,z_0)$

A solution to this equation is:
$$\begin{cases} F(x,y)=0 \\ G(x,y)=-\int_{y_0}^{y}B_z(x,y,z_0)dy \end{cases}$$

$$\overline{A}^*=\left(\int_{z_0}^{z}B_y(x,y,z)dz-\int_{y_0}^{y}B_z(x,y,z_0)dy,\quad -\int_{z_0}^{z}B_x(x,y,z)dz,\quad 0\right)$$

The general solution can be found using $\overline{B}=rot\overline{A}$:

$$rot\left(\overline{A}-\overline{A}^*\right)=\overline{B}-\overline{B}=0 \quad \Rightarrow \quad \overline{A}-\overline{A}^*=grad\psi \quad \Rightarrow \quad \overline{A}=\overline{A}^*+grad\psi$$

WHICH STATEMENT IS WRONG?

1- The curl of a vector field is a scalar

2- The curl is related to the line integral of a field along a closed curve

3- Stokes’ theorem translates a line integral into a surface integral

4- The Stokes’ theorem can be applied only to a closed curve