

VEKTORANALYS

HT 2022

CELTE / CENMI

ED1110

GAUSS' THEOREM and STOKES' THEOREM: ÖVNINGAR

Kursvecka 3

Kapitel 8-9 (*Vektoranalys*, 1:e uppl, Frassinetti/Scheffel)



PROBLEM 1

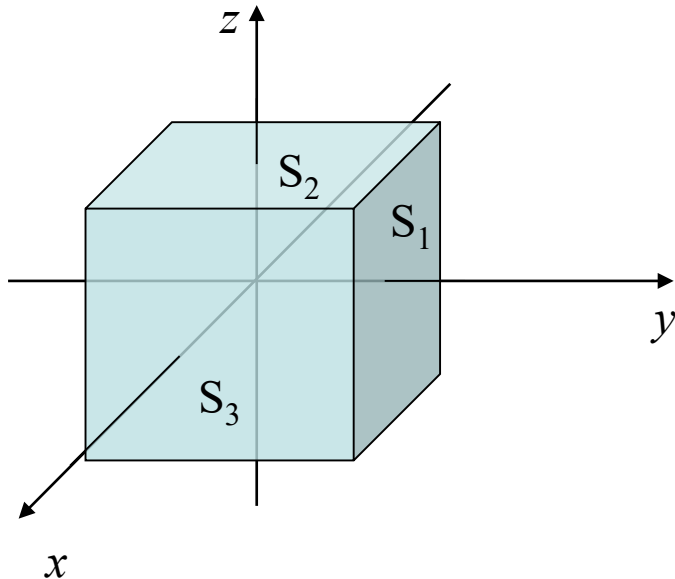
Calculate: $\iint_S \vec{A} \cdot d\vec{S}$ where the vector field is: $\vec{A} = (x, y, z)$

and S is a cube (length 2 each side) centred in the origin.

The normal to the surface points out from the cube

- (a) In a direct way (using the parameterization of the surface).
- (b) Using Gauss' theorem

SOLUTION



$$(a) \quad \iint_S \vec{A} \cdot d\vec{S} = \sum_i \iint_{S_i} \vec{A} \cdot d\vec{S}$$

Let's start with S_1

1- parameterization of S_1 :

$$\left. \begin{array}{l} y = 1 \\ |x| < 1 \\ |z| < 1 \end{array} \right\} \Rightarrow \vec{r}(u, v) = (u, 1, v) \quad \begin{array}{l} u: -1 \rightarrow +1 \\ v: -1 \rightarrow +1 \end{array}$$

2- Integral calculation:

$$\int_{S_1} \bar{A} \cdot d\bar{S} = \int_u \int_v \bar{A}(\bar{r}(u, v)) \cdot \left(\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} \right) dudv$$

$$\left. \begin{array}{l} \frac{\partial \bar{r}}{\partial u} = (1, 0, 0) \\ \frac{\partial \bar{r}}{\partial v} = (0, 0, 1) \end{array} \right\} \Rightarrow \left(\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} \right) = (0, -1, 0) \quad \text{The normal is pointing inside the volume} \rightarrow \text{remember to change the sign of the flow in the final answer.}$$

$$\int_{-S_1} \bar{A} \cdot d\bar{S} = \int_{-1}^1 \int_{-1}^1 (u, 1, v) \cdot (0, -1, 0) dudv = \int_{-1}^1 \int_{-1}^1 dudv = -4$$

Due to the symmetry of the problem we have: $\int_{-S_i} \bar{A} \cdot d\bar{S} = -4$

$$\Rightarrow \iint_{-S} \bar{A} \cdot d\bar{S} = \sum_i \iint_{-S_i} \bar{A} \cdot d\bar{S} = 6 \cdot (-4) = -24$$

We need to change the sign of the flow: $\Rightarrow \iint_S \bar{A} \cdot d\bar{S} = 24$

(b) S is a closed surface and $\bar{A}$ is continuous and differentiable $\Rightarrow$ we can apply Gauss' theorem

$$\left. \begin{array}{l} \iint_S \bar{A} \cdot d\bar{S} = \iiint_V \text{div} \bar{A} dV \\ \text{div} \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 1 + 1 + 1 = 3 \end{array} \right\} \Rightarrow \iint_S \bar{A} \cdot d\bar{S} = \iiint_V 3 dV = 3V = 3 \cdot 2^3 = 24$$

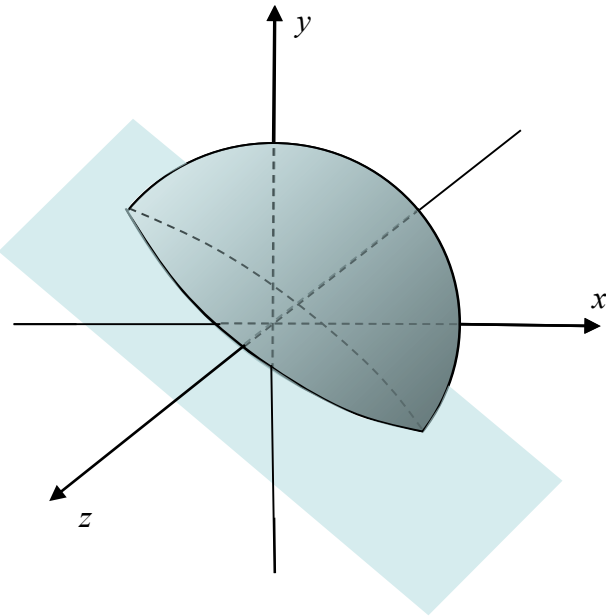
PROBLEM 2

Calculate $\iint_S \vec{A} \cdot d\vec{S}$ using Gauss' theorem

where the vector field is: $\vec{A} = (x^3, y^3, z^3)$

with S an open surface defined as the half sphere:
$$\begin{cases} x^2 + y^2 + z^2 = R^2 \\ x + y > 0 \end{cases}$$

SOLUTION



But S is NOT a closed surface!
So we can consider the surface

$$S_{tot} = S + S_{plane}$$

$$\oiint_{S_{tot}} \vec{A} \cdot d\vec{S} = \iiint_V \text{div} \vec{A} dV$$

$$\iint_S \vec{A} \cdot d\vec{S} = \oiint_{S_{tot}} \vec{A} \cdot d\vec{S} - \iint_{S_{plane}} \vec{A} \cdot d\vec{S}$$

$$\iint_S \bar{A} \cdot d\bar{S} = \iiint_V \text{div} \bar{A} dV - \iint_{S_{\text{plane}}} \bar{A} \cdot d\bar{S}$$

So we have transformed a surface integral
into a volume integral minus another surface integral
What is the advantage?
They can be calculated much easier!!

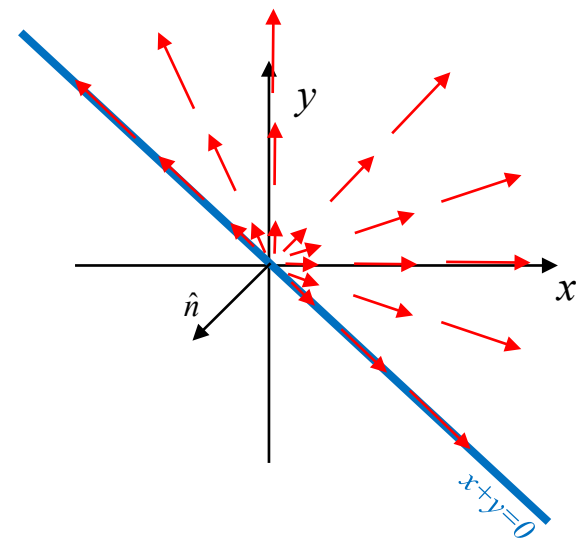
Let's consider the second integral.

$$S_{\text{plane}} \quad \begin{cases} x^2 + y^2 + z^2 \leq R^2 \\ x + y = 0 \end{cases}$$

$$\text{On } S_{\text{plane}} \quad x = -y$$

$$\bar{A} = (x^3, y^3, z^3) \quad \Rightarrow \quad \bar{A} = (x^3, -x^3, z^3)$$

$$\text{On } S_{\text{plane}} \quad \text{the vector is perpendicular to } \hat{n} \quad \Rightarrow \quad \iint_{S_{\text{plane}}} \bar{A} \cdot d\bar{S} = 0$$



Let's consider the first integral.

$$\iiint_V \text{div} \bar{A} dV \quad \text{with} \quad \text{div} \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 3x^2 + 3y^2 + 3z^2 = 3r^2$$

Spherical coordinates



$$\text{since } \operatorname{div} \bar{A} = 3r^2 \quad \begin{array}{c} \text{due to symmetry} \\ \Rightarrow \end{array} \quad \iiint_V \operatorname{div} \bar{A} dV = \frac{1}{2} \iiint_{V_{\text{sphere}}} \operatorname{div} \bar{A} dV$$

$$\iiint_{V_{\text{sphere}}} \operatorname{div} \bar{A} dV = \int_0^{2\pi} \int_0^\pi \int_0^R 3r^2 \cdot r^2 \sin \theta dr d\theta d\varphi = 3 \int_0^{2\pi} d\varphi \int_0^\pi \sin \theta d\theta \int_0^R r^4 dr = \frac{12\pi R^5}{5}$$

$$\Rightarrow \quad \iiint_V \operatorname{div} \bar{A} dV = \frac{1}{2} \iiint_{V_{\text{sphere}}} \operatorname{div} \bar{A} dV = \frac{6\pi R^5}{5}$$

$$\iint_S \bar{A} \cdot d\bar{S} = \iiint_V \operatorname{div} \bar{A} dV - \iint_{S_{\text{plane}}} \bar{A} \cdot d\bar{S} = \frac{6\pi R^5}{5}$$

PROBLEM 3 (example 9.2 in the book)

Med hjälp av Stokes sats, beräkna linjeintegralen av vektorfältet

$$\bar{A}(\bar{r}) = x\hat{e}_y$$

längs cirkeln C med radie 1, centrerad i punkten $(-1,1,0)$. Cirkeln ligger i xy -planet och har en kontur som är orienterad moturs, sett från positiva z -axeln.

SOLUTION

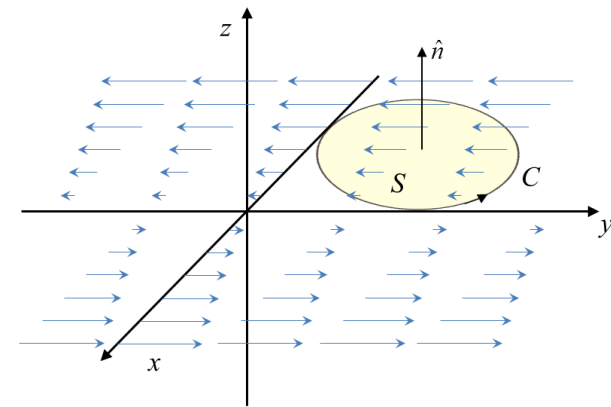
$$\oint_C \bar{A} \cdot d\bar{r} = \iint_S \text{rot} \bar{A} \cdot d\bar{S}$$

$$\text{rot} \bar{A} = \text{rot}(x\hat{e}_y) = \left(-\frac{\partial x}{\partial z}\right)\hat{e}_x + \left(\frac{\partial x}{\partial x}\right)\hat{e}_z = \hat{e}_z$$

For S we can choose any surface that has the boundary on C . The simplest choice is the surface corresponding to the area that lies in xy -plane inside C .

According to the right hand rule, the normal will be in the $\hat{e}_z$ direction

$$\oint_C \bar{A} \cdot d\bar{r} = \iint_S \text{rot} \bar{A} \cdot d\bar{S} = \iint_S \hat{e}_z \cdot d\bar{S} = \iint_S \hat{e}_z \cdot \hat{e}_z dS = \iint_S dS = \pi$$



PROBLEM 4

Calculate the line integral of the vector field: $\vec{A} = (y + 2x, x^2 + z, y)$

along the closed curve: $\begin{cases} \vec{r}(u) = (\cos u, \sin u, f(u)) & \text{with } f(u) = f(u + 2\pi) \\ u: 0 \rightarrow 2\pi \end{cases}$

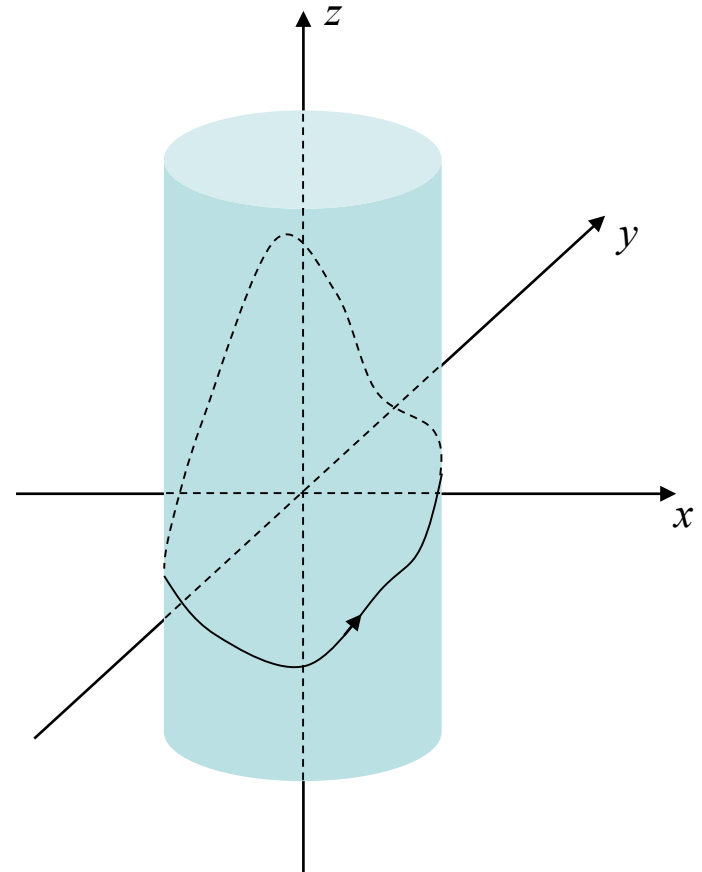
(a) directly

(b) using Stokes' theorem

SOLUTION

The curve is on the cylinder defined by $(\cos u, \sin u, z)$

On the cylinder the curve is defined by $z = f(u)$



SOLUTION (A)

We will calculate $\int_L \bar{A} \cdot d\bar{r} = \int_a^b \bar{A}(\bar{r}(u)) \cdot \frac{d\bar{r}}{du} du$

$$\frac{d\bar{r}}{du} = \left(-\sin u, \cos u, \frac{df}{du} \right)$$

$$\bar{A}(\bar{r}(u)) = (\sin u + 2 \cos u, \cos^2 u + f(u), \sin u)$$

$$\begin{aligned} \Rightarrow \int_L \bar{A} \cdot d\bar{r} &= \int_0^{2\pi} (\sin u + 2 \cos u, \cos^2 u + f(u), \sin u) \cdot \left(-\sin u, \cos u, \frac{df}{du} \right) du = \\ &= -\int_0^{2\pi} \sin^2 u du - 2 \int_0^{2\pi} \sin u \cos u du + \int_0^{2\pi} \cos^3 u du + \int_0^{2\pi} \left(f(u) \cos u + \frac{df}{du} \sin u \right) du = \\ &\quad \begin{array}{cccc} \vdots & \vdots & \vdots & \vdots \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ &= -\left[\frac{u}{2} - \frac{\sin 2u}{4} \right]_0^{2\pi} - \left[\sin^2 u \right]_0^{2\pi} + \left[\sin u - \frac{1}{3} \sin^3 u \right]_0^{2\pi} + \left[f(u) \sin u \right]_0^{2\pi} = -\pi \end{aligned}$$

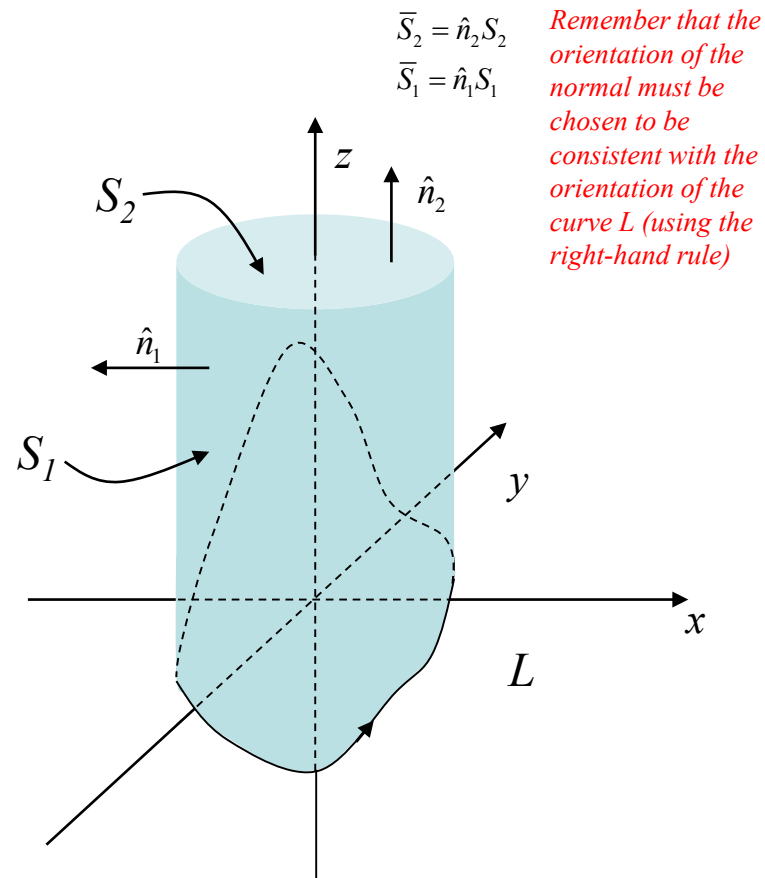
SOLUTION (B)

$$\int_L \bar{A} \cdot d\bar{r} = \iint_S \text{rot } \bar{A} \cdot d\bar{S}$$

$$S = S_1 + S_2$$

$$\text{rot } \bar{A} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+2x & x^2+z & y \end{vmatrix} = (1-1, 0-0, 2x-1) = (0, 0, 2x-1)$$

$$\text{rot } \bar{A} \text{ is in the z-direction} \Rightarrow \iint_{S_1} \text{rot } \bar{A} \cdot d\bar{S} = 0$$



$$\int_L \bar{A} \cdot d\bar{r} = \iint_S \text{rot } \bar{A} \cdot d\bar{S} = \iint_{S_2} \text{rot } \bar{A} \cdot d\bar{S}_2 = \iint_{S_2} (0, 0, 2x-1) \cdot \hat{e}_z dx dy = \iint_{S_2} (2x-1) dx dy$$

cylindrical coord.

$$\downarrow$$

$$= \int_0^{2\pi} \int_0^1 (2\rho \cos \varphi - 1) \rho d\rho d\varphi = 2 \int_0^{2\pi} \cos \varphi d\varphi \int_0^1 \rho^2 d\rho - \int_0^{2\pi} d\varphi \int_0^1 \rho d\rho = -2\pi \left[\frac{\rho^2}{2} \right]_0^1 = -\pi$$

$\hat{e}_z$ is the direction of $\hat{n}_2$