

VEKTORANALYS /ED1110

HT 2022

CELTE / CENMI

**PRACTICAL EXAMPLES USEFUL
FOR VECTOR ANALYSIS**



VOLUMETRIC DENSITY

Probably, you are familiar with the mass density:

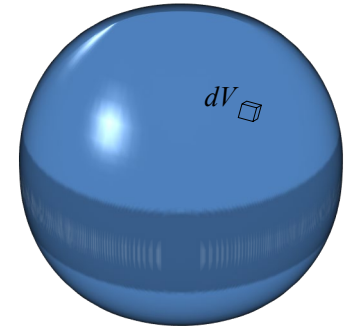
mass M
volume V

- The volumetric mass density of a substance is its mass per unit volume.
- If the substance is uniform, the volumetric mass density is defined as: $\rho = M/V$ where M is the mass and V the volume
- Example: the mass density of the water is $\rho=1000\text{kg/m}^3$.
- If the material is not uniform, the mass density is not constant. In this case, it is defined as:

$$\rho(x, y, z) = dM / dV$$

- If you know the volumetric mass density of a body, you can calculate its mass with a volume integral:

$$M = \int_V \rho dV$$

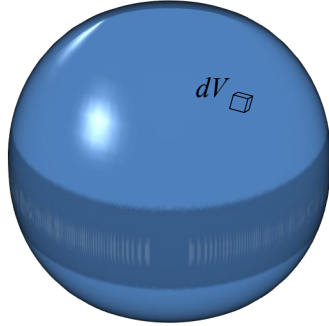


In analogy with the mass, it is possible to define the density of other quantities, for example, the electric charge.

ELECTRIC CHARGE DENSITY

The **volumetric charge density** it is the electric charge per unit volume:

total electric charge Q
volume V



$$\rho_V = \frac{dQ}{dV}$$

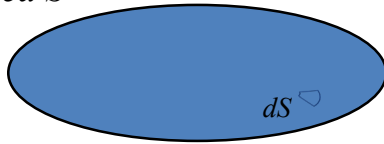
$$dQ = \rho_V dV$$
$$Q = \int dQ$$

$$Q = \int_V \rho_V dV$$

dV is an infinitesimal volume element in the volume V

The **surface charge density** is the electric charge per unit area:

total electric charge Q
area S



$$\rho_S = \frac{dQ}{dS}$$

$$dQ = \rho_S dS$$
$$Q = \int dQ$$

$$Q = \int_S \rho_S dS$$

dS is an infinitesimal surface element on the surface S

The **linear charge density** is the electric charge per unit length:

total electric charge Q
curve L



$$\rho_l = \frac{dQ}{dl}$$

$$dQ = \rho_l dl$$
$$Q = \int dQ$$

$$Q = \int_L \rho_l dl$$

dl is an infinitesimal line element along the curve L

LINEAR ELECTRIC CHARGE DENSITY: EXAMPLE

Consider a circular arc with radius R , centered in the origin and from $\varphi = -\pi/2$ to $\varphi = +\pi/2$

EXAMPLE 1:

The arc is electrically charged. The line charge density in the arc is not uniform:

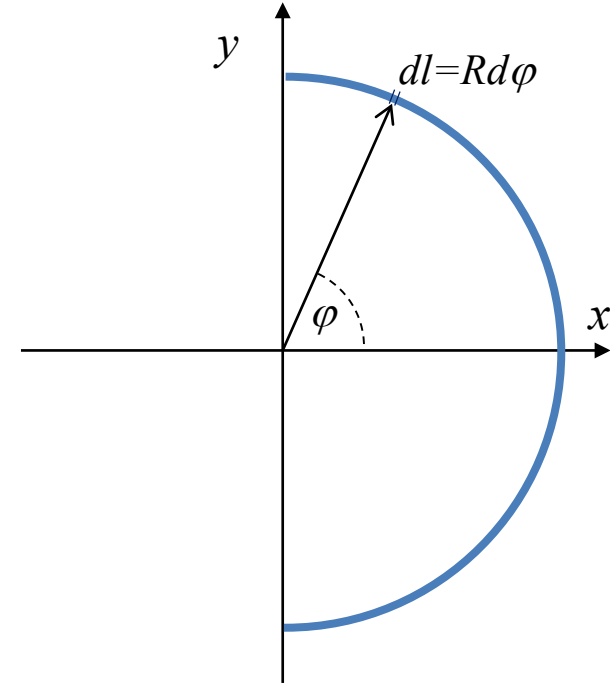
$$\rho_l = \rho_0 \cos \varphi$$

where ρ_0 is constant.

Calculate the total electric charge in the arc.

$$\left. \begin{array}{l} dq = \rho_l dl \\ dl = R d\varphi \end{array} \right\} \Rightarrow dq = \rho_0 \cos \varphi R d\varphi$$

$$Q = \int dq = \int_{-\pi/2}^{\pi/2} \rho_0 \cos \varphi R d\varphi = \rho_0 R [\sin \varphi]_{-\pi/2}^{\pi/2} = 2\rho_0 R$$



SURFACE ELECTRIC CHARGE DENSITY: EXAMPLE

Consider a half spherical shell with radius R and centered in the origin and with base parallel to the x - z plane

EXAMPLE 2:

The shell is electrically charged. The surface charge density in the shell is not uniform:

$$\rho_s = \rho_0 \sin \varphi$$

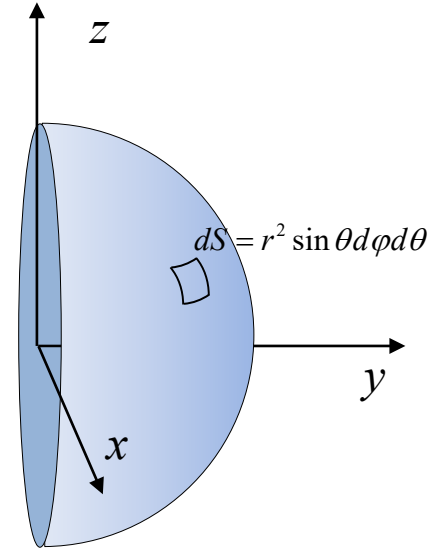
where ρ_0 is constant.

Calculate the total electric charge in the shell.

$$\left. \begin{array}{l} dq = \rho_s dS \\ dS = r^2 \sin \theta d\varphi d\theta \end{array} \right\} \Rightarrow dq = \rho_0 \sin \varphi r^2 \sin \theta d\varphi d\theta$$

$$\text{The radius is } R \text{ (and it is constant)} \Rightarrow dq = \rho_0 R^2 \sin \varphi \sin \theta d\varphi d\theta$$

$$Q = \int dq = \int_0^\pi \int_0^\pi \rho_0 R^2 \sin \varphi d\varphi \sin \theta d\theta = \rho_0 R^2 \left[-\cos \varphi \right]_0^\pi \left[-\cos \theta \right]_0^\pi = 4\rho_0 R^2$$



VOLUMETRIC ELECTRIC CHARGE DENSITY: EXAMPLE

Consider a cylinder with radius R , height L , with axis along the z -axis and base on the x - y plane

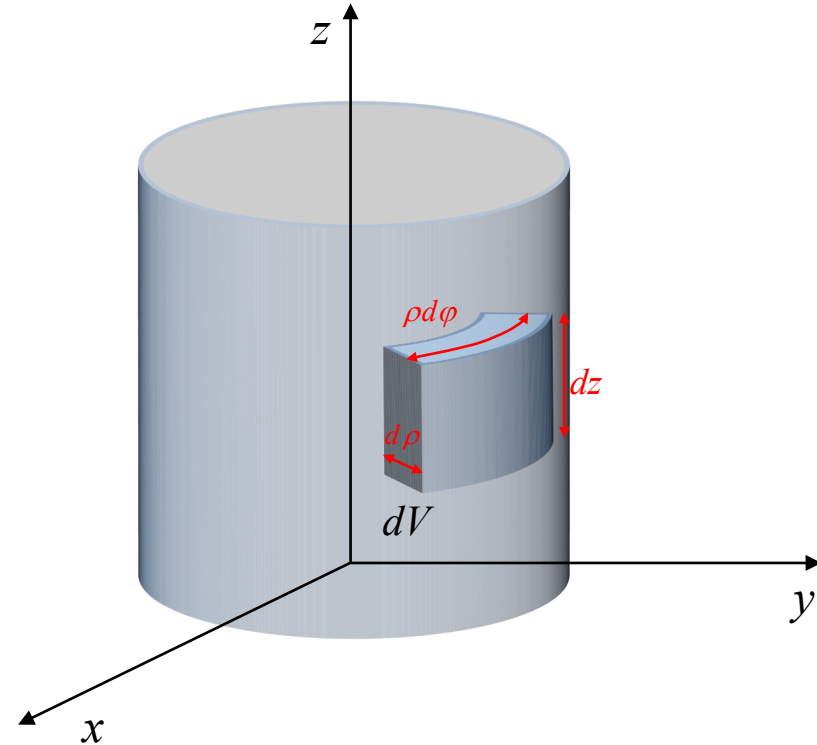
EXAMPLE 3:

The volume charge density in the cylinder is uniform: $\rho_V = \rho_0$
where ρ_0 is constant.
Calculate the total charge in the cylinder.

$$dq = \rho_V dV$$

$$dV = \rho d\phi d\rho dz$$

$$Q = \int dq = \int_0^L \int_0^R \int_0^{2\pi} \rho_V \rho d\phi d\rho dz = \int_0^L \int_0^R \int_0^{2\pi} \rho_V \rho d\phi d\rho dz = \pi \rho_0 R^2 L$$



EXAMPLE 4:

The volume charge density in the cylinder is not uniform: $\rho_V = \rho_0(1 - \rho/R)$
where ρ_0 is constant.
Calculate the total charge in the cylinder.

$$\left. \begin{array}{l} dq = \rho_V dV \\ dV = \rho d\phi d\rho dz \end{array} \right\} \Rightarrow dq = \rho_0 \left(1 - \frac{\rho}{R}\right) \rho d\phi d\rho dz$$

$$\begin{aligned} Q &= \int dq = \int_0^L \int_0^R \int_0^{2\pi} \rho_0 \left(1 - \frac{\rho}{R}\right) \rho d\phi d\rho dz = 2\pi L \rho_0 \int_0^R \left(1 - \frac{\rho}{R}\right) \rho d\rho = 2\pi L \rho_0 \int_0^R \left(\rho - \frac{\rho^2}{R}\right) d\rho = \\ &= 2\pi L \rho_0 \left[\frac{\rho^2}{2} - \frac{\rho^3}{3R} \right]_0^R = 2\pi L \rho_0 \frac{R^2}{6} = \frac{\pi}{3} \rho_0 L R^2 \end{aligned}$$

ELECTRIC FIELD GENERATED BY SEVERAL POINT CHARGES

The electric field in P generated by one point charge q is:

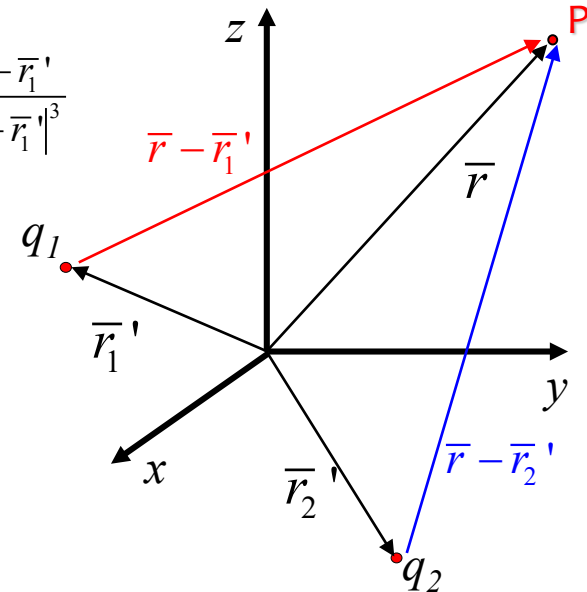
$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\vec{r} - \vec{r}_1'}{|\vec{r} - \vec{r}_1'|^3}$$

The electric field generated by two point charges q_1 and q_2 is:

$$\vec{E} = \frac{q_1}{4\pi\epsilon_0} \frac{\vec{r} - \vec{r}_1'}{|\vec{r} - \vec{r}_1'|^3} + \frac{q_2}{4\pi\epsilon_0} \frac{\vec{r} - \vec{r}_2'}{|\vec{r} - \vec{r}_2'|^3}$$

The electric field generated by N point charges is:

$$\vec{E} = \sum_{i=1}^N \frac{q_i}{4\pi\epsilon_0} \frac{\vec{r} - \vec{r}_i'}{|\vec{r} - \vec{r}_i'|^3}$$



ELECTRIC FIELD GENERATED BY A CHARGE DISTRIBUTION

Let's consider an electrically charge object with a specific charge distribution. We want the total electric field in a point P .

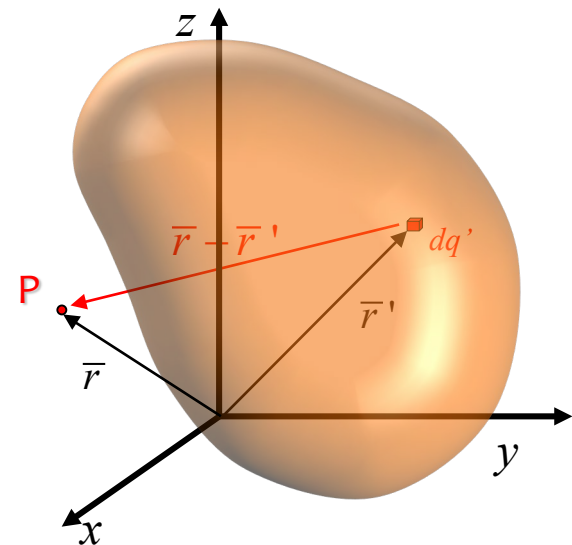
The electric field produced in the point P by an infinitesimal charge is:

$$d\vec{E} = \frac{dq'}{4\pi\epsilon_0} \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

where $\vec{r}'$ is the position vector of dq' and $\vec{r}$ is the position vector of P .

Then the total electric field is the integral of $d\vec{E}$:

$$\vec{E} = \int \frac{dq'}{4\pi\epsilon_0} \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$



ELECTRIC FIELD GENERATED BY A CHARGE DISTRIBUTION

If the charge is spread in a **volume**: $dq = \rho_V dV$

$$\vec{E} = \int_V \frac{\rho_V}{4\pi\epsilon_0} \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dV'$$

If the charge is spread on a **surface**: $dq = \rho_S dS$

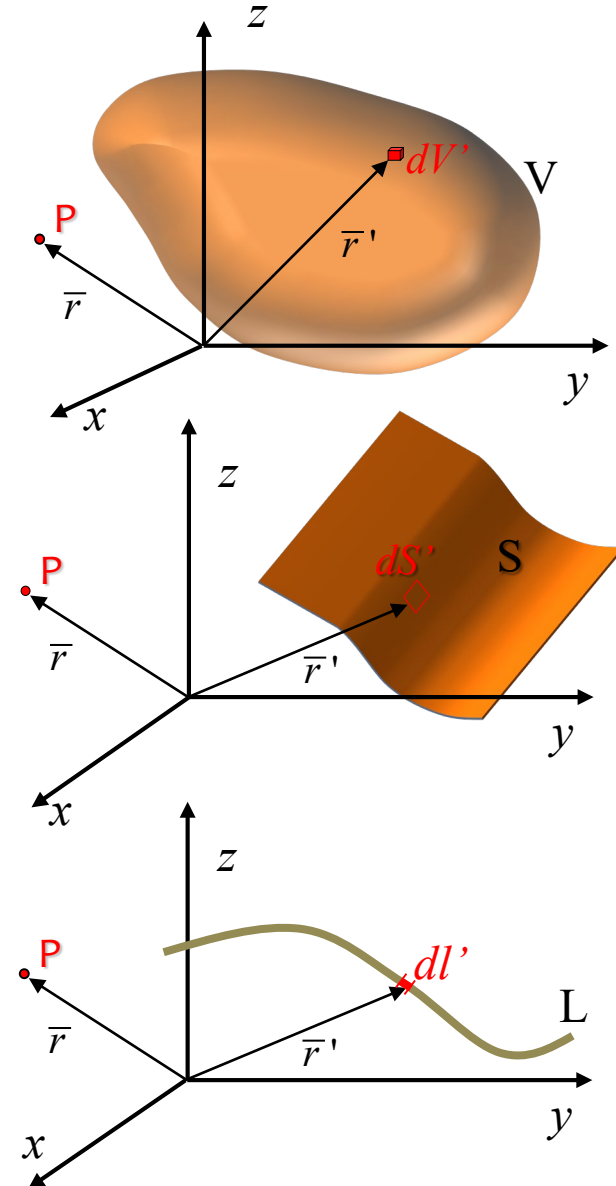
$$\vec{E} = \int_S \frac{\rho_S}{4\pi\epsilon_0} \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dS'$$

If the charge is spread on a **curve**: $dq = \rho_l dl$

$$\vec{E} = \int_L \frac{\rho_l}{4\pi\epsilon_0} \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dl'$$

$\vec{r}$ is the position vector of the P (where we want to calculate the field)

$\vec{r}'$ is the position vector of the infinitesimal charge



EXAMPLE: the electric field generated by a straight wire

A straight wire "L" with charge density ρ_l (constant) has length $2L_0$, is located along the z-axis and centered at $z=0$.

Calculate the electric field on the plane $z=0$ produced by the wire L.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_L \rho_l \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dl'$$

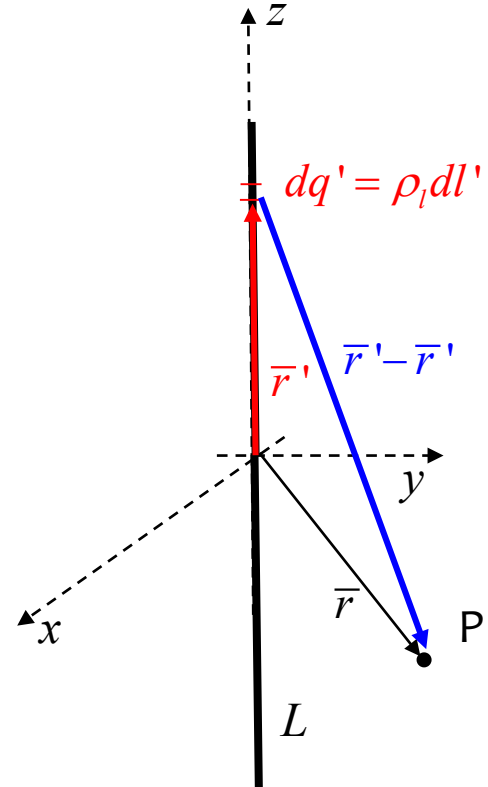
$$\vec{r}' = (0, 0, z') = z' \hat{e}_z$$

$$dl' = dz'$$

$$\vec{r} = \rho \hat{e}_\rho + z \hat{e}_z = \rho \hat{e}_\rho \Rightarrow \vec{r} - \vec{r}' = \rho \hat{e}_\rho - z' \hat{e}_z \Rightarrow |\vec{r} - \vec{r}'| = \sqrt{\rho^2 + z'^2}$$

$$\frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dl' = \frac{\rho \hat{e}_\rho - z' \hat{e}_z}{\sqrt{\rho^2 + z'^2}^3} dz'$$

$$\begin{aligned} \vec{E}(\vec{r}) &= \frac{\rho_l}{4\pi\epsilon_0} \int_L \frac{\rho \hat{e}_\rho - z' \hat{e}_z}{\sqrt{\rho^2 + z'^2}^3} dz' = \frac{\rho_l}{4\pi\epsilon_0} \int_L \frac{\rho \hat{e}_\rho}{\sqrt{\rho^2 + z'^2}^3} dz' - \frac{\rho_l}{4\pi\epsilon_0} \int_L \frac{z' \hat{e}_z}{\sqrt{\rho^2 + z'^2}^3} dz' = \\ &= \frac{\rho_l}{4\pi\epsilon_0} \rho \hat{e}_\rho \left[\frac{z'}{\rho^2 \sqrt{\rho^2 + z'^2}} \right]_{-L_0}^{L_0} - \frac{\rho_l}{4\pi\epsilon_0} \hat{e}_z \left[-\frac{1}{\sqrt{\rho^2 + z'^2}} \right]_{-L_0}^{L_0} = \frac{\rho_l}{4\pi\epsilon_0} \frac{2L_0}{\rho \sqrt{\rho^2 + L_0^2}} \hat{e}_\rho \end{aligned}$$



$\hat{e}_\rho$ does not depend on z , so you can move it out from the integration.

But, in general, $\hat{e}_\rho$ is not constant so you need to integrate it.

$\hat{e}_z$ is constant, so you can always move it out from the integration.

EXAMPLE: the electric field generated by half sphere

A half sphere with radius R is centered in the origin and has the base on the x - y plane. The volume charge density in the sphere is:

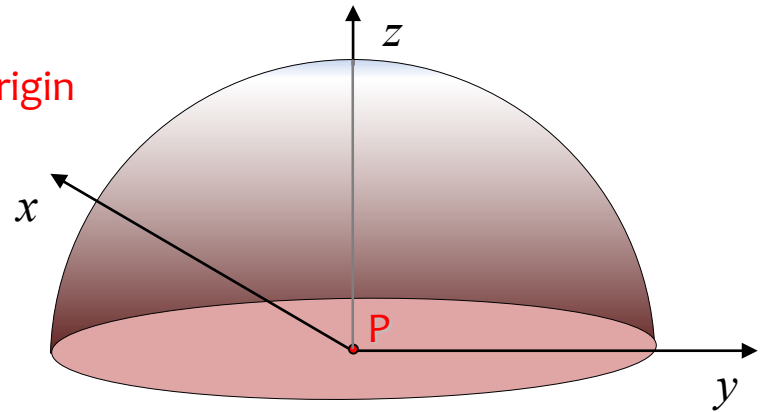
$$\rho_V(r) = \rho_0 \left(1 - \frac{r}{R}\right)$$

Calculate the electric field in the point P located in the origin

$$\vec{E} = \int_V \frac{\rho_V}{4\pi\epsilon_0} \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dV'$$

$$\left. \begin{aligned} \vec{r} &= (0,0,0) = \vec{0} \\ \vec{r}' &= r' \hat{e}_r \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} \vec{r} - \vec{r}' &= -r' \hat{e}_r \\ |\vec{r} - \vec{r}'| &= r' \end{aligned} \right.$$

$$dV' = r'^2 \sin \theta' d\varphi' d\theta' dr'$$



$$\begin{aligned} \vec{E} &= -\frac{\rho_0}{4\pi\epsilon_0} \int_V \left(1 - \frac{r'}{R}\right) \frac{r' \hat{e}_r}{r'^3} r'^2 \sin \theta' d\varphi' d\theta' dr' = -\frac{\rho_0}{4\pi\epsilon_0} \int_0^R \int_0^{\pi/2} \int_0^{2\pi} \left(1 - \frac{r'}{R}\right) \hat{e}_r \sin \theta' d\varphi' d\theta' dr' \\ &= -\frac{\rho_0}{4\pi\epsilon_0} \int_0^R \int_0^{\pi/2} \int_0^{2\pi} \left(1 - \frac{r'}{R}\right) (\sin \theta' \cos \varphi' \hat{e}_x + \sin \theta' \sin \varphi' \hat{e}_y + \cos \theta' \hat{e}_z) \sin \theta' d\varphi' d\theta' dr' \\ &= -\frac{\rho_0}{4\pi\epsilon_0} \int_0^R \int_0^{\pi/2} \int_0^{2\pi} \left(1 - \frac{r'}{R}\right) (\sin^2 \theta' \cos \varphi' \hat{e}_x) d\varphi' d\theta' dr' - \frac{\rho_0}{4\pi\epsilon_0} \int_0^R \int_0^{\pi/2} \int_0^{2\pi} \left(1 - \frac{r'}{R}\right) (\sin^2 \theta' \sin \varphi' \hat{e}_y) d\varphi' d\theta' dr' \\ &\quad - \frac{\rho_0}{4\pi\epsilon_0} \int_0^R \int_0^{\pi/2} \int_0^{2\pi} \left(1 - \frac{r'}{R}\right) (\sin \theta' \cos \theta' \hat{e}_z) d\varphi' d\theta' dr' = \\ &= -\frac{\rho_0}{4\pi\epsilon_0} 2\pi \hat{e}_z \int_0^{\pi/2} (\sin \theta' \cos \theta') d\theta' \int_0^R \left(1 - \frac{r'}{R}\right) dr' = -\frac{\rho_0}{2\epsilon_0} \hat{e}_z \left[-\frac{\cos^2 \theta'}{2} \right]_0^{\pi/2} \left[r' - \frac{r'^2}{2R} \right]_0^R = -\frac{\rho_0}{2\epsilon_0} \hat{e}_z \frac{1}{2} \frac{R}{2} = -\frac{\rho_0 R}{8\epsilon_0} \hat{e}_z \end{aligned}$$

EXAMPLE: electrostatic potential generated by a straight wire (using cartesian coordinates)

The electrostatic potential ϕ in the point P (x_p, y_p, z_p) produced by an electrically charged wire "C" with constant charge density λ is:

$$\phi(\vec{r}) = \int_C \frac{\lambda}{4\pi\epsilon_0} \frac{|d\vec{r}'|}{|\vec{r} - \vec{r}'|}$$

where:

$\vec{r}$ is the position vector of the point
 $d\vec{r}'$ is an infinitesimal vector on the curve C
 $\vec{r}'$ is the vector from the origin to $d\vec{r}'$

Exercise: calculate the potential produced by a straight wire located along the z-axis, centered in the origin and with length 2L

Step 1. Calculate the term $\frac{|d\vec{r}'|}{|\vec{r} - \vec{r}'|}$

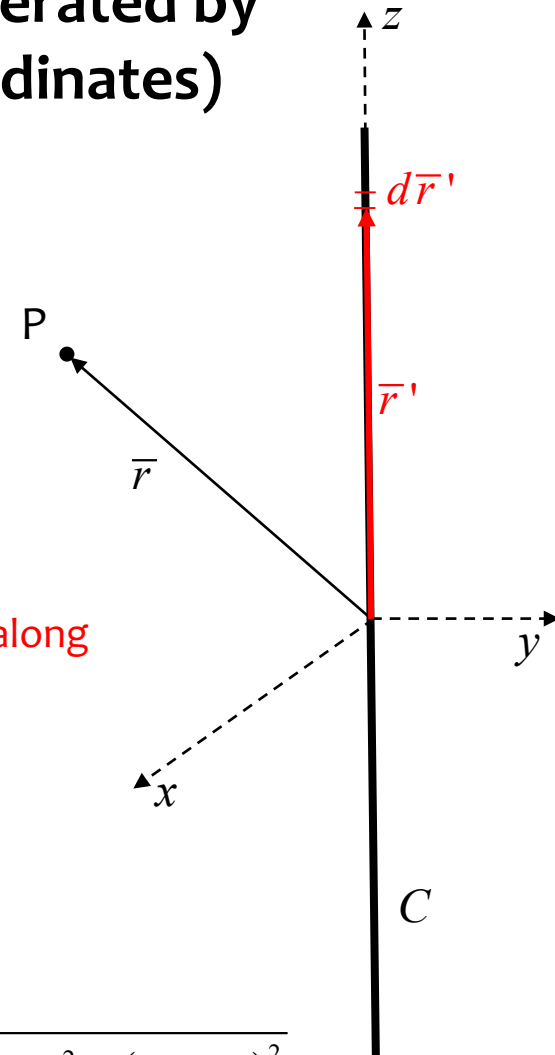
$$\vec{r}' = (0, 0, z') = z' \hat{e}_z$$

$$d\vec{r}' = (0, 0, dz') = dz' \hat{e}_z \Rightarrow |d\vec{r}'| = dz'$$

$$\vec{r} = (x_p, y_p, z_p) \Rightarrow \vec{r} - \vec{r}' = (x_p, y_p, z_p - z') \Rightarrow |\vec{r} - \vec{r}'| = \sqrt{x_p^2 + y_p^2 + (z_p - z')^2}$$

$$\frac{|d\vec{r}'|}{|\vec{r} - \vec{r}'|} = \frac{dz'}{\sqrt{\rho_p^2 + (z_p - z')^2}} \quad \text{where } \rho_p \text{ is the distance of P from the curve C}$$

Step 2. Calculate the integral
$$\phi(\vec{r}_p) = \frac{\lambda}{4\pi\epsilon_0} \int_{-L}^{+L} \frac{dz'}{\sqrt{\rho_p^2 + (z_p - z')^2}}$$



EXAMPLE: the force on a wire in a magnetic field

The force on a wire L carrying current I , in a magnetic field $\vec{B}$ is: $\vec{F} = I \int_L (d\vec{l} \times \vec{B})$

Calculate the force on a circular coil L (with radius R and center in the origin). L lies on xy -plane ($z=0$) and has only one turn around the z -axis. The magnetic field $\vec{B}$ is defined in cylindrical coordinates by the expression: $\vec{B} = B_0 \rho (\cos \varphi \hat{e}_z + \sin \varphi \hat{e}_\varphi)$

Use the following steps:

- (a) express $d\vec{l}$ in a cylindrical coordinate system
- (b) calculate $d\vec{l} \times \vec{B}$ in a cylindrical coordinate system
- (c) Integrate and calculate $\vec{F}$ (here you can use a cartesian coordinate system)

$$d\vec{l} = \rho d\varphi \hat{e}_\varphi$$

$$d\vec{l} \times \vec{B} = \rho d\varphi \hat{e}_\varphi \times B_0 \rho (\cos \varphi \hat{e}_z + \sin \varphi \hat{e}_\varphi) = B_0 \rho^2 \cos \varphi d\varphi \hat{e}_\rho$$

$$\vec{F} = I \int_0^{2\pi} B_0 \rho^2 \cos \varphi d\varphi \hat{e}_\rho = IB_0 R^2 \int_0^{2\pi} \cos \varphi \hat{e}_\rho d\varphi =$$

$$= IB_0 R^2 \int_0^{2\pi} \cos \varphi (\cos \varphi \hat{e}_x + \sin \varphi \hat{e}_y) d\varphi =$$

$$IB_0 R^2 \int_0^{2\pi} \cos^2 \varphi \hat{e}_x d\varphi + IB_0 R^2 \int_0^{2\pi} \cos \varphi \sin \varphi \hat{e}_y d\varphi = IB_0 R^2 \hat{e}_x \int_0^{2\pi} \cos^2 \varphi d\varphi =$$

$$= IB_0 R^2 \hat{e}_x \left[\frac{\varphi}{2} + \frac{\sin 2\varphi}{4} \right]_0^{2\pi} = IB_0 R^2 \pi \hat{e}_x$$