

VEKTORANALYS

HT 2022

CELTE / CENMI

ED1110

LINE INTEGRALS and FLUX INTEGRALS: ÖVNINGAR

Kursvecka 2

Kapitel 6-7 (*Vektoranalys*, 1:e uppl, Frassinetti/Scheffel)



PROBLEM 1

Calculate the potential ϕ for the following vector field: $\vec{A} = 2xz\hat{e}_x + 2z^2\hat{e}_y + (x^2 + 4yz)\hat{e}_z$

SOLUTION

Step 1. Verify that the vector field can have a potential: $\frac{\partial A_x}{\partial y} = \frac{\partial A_y}{\partial x}$ $\frac{\partial A_y}{\partial z} = \frac{\partial A_z}{\partial y}$ $\frac{\partial A_z}{\partial x} = \frac{\partial A_x}{\partial z}$

Theorem 5.1

$$\left. \begin{array}{l} \frac{\partial A_x}{\partial y} = 0 \quad \frac{\partial A_y}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial A_x}{\partial y} = \frac{\partial A_y}{\partial x} \\ \frac{\partial A_y}{\partial z} = 4z \quad \frac{\partial A_z}{\partial y} = 4z \quad \Rightarrow \quad \frac{\partial A_y}{\partial z} = \frac{\partial A_z}{\partial y} \\ \frac{\partial A_z}{\partial x} = 2x \quad \frac{\partial A_x}{\partial z} = 2x \quad \Rightarrow \quad \frac{\partial A_z}{\partial x} = \frac{\partial A_x}{\partial z} \end{array} \right\} \Rightarrow \vec{A} \text{ might have a potential}$$

Step 2. Calculate the potential

$$\vec{A} = (2xz, 2z^2, x^2 + 4yz) \quad (a)$$

$$\vec{A} = \text{grad}\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) \quad (b)$$

$$\frac{\partial \phi}{\partial x} = 2xz \Rightarrow \phi = x^2 z + F(y, z) \quad (c)$$

$$\left. \begin{array}{l} \text{From (a) and (b)} \\ \frac{\partial \phi}{\partial y} = 2z^2 \\ \text{From (c)} \\ \frac{\partial \phi}{\partial y} = \frac{\partial F(y, z)}{\partial y} \end{array} \right\} \Rightarrow \frac{\partial F(y, z)}{\partial y} = 2z^2 \Rightarrow F(y, z) = 2yz^2 + G(z)$$

$$\phi = x^2 z + 2yz^2 + G(z) \quad (d)$$

$$\left. \begin{array}{l} \text{From (a) and (b)} \\ \frac{\partial \phi}{\partial z} = x^2 + 4yz \\ \text{From (d)} \\ \frac{\partial \phi}{\partial z} = x^2 + 4yz + \frac{\partial G(z)}{\partial z} \end{array} \right\} \Rightarrow \frac{\partial G(z)}{\partial z} = 0 \Rightarrow G(z) = \text{const.}$$

$$\boxed{\phi = x^2 z + 2yz^2 + \text{const.}}$$

PROBLEM 2

Calculate the line integral of the vector field: $\vec{A} = (2 - y)\hat{e}_x - xy\hat{e}_y + \hat{e}_z$

along: (A) the curve L
$$\begin{cases} 4x - y^2 = 0 \\ x^2 + y^2 - z = 0 \end{cases}$$

from the point (0,0,0) to the point (1,2,5)

(B) a straight line from point (0,0,0) to (1,2,0) and then from (1,2,0) to (1,2,5)

SOLUTION to problem 2 (part A)

The line integral is: $\int_L \vec{A}(\vec{r}) \cdot d\vec{r} = \int_a^b \vec{A}(\vec{r}(u)) \cdot \frac{d\vec{r}}{du} du$

STEP 1: we need to find a parameterization of the curve L in order to have $\vec{r} = \vec{r}(u)$

If $u=y$ we have:

$$x(u) = \frac{u^2}{4} \quad y(u) = u \quad z(u) = \frac{u^4}{16} + u^2 \quad \vec{r}(u) = \left(\frac{u^2}{4}, u, \frac{u^4}{16} + u^2 \right) \quad (\text{See problem 1 week 1})$$

from point (0,0,0) till the point (1,2,5) $\Rightarrow$ $u: a \rightarrow b$ with $a=0$ and $b=2$

STEP 2: we calculate the integral

$$\left. \begin{aligned} \vec{A}(\vec{r}(u)) &= (2 - y(u), -x(u)y(u), 1) = \left(2 - u, -\frac{u^3}{4}, 1 \right) \\ \frac{d\vec{r}}{du} &= \left(\frac{2u}{4}, 1, \frac{4u^3}{16} + 2u \right) = \left(\frac{u}{2}, 1, \frac{u^3}{4} + 2u \right) \end{aligned} \right\} \Rightarrow \int_L \vec{A}(\vec{r}) \cdot d\vec{r} = \int_0^2 \left(2 - u, -\frac{u^3}{4}, 1 \right) \cdot \left(\frac{u}{2}, 1, \frac{u^3}{4} + 2u \right) du =$$

$$\int_0^2 \left(u - \frac{u^2}{2} - \frac{u^3}{4} + \frac{u^3}{4} + 2u \right) du = \left[3\frac{u^2}{2} - \frac{u^3}{6} \right]_0^2 = \frac{14}{3}$$

SOLUTION to problem 2 (part B)

straight line $\underbrace{\text{from point } (0,0,0) \text{ till } (1,2,0)}_{L_1}$ and $\underbrace{\text{then from } (1,2,0) \text{ till } (1,2,5)}_{L_2} \Rightarrow \int_L \bar{A}(\bar{r}) \cdot d\bar{r} = \int_{L_1} \bar{A}(\bar{r}) \cdot d\bar{r} + \int_{L_2} \bar{A}(\bar{r}) \cdot d\bar{r}$

- We start with the first integral.

STEP 1: parameterization of L_1 : $\bar{r}(u) = (u, 2u, 0)$

$$\frac{d\bar{r}}{du} = (1, 2, 0) \quad u: 0 \rightarrow 1$$

STEP 2: line integral calculation

$$\begin{aligned} \int_{L_1} \bar{A}(\bar{r}) \cdot d\bar{r} &= \int_a^b \bar{A}(\bar{r}(u)) \cdot \frac{d\bar{r}}{du} du = \int_0^1 (2-2u, -2u^2, 1) \cdot (1, 2, 0) du = \\ &= \int_0^1 (2-2u-4u^2) du = \left[2u - u^2 - \frac{4}{3}u^3 \right]_0^1 = \left(2 - 1 - \frac{4}{3} \right) = -\frac{1}{3} \end{aligned}$$

- We continue with the second integral.

STEP 1: parameterization of L_2 : $\bar{r}(u) = (1, 2, u)$

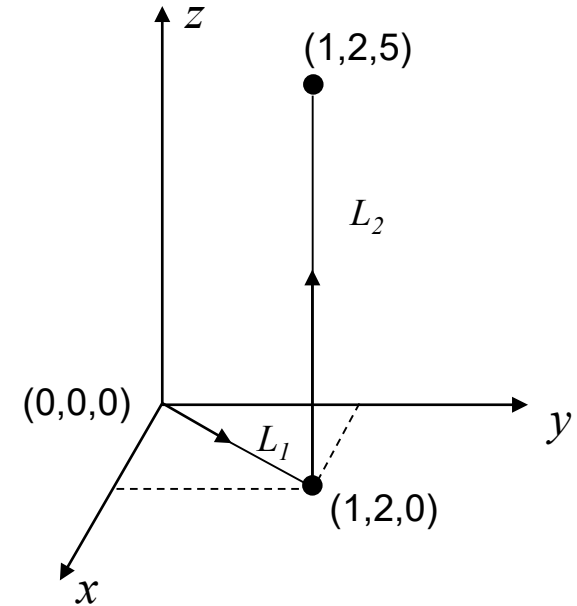
$$\frac{d\bar{r}}{du} = (0, 0, 1) \quad u: 0 \rightarrow 5$$

STEP 2: line integral calculation

$$\int_{L_2} \bar{A}(\bar{r}) \cdot d\bar{r} = \int_a^b \bar{A}(\bar{r}(u)) \cdot \frac{d\bar{r}}{du} du = \int_0^5 (2-2, -2, 1) \cdot (0, 0, 1) du = \int_0^5 du = 5$$

• Finally:

$$\int_L \bar{A}(\bar{r}) \cdot d\bar{r} = \int_{L_1} \bar{A}(\bar{r}) \cdot d\bar{r} + \int_{L_2} \bar{A}(\bar{r}) \cdot d\bar{r} = -\frac{1}{3} + 5 = \frac{14}{3}$$



PROBLEM 3

Calculate the line integral of the vector field: $\bar{A} = \frac{x}{\sqrt{x^2 + y^2 + 5z^2}} \hat{e}_x + \frac{\sqrt{5}yz}{\sqrt{x^2 + y^2 + z^2}} \hat{e}_y + \frac{e^x \sin y}{1 + \ln(xyz)} \hat{e}_z$

Along the path L defined by:

$$L = \begin{cases} x^2 + y^2 = 4 \\ z = 1 \end{cases}$$

From the point P_1 : (2,0,1) to the point P_2 : (0,2,1)

SOLUTION

Parameterization of L:

$$\left. \begin{aligned} x &= 2 \cos \varphi \\ y &= 2 \sin \varphi \\ z &= 1 \end{aligned} \right\} \Rightarrow L: \bar{r}(\varphi) = (2 \cos \varphi, 2 \sin \varphi, 1) \Rightarrow \frac{d\bar{r}(\varphi)}{d\varphi} = (-2 \sin \varphi, 2 \cos \varphi, 0)$$

$$P1=(2,0,1) \Rightarrow \varphi_1 = 0$$

$$P2=(0,2,1) \Rightarrow \varphi_2 = \frac{\pi}{2}$$

$$\int_L \bar{A} \cdot d\bar{r} = \int_{\varphi_1}^{\varphi_2} \bar{A}(\bar{r}(\varphi)) \cdot \frac{d\bar{r}}{d\varphi} d\varphi$$

$$\bar{A} = \left(\frac{x}{\sqrt{x^2 + y^2 + 5z^2}}, \frac{\sqrt{5}yz}{\sqrt{x^2 + y^2 + z^2}}, \frac{e^x \sin y}{1 + \ln(xyz)} \right) = \left(\frac{2 \cos \varphi}{\sqrt{(2 \cos \varphi)^2 + (2 \sin \varphi)^2 + 5(1)^2}}, \frac{2 \sin \varphi \sqrt{5}}{\sqrt{(2 \cos \varphi)^2 + (2 \sin \varphi)^2 + (1)^2}}, \frac{e^x \sin y}{1 + \ln(xyz)} \right) = \left(\frac{2 \cos \varphi}{3}, 2 \sin \varphi, \frac{e^x \sin y}{1 + \ln(xyz)} \right)$$

$$\int_L \bar{A} \cdot d\bar{r} = \int_0^{\pi/2} \left(\frac{2 \cos \varphi}{3}, 2 \sin \varphi, \frac{e^x \sin y}{1 + \ln(xyz)} \right) \cdot (-2 \sin \varphi, 2 \cos \varphi, 0) d\varphi = \int_0^{\pi/2} \left(-\frac{4}{3} \sin \varphi \cos \varphi + 4 \sin \varphi \cos \varphi \right) d\varphi = \int_0^{\pi/2} \left(\frac{8}{3} \sin \varphi \cos \varphi \right) d\varphi =$$

$$\int_0^{\pi/2} \left(\frac{4}{3} \sin 2\varphi \right) d\varphi = -\frac{4}{3} \left[\frac{\cos 2\varphi}{2} \right]_0^{\pi/2} = -\frac{2}{3} [-1 - 1] = \frac{4}{3}$$

PROBLEM 4

Calculate the line integral of the vector field: $\vec{A} = x^2 \hat{e}_x + \sinh(yz) \hat{e}_y + z^2 \hat{e}_z$

$$\left(\sinh x = \frac{e^x - e^{-x}}{2} \right)$$

Along the path L defined by:

$$L = \begin{cases} x^2 - z = 0 \\ y = 2 \end{cases}$$

From the point P_1 : $(-1, 2, 1)$ to the point P_2 : $(1, 2, 1)$.

SOLUTION

Parameterization of L:

$$\left. \begin{array}{l} x = u \\ y = 2 \\ z = u^2 \end{array} \right\} \Rightarrow L: \vec{r}(u) = (u, 2, u^2) \Rightarrow \frac{d\vec{r}(u)}{du} = (1, 0, 2u)$$

$$P_1 = (-1, 2, 1) \Rightarrow u_1 = -1$$

$$P_2 = (1, 2, 1) \Rightarrow u_2 = +1$$

$$\int_L \vec{A} \cdot d\vec{r} = \int_{u_1}^{u_2} \vec{A}(\vec{r}(u)) \cdot \frac{d\vec{r}}{du} du$$

$$\vec{A} = (x^2, \sinh(yz), z^2) = (u^2, \sinh(2u^2), u^4)$$

$$\int_L \vec{A} \cdot d\vec{r} = \int_{-1}^1 (u^2, \sinh(2u^2), u^4) \cdot (1, 0, 2u) du =$$

$$\int_{-1}^1 (u^2 + 2u^5) du = \left[\frac{1}{3}u^3 + \frac{1}{3}u^6 \right]_{-1}^1 = \frac{2}{3}$$

PROBLEM 5

Calculate the flux of the vector field: $\vec{A} = (x^2 - y^2)\hat{e}_x + (x + y)^2\hat{e}_y + (x - y)^2\hat{e}_z$

through the surface S: $\vec{r}(u, v) = (u + v, u - v, uv)$

$$\begin{cases} u: & -1 \rightarrow 1 \\ v: & -1 \rightarrow 1 \end{cases}$$

$$\hat{n} \cdot \hat{e}_z > 0$$

SOLUTION

The flux is the integral: $\iint_S \vec{A} \cdot d\vec{S} = \int_v \int_u \vec{A}(\vec{r}(u, v)) \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) du dv$

$$\bullet \quad \vec{A}(\vec{r}(u, v)) = ((u + v)^2 - (u - v)^2, (u + v + u - v)^2, (u + v - u + v)^2) = (u^2 + v^2 + 2uv - (u^2 + v^2 - 2uv), 4u^2, 4v^2) = (4uv, 4u^2, 4v^2) = 4(uv, u^2, v^2)$$

$$\bullet \quad \left. \begin{array}{l} \frac{d\vec{r}}{du} = (1, 1, v) \\ \frac{d\vec{r}}{dv} = (1, -1, u) \end{array} \right\} \Rightarrow \frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & v \\ 1 & -1 & u \end{vmatrix} = (u + v, v - u, -2) \quad \hat{n} \cdot \hat{e}_z > 0 \Rightarrow -S$$

$$\bullet \quad \iint_{-S} \vec{A} \cdot d\vec{S} = 4 \int_{-1}^1 \int_{-1}^1 (uv, u^2, v^2) \cdot (u + v, v - u, -2) du dv = 4 \int_{-1}^1 \int_{-1}^1 (u^2 v + uv^2 + u^2 v - u^3 - 2v^2) du dv =$$

$$4 \int_{-1}^1 \left[\frac{u^3}{3} v + \frac{u^2}{2} v^2 + \frac{u^3}{3} v - \frac{u^4}{4} - 2uv^2 \right]_{-1}^1 dv = 4 \int_{-1}^1 \left(\frac{2v}{3} + \frac{2v}{3} - 4v^2 \right) dv = 4 \left[\frac{4v^2}{6} - \frac{4v^3}{3} \right]_{-1}^1 = -\frac{32}{3}$$

$$\Rightarrow \iint_S \vec{A} \cdot d\vec{S} = - \iint_{-S} \vec{A} \cdot d\vec{S} = \frac{32}{3}$$

PROBLEM 6

Calculate the flux of the vector field: $\vec{A} = xy\hat{e}_x + yz\hat{e}_y + (xz + 1)\hat{e}_z$

through the surface S:

$$\begin{cases} z = 4 - x^2 - y^2 \\ x^2 + y^2 \leq 4 \\ \hat{n} \cdot \hat{e}_z \leq 0 \end{cases} \quad \text{on S}$$

SOLUTION

The flux is the integral: $\iint_S \vec{A} \cdot d\vec{S} = \int_v \int_u \vec{A}(\vec{r}(u, v)) \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) du dv$

Parameterization of S:

$$\begin{cases} x = \rho \cos \theta \\ y = \rho \sin \theta \\ z = 4 - \rho^2 \end{cases} \Rightarrow S: \vec{r}(\rho, \theta) = (\rho \cos \theta, \rho \sin \theta, 4 - \rho^2)$$

$$\begin{cases} \frac{d\vec{r}(\rho, \theta)}{d\rho} = (\cos \theta, \sin \theta, -2\rho) \\ \frac{d\vec{r}(\rho, \theta)}{d\theta} = (-\rho \sin \theta, \rho \cos \theta, 0) \end{cases} \Rightarrow \left(\frac{d\vec{r}(\rho)}{d\rho} \times \frac{d\vec{r}(\rho, \theta)}{d\theta} \right) = (2\rho^2 \cos \theta, 2\rho^2 \sin \theta, \rho)$$

$$\begin{aligned} \iint_S \vec{A} \cdot d\vec{S} &= \int_0^{2\pi} \int_0^2 \left(\rho^2 \sin \theta \cos \theta, (4 - \rho^2) \rho \sin \theta, (4 - \rho^2) \rho \cos \theta + 1 \right) \cdot (2\rho^2 \cos \theta, 2\rho^2 \sin \theta, \rho) d\rho d\theta = \\ &= \int_0^{2\pi} \int_0^2 \left(2\rho^4 \sin \theta \cos^2 \theta + 2(4 - \rho^2) \rho^3 \sin^2 \theta + (4 - \rho^2) \rho^2 \cos \theta + \rho \right) d\rho d\theta = \int_0^2 \left(2(4 - \rho^2) \rho^3 \left[\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right) \right]_0^{2\pi} + \rho [\theta]_0^{2\pi} \right) d\rho = \end{aligned}$$

odd functions $\Rightarrow$ the integral in θ over 2π is zero

$$= \int_0^2 \left(2\pi(4 - \rho^2) \rho^3 + 2\pi\rho \right) d\rho = 2\pi \left[4 \frac{\rho^4}{4} - \frac{\rho^6}{6} \right]_0^2 + \pi [\rho^2]_0^2 = 2\pi \left(16 - \frac{32}{3} \right) + 4\pi = \frac{44}{3} \pi$$

$$\iint_S \vec{A} \cdot d\vec{S} = -\frac{44}{3} \pi$$

The normal is defined so that its z-component is negative.
Instead, with this parameterization we get a positive the z-component
 $\rightarrow$ we need to change sign to the final result