VEKTORANALYS / ED1110

HT 2022

CELTE / CENMI

BASICS OF VECTOR ALGEBRA AND SOME APPLICATIONS

version: 24-aug-2022

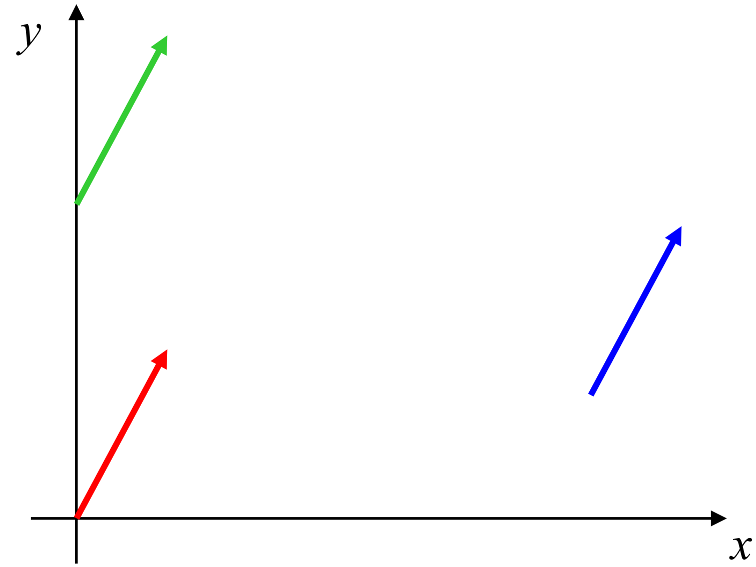
VECTORS

A vector is a quantity with magnitude and direction

Let's consider a vector in
Cartesian coordinates: $\overline{v} = (1, 2, 0)$

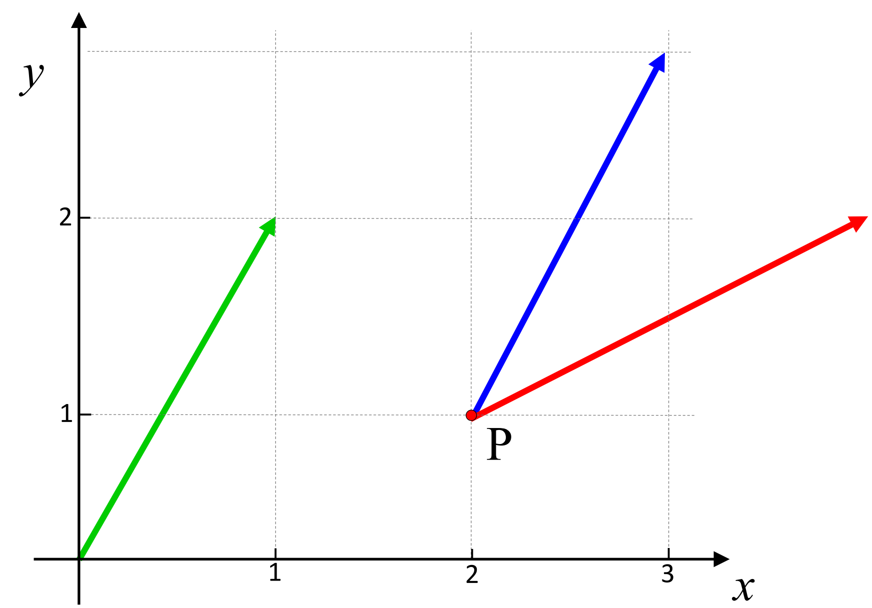
which arrow in the figure
represents best the vector $\overline{v}$?

- the red
- the blue
- the green
- all of them



Plot the vector $\overline{v} = (1, 2, 0)$ *(in a Cartesian coord. sis.)*
in the point P

Plot the position vector $\overline{r} = (2, 1, 0)$
(with the components in cartesian coordinates)



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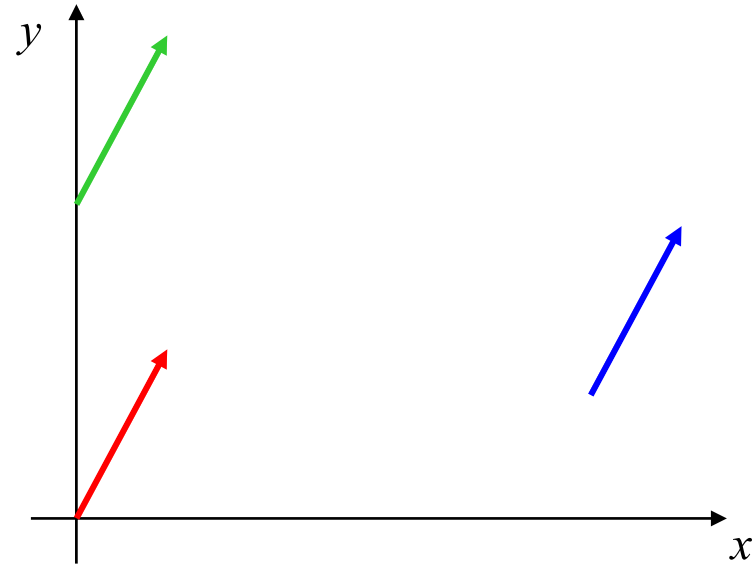

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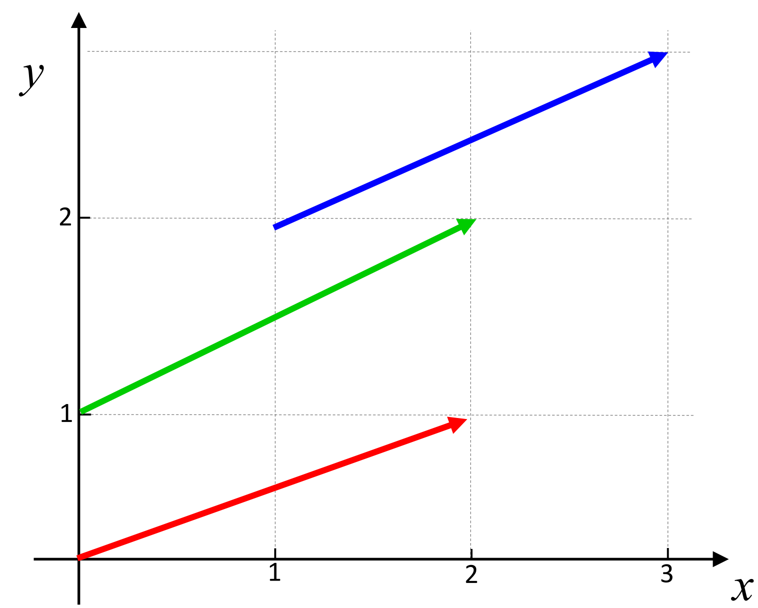
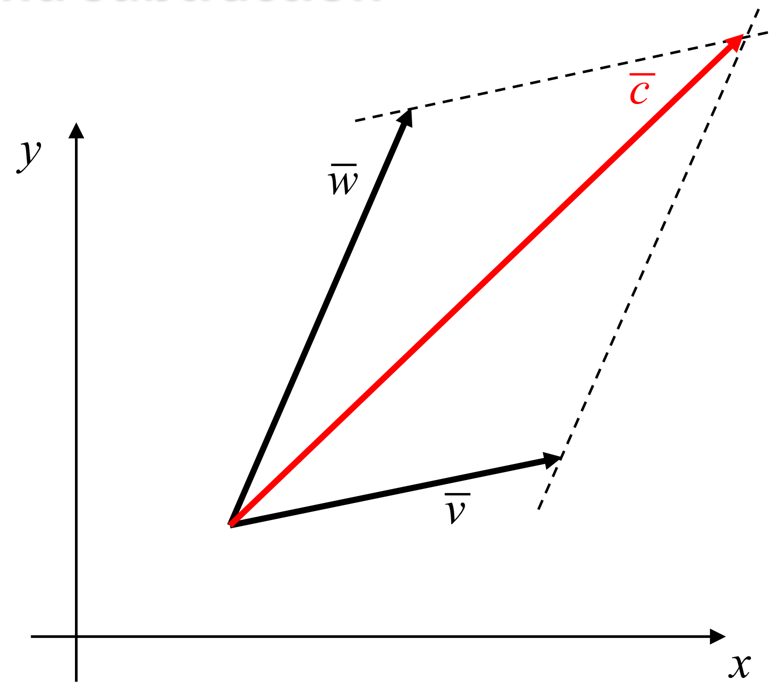


VECTORS: addition and subtraction

Let's consider two vectors in Cartesian coordinates:

$$\overline{v} = \left(v_x, v_y, v_z\right)$$

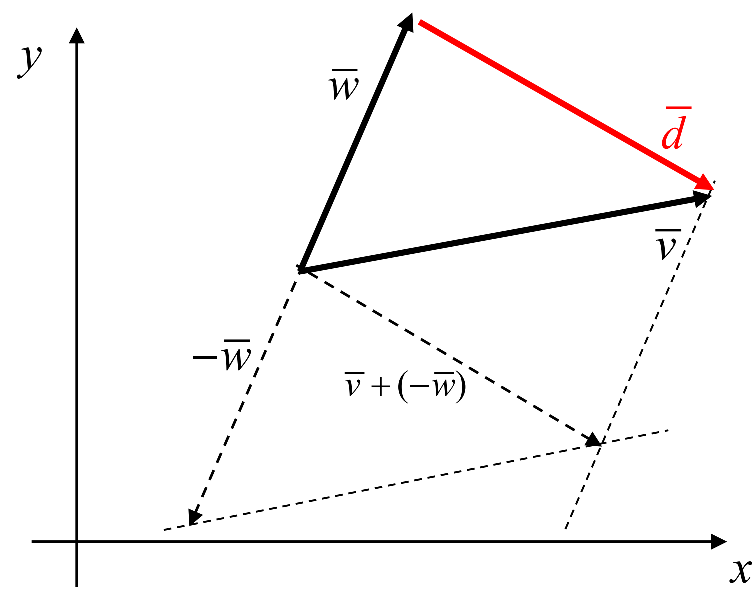
$$\overline{w} = \left(w_x, w_y, w_z\right)$$



Addition:

$$\overline{c} = \overline{v} + \overline{w} = \left(v_x + w_x, v_y + w_y, v_z + w_z\right)$$

Subtraction:

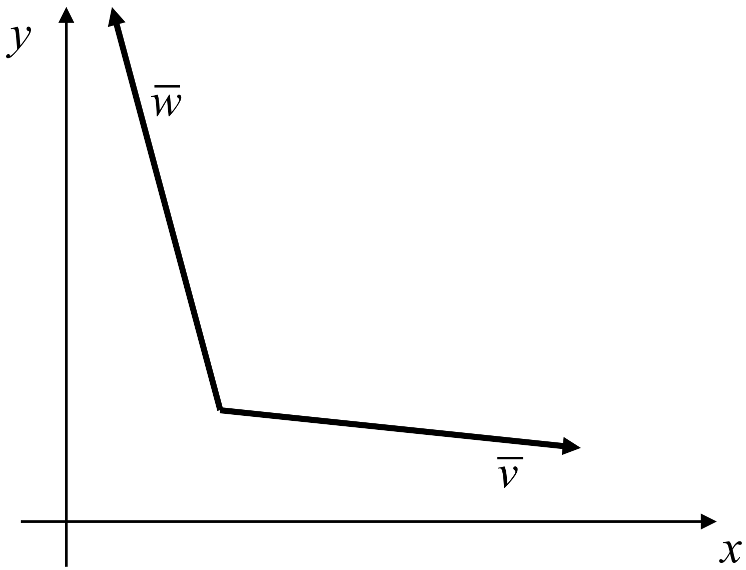


$$\overline{d} = \overline{v} - \overline{w} = \left(v_x - w_x, v_y - w_y, v_z - w_z\right)$$

$$\overline{d} = \overline{v} + (-\overline{w})$$

VECTORS: addition and subtraction

Plot $\overline{v} + \overline{w}$



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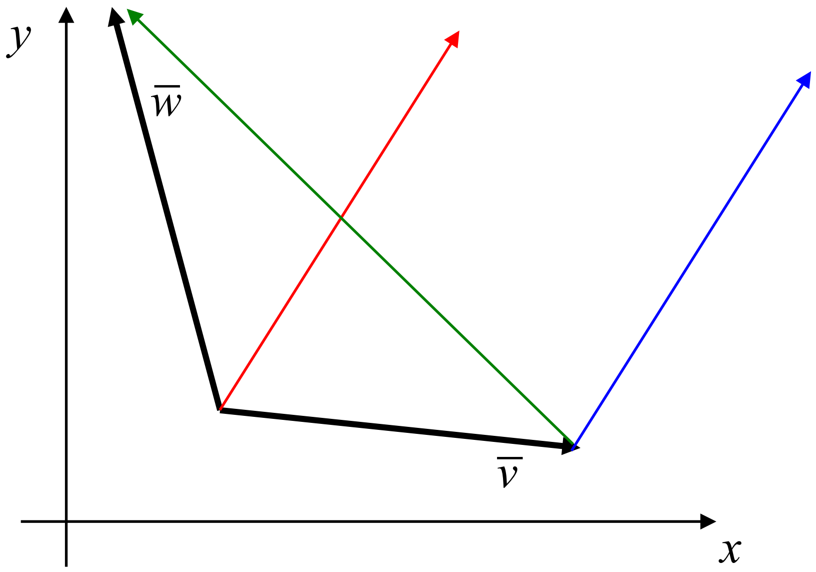

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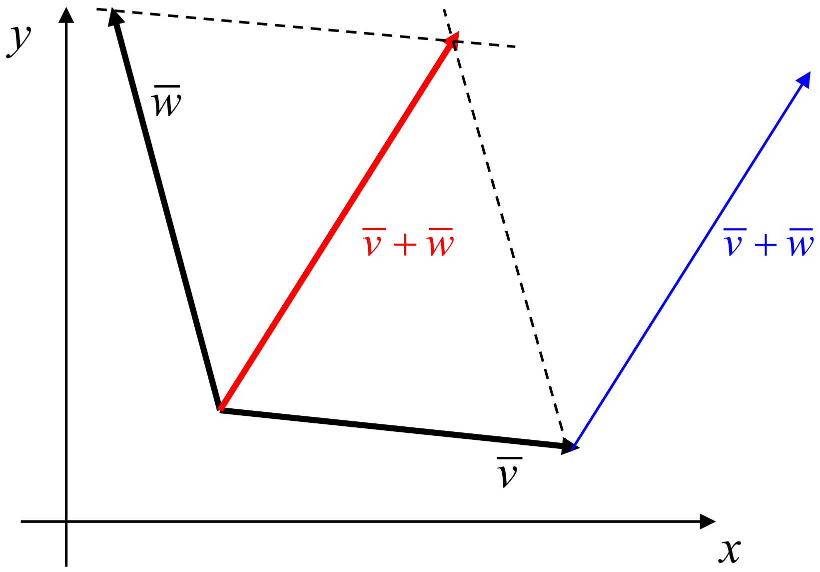

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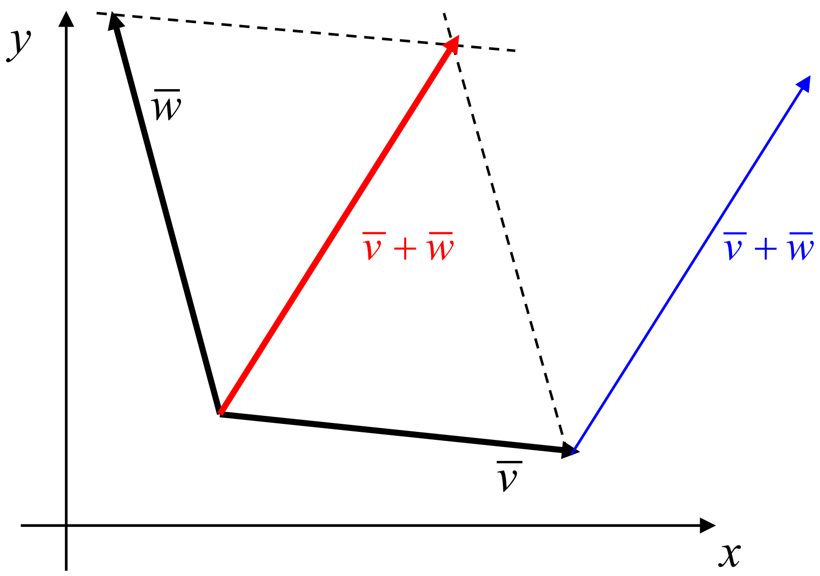

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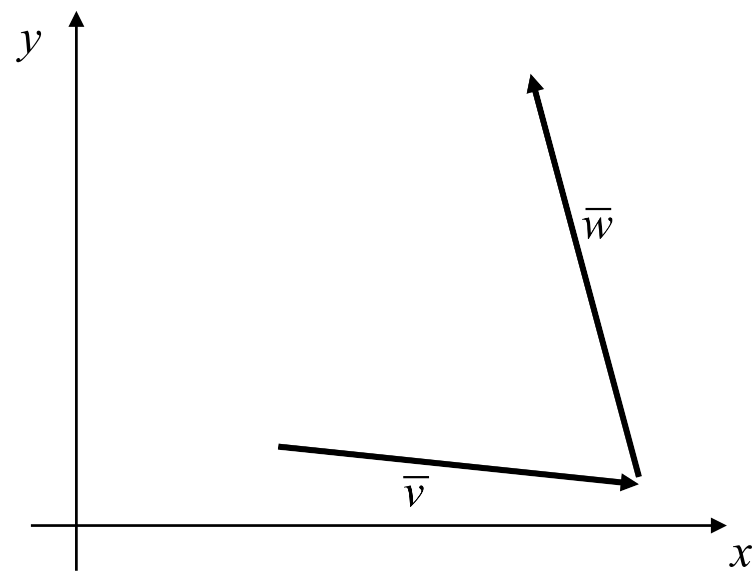

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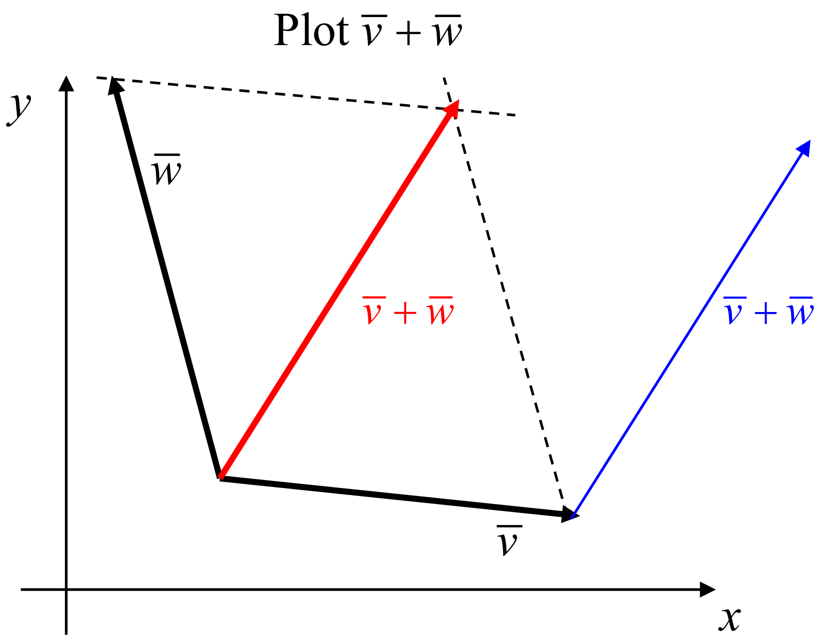

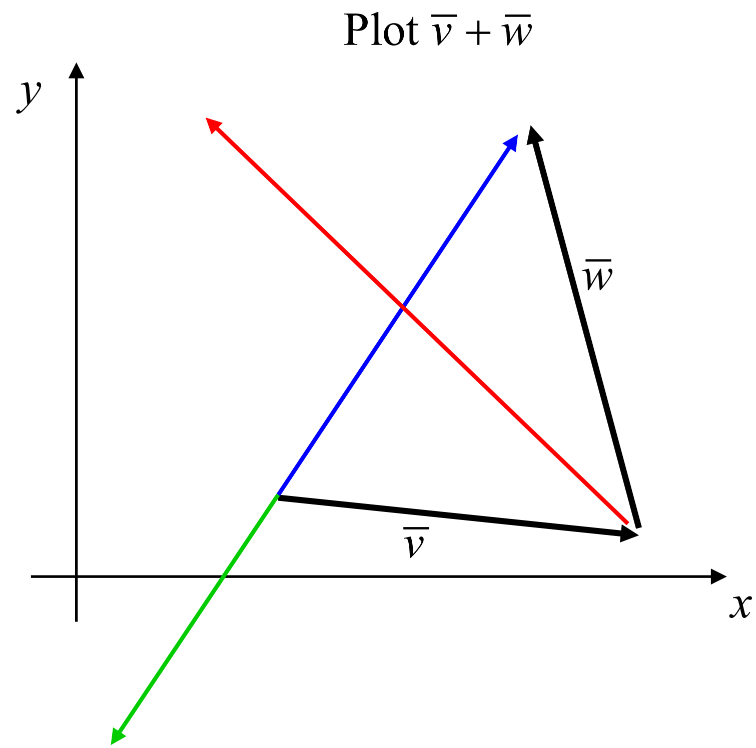

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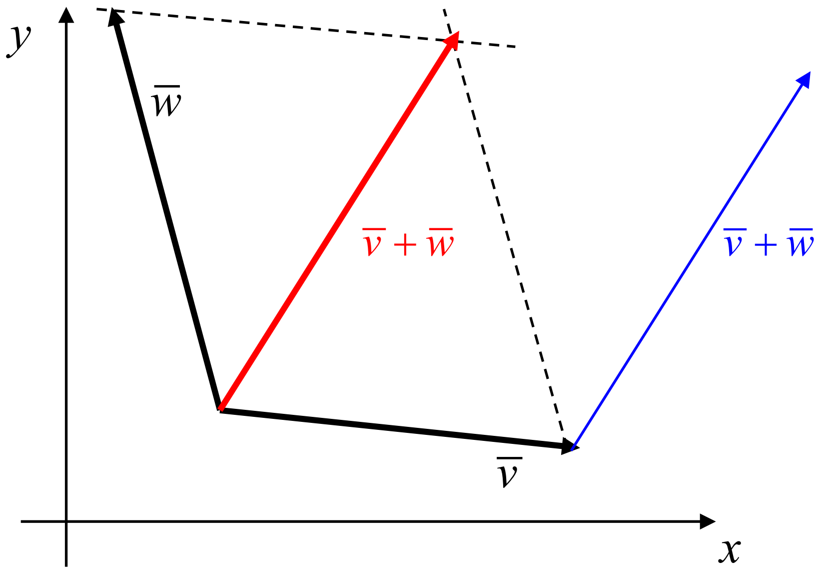

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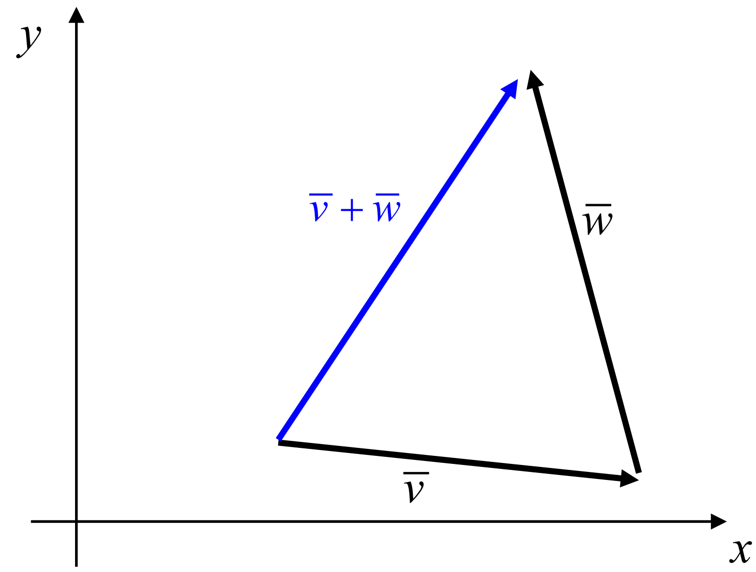

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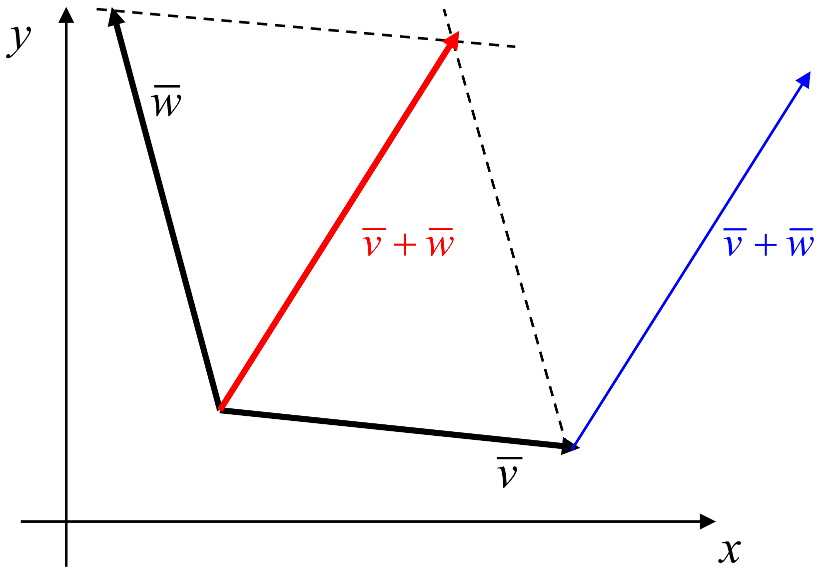

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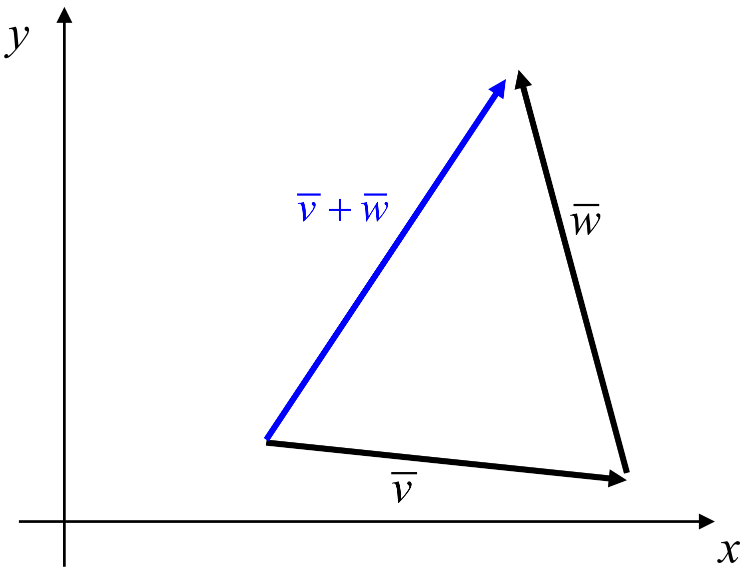
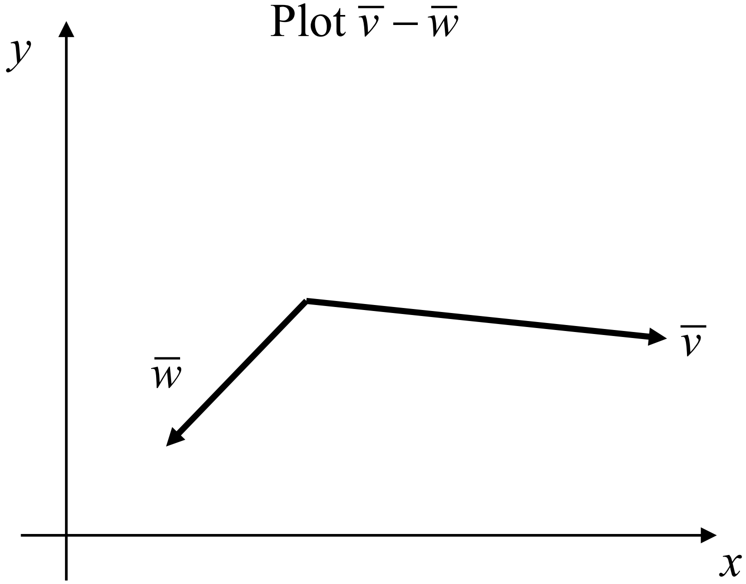


Plot $\overline{v} - \overline{w}$



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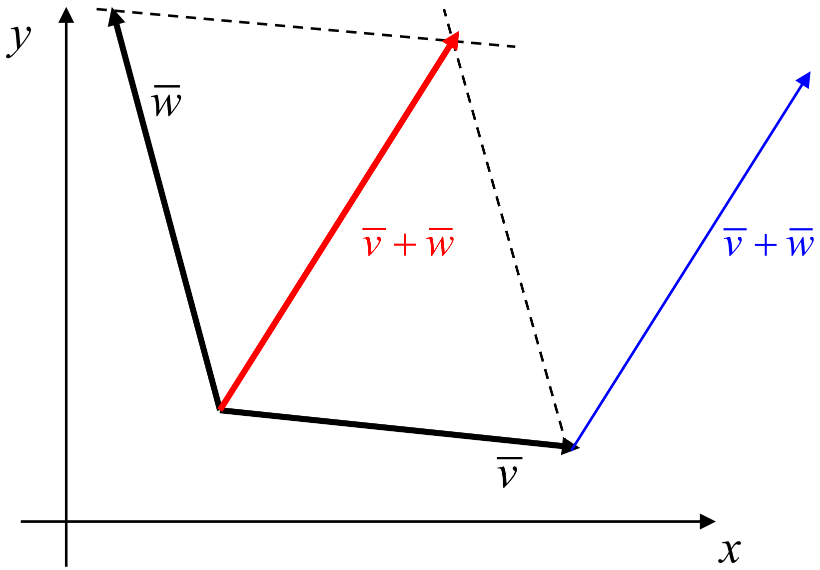

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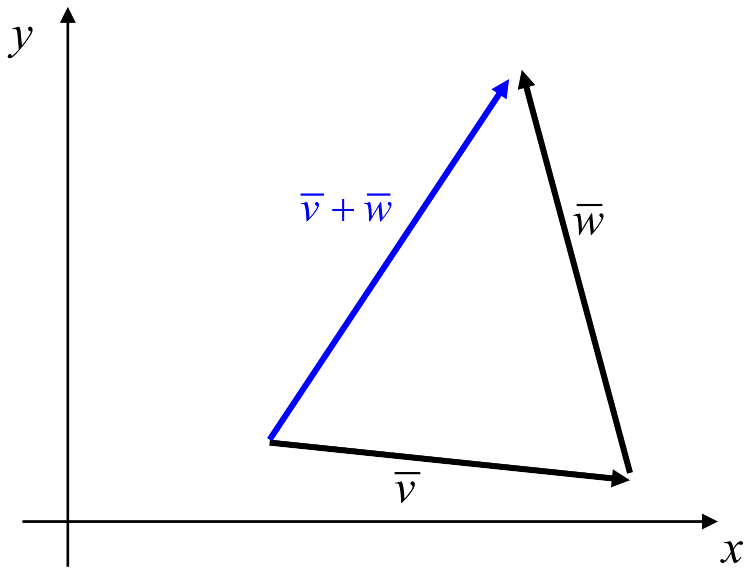
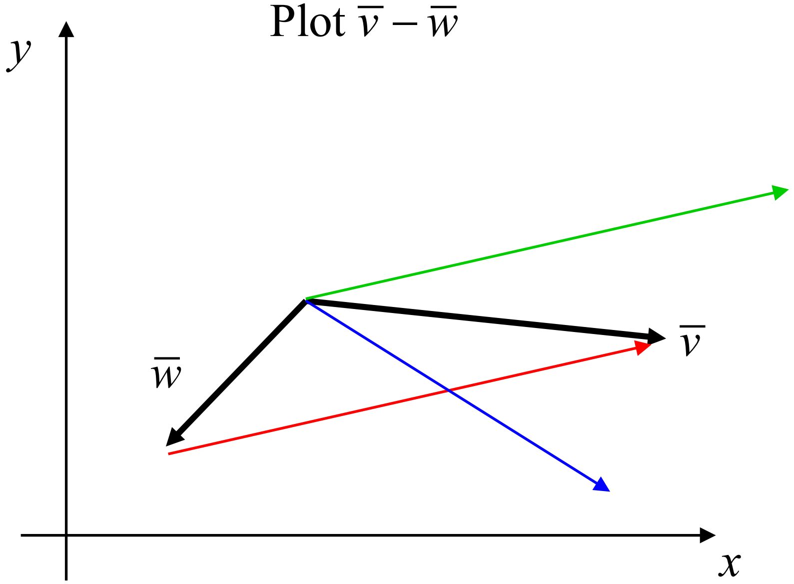


Plot $\overline{v}-\overline{w}$



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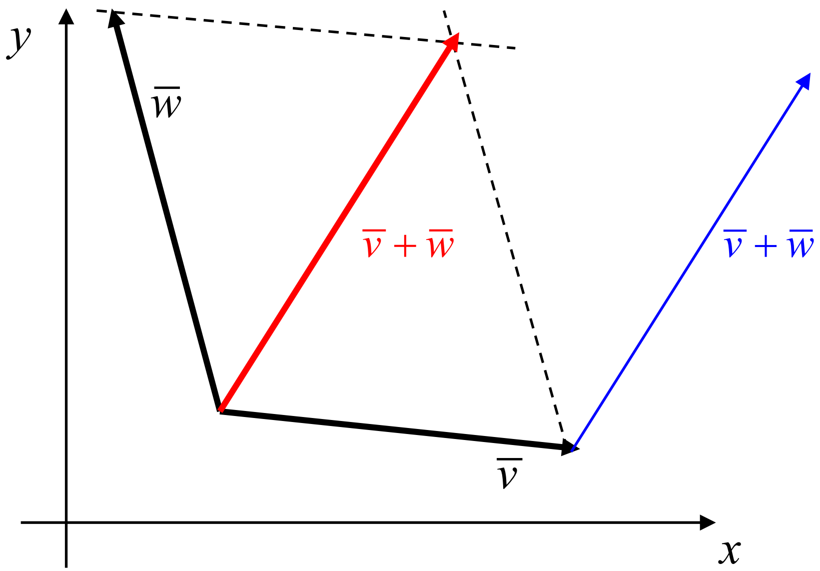

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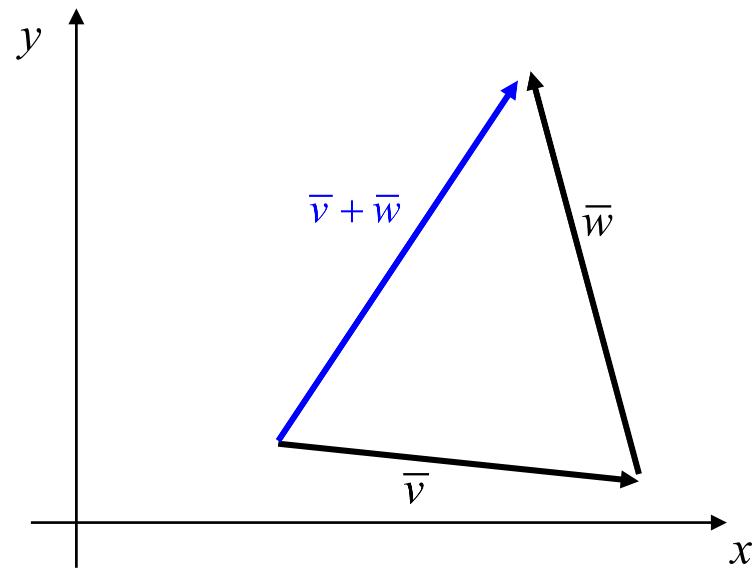

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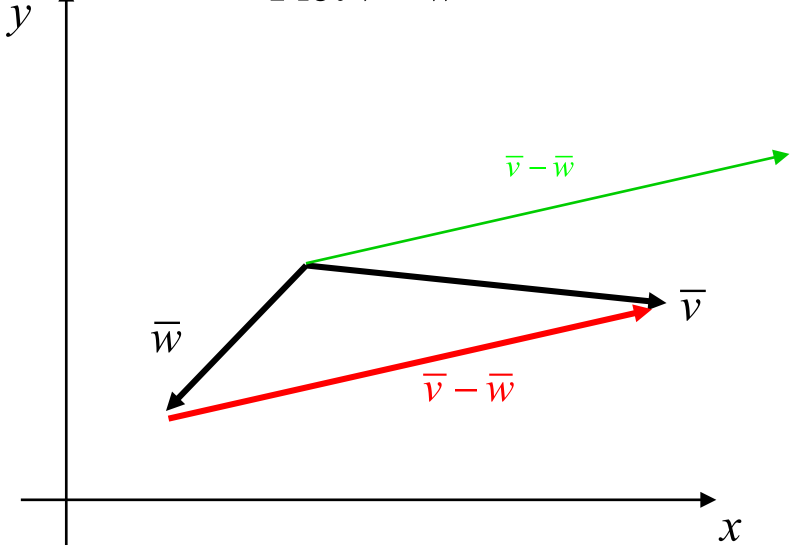

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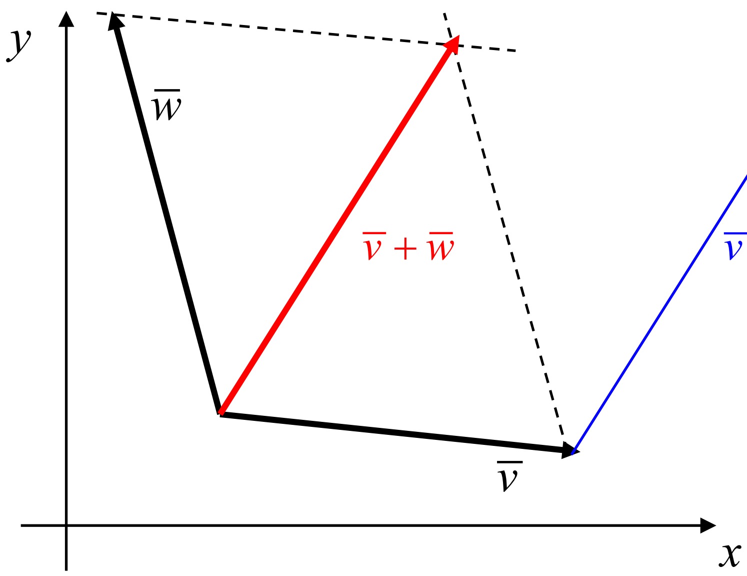

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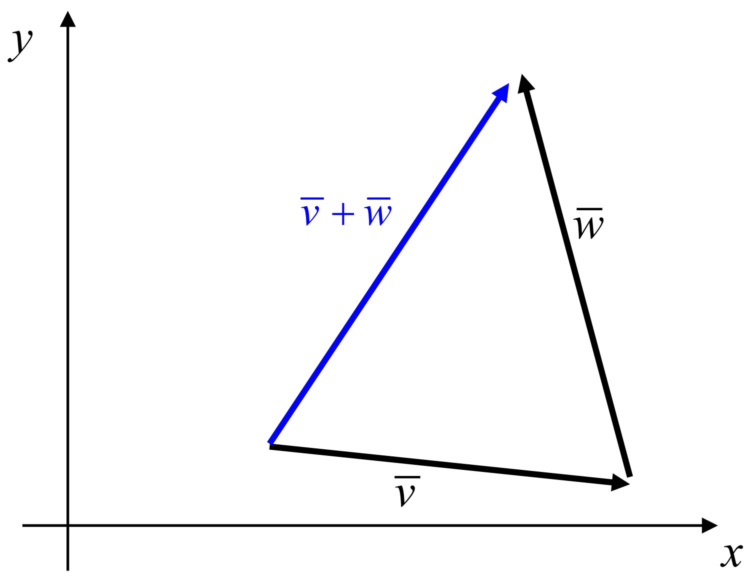

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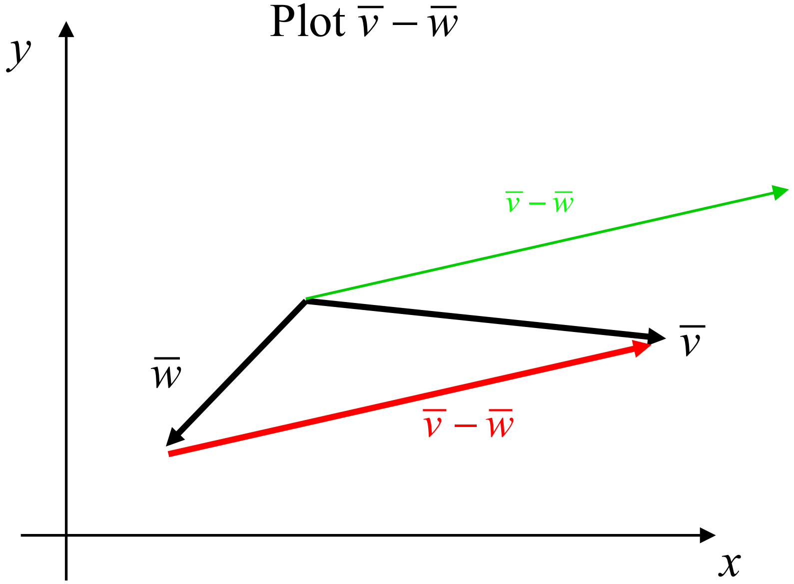

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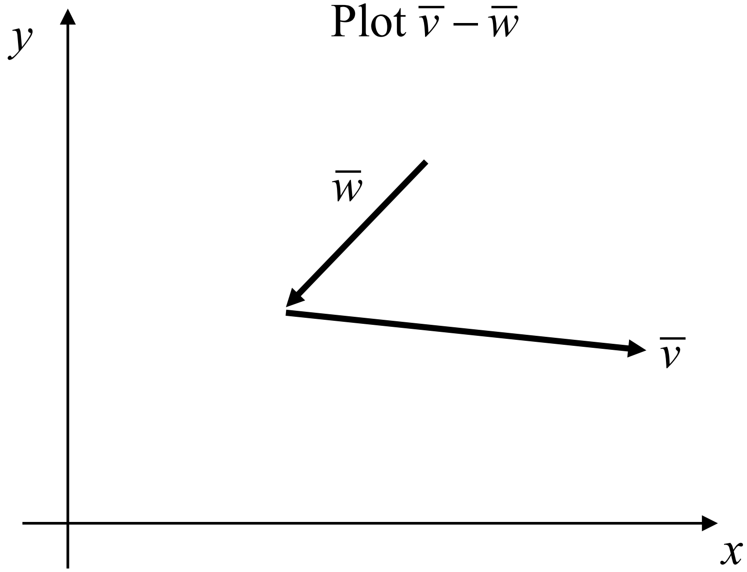

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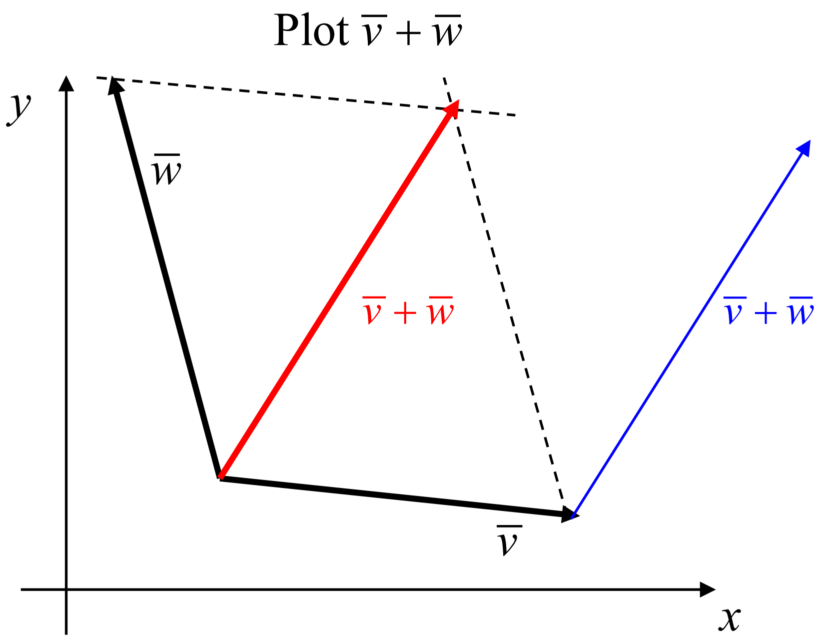

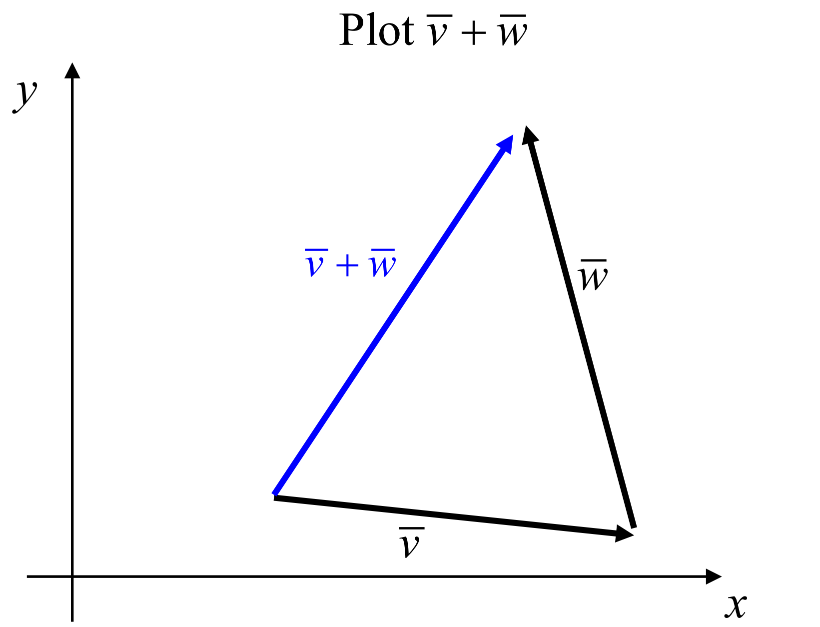

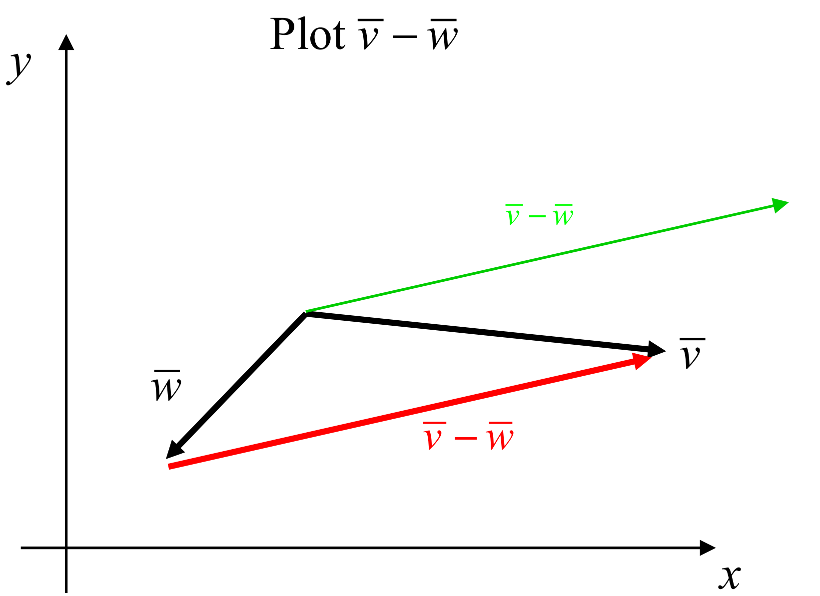

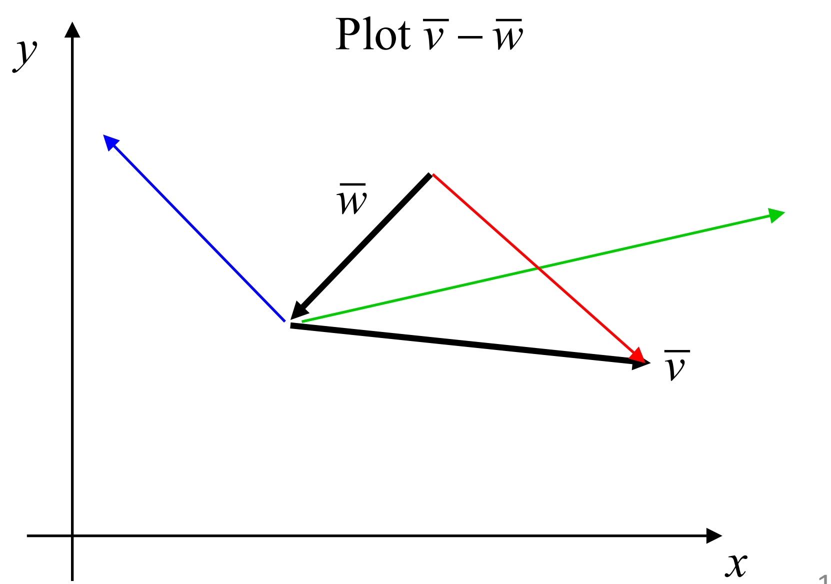

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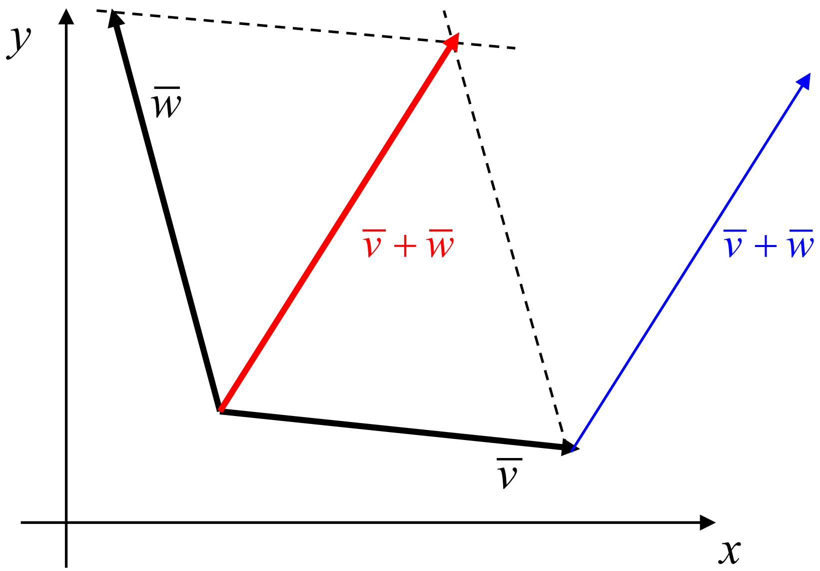

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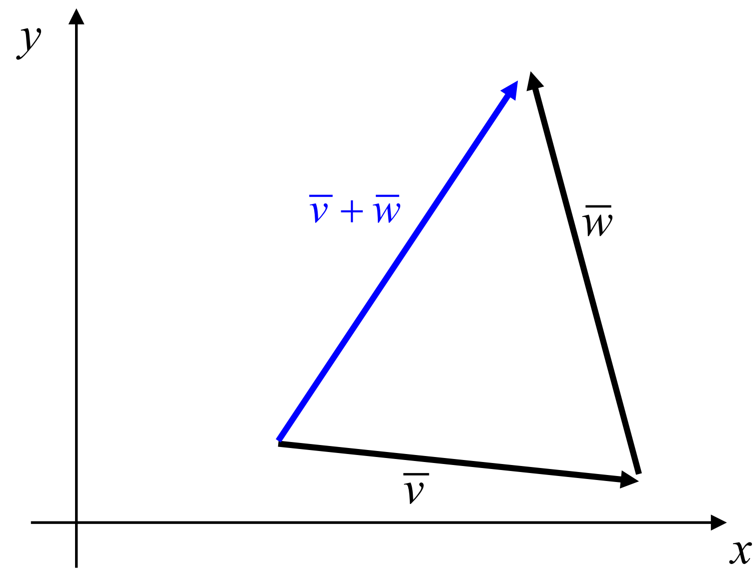

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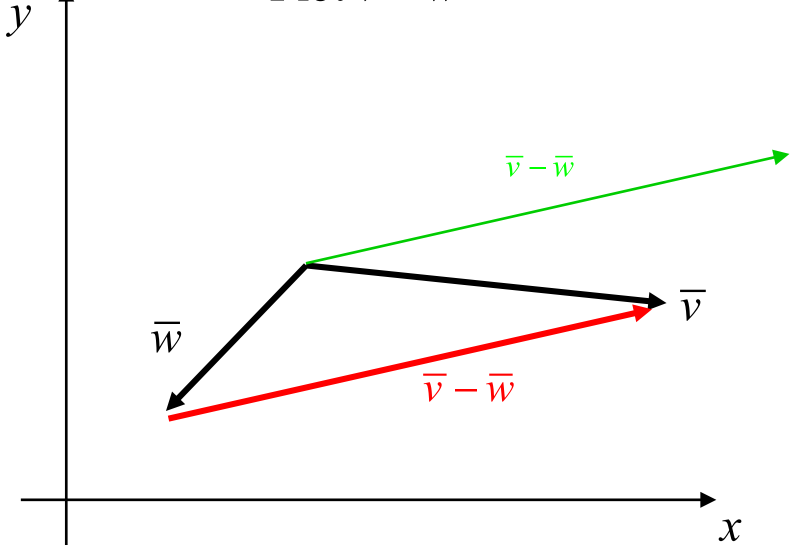

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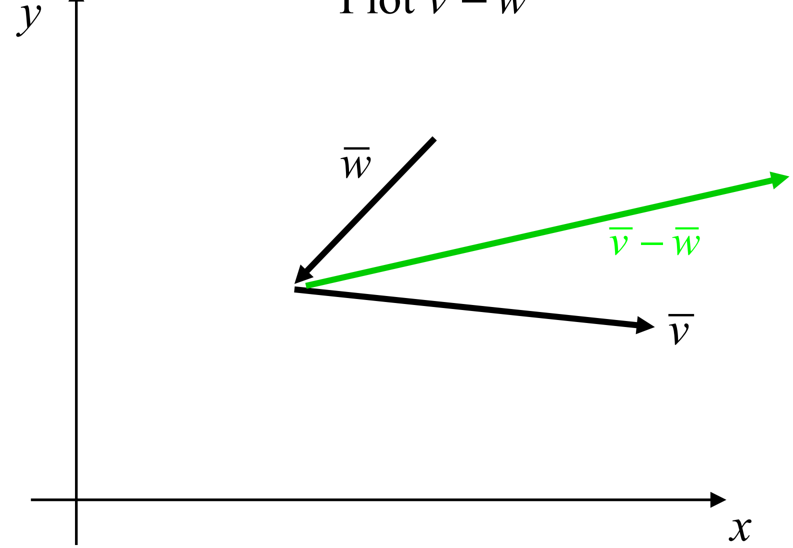


BASIS VECTORS IN CARTESIAN COORDINATES

The basis vectors are vectors of length 1 and direction along the axes.

In a Cartesian coordinate system, the basis vectors are:

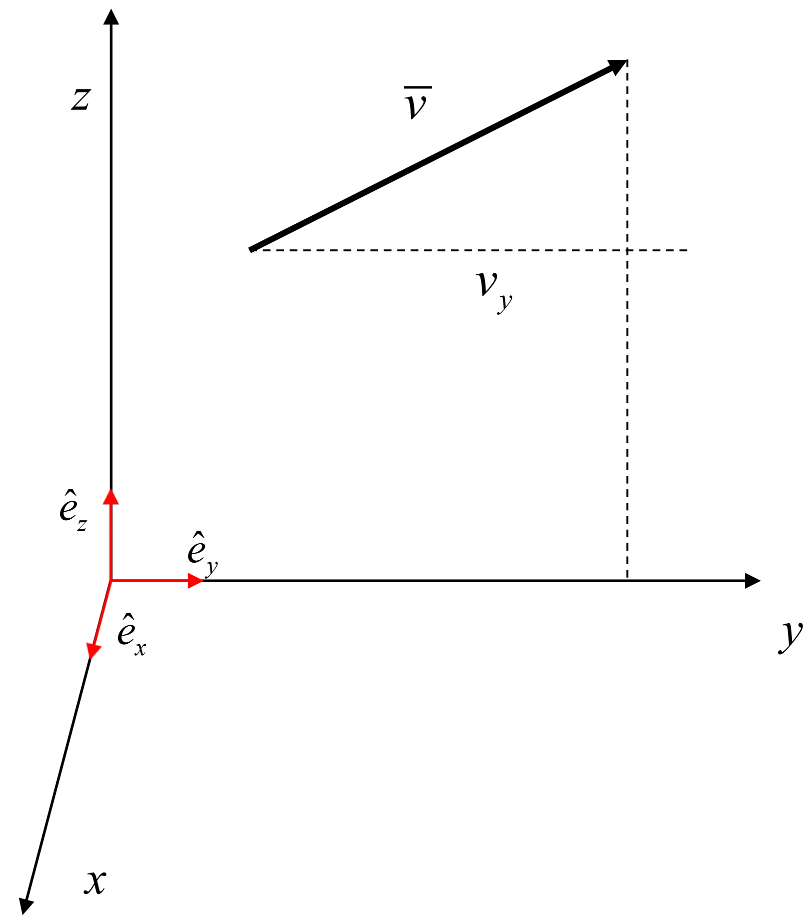
$$\hat{e}_x = (1,0,0) \quad \hat{e}_y = (0,1,0) \quad \hat{e}_z = (0,0,1)$$

Let's consider the vector $\overline{v} = (2,4,3)$ in Cartesian coordinates:

$$\overline{v} = (2,4,3) = (2,0,0) + (0,4,0) + (0,0,3) =$$
$$2(1,0,0) + (0,4,0) + 3(0,0,1) = 2\hat{e}_x + 4\hat{e}_y + 3\hat{e}_z$$

In general, **any vector can be represented using the basis vectors of the coordinate system:**

$$\boxed{\overline{w} = (\mathrm{a},\mathrm{b},\mathrm{c}) = a\hat{e}_x + b\hat{e}_y + c\hat{e}_z}$$



__Exercise:__

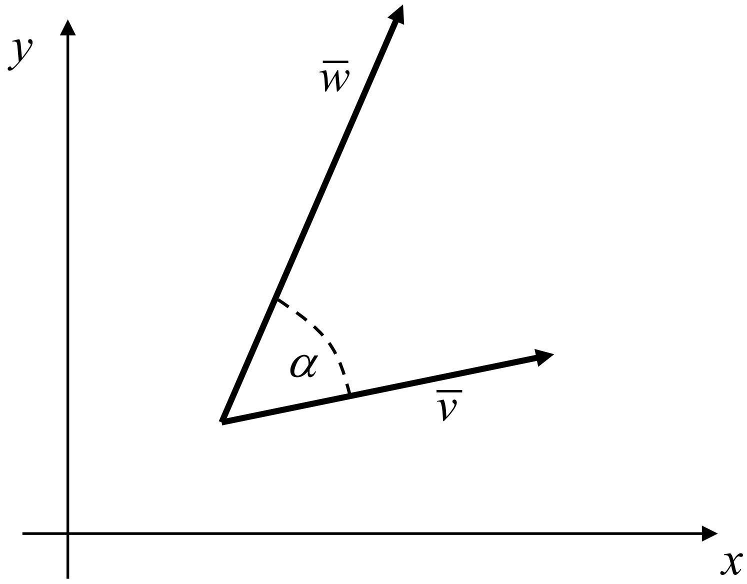
Use the scalar product and the basis vectors to express the y-component v_y of a vector $\overline{v}$:

$v_y =$

VECTORS: absolute value, scalar product, and cross product

Let's consider two vectors in Cartesian coordinates:

$$\overline{v} = v_x \hat{e}_x + v_y \hat{e}_y + v_z \hat{e}_z \qquad \overline{w} = w_x \hat{e}_x + w_y \hat{e}_y + w_z \hat{e}_z$$



Absolute value: $\left|\overline{v}\right| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

Scalar product:

$$c = \overline{v} \cdot \overline{w} = v_x w_x + v_y w_y + v_z w_z$$

$$c = \left|\overline{v}\right| \left|\overline{w}\right| \cos\alpha$$

therefore,

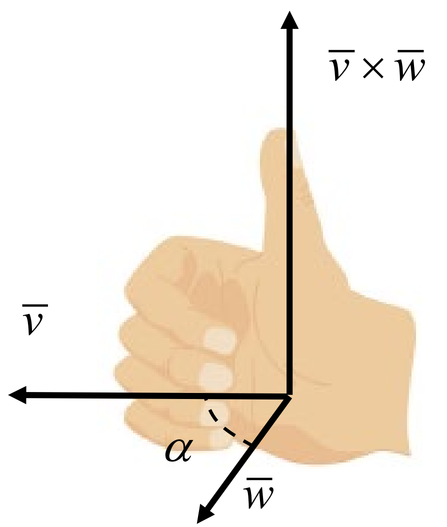
- the angle between two vectors can be calculated from: $\cos\alpha = \dfrac{\overline{v} \cdot \overline{w}}{\left|\overline{v}\right| \left|\overline{w}\right|}$
- the absolute value can be calculated as: $\left|\overline{v}\right| = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\overline{v} \cdot \overline{v}}$

Warning: never write $\bar{v}^2$. It is not clear which product you are using

Cross product:

$$\overline{v} \times \overline{w} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \left(v_y w_z - v_z w_y\right)\hat{e}_x + \left(v_z w_x - v_x w_z\right)\hat{e}_y + \left(v_x w_y - v_y w_x\right)\hat{e}_z$$

$$\left|\overline{v} \times \overline{w}\right| = \left|\overline{v}\right| \left|\overline{w}\right| \sin\alpha$$

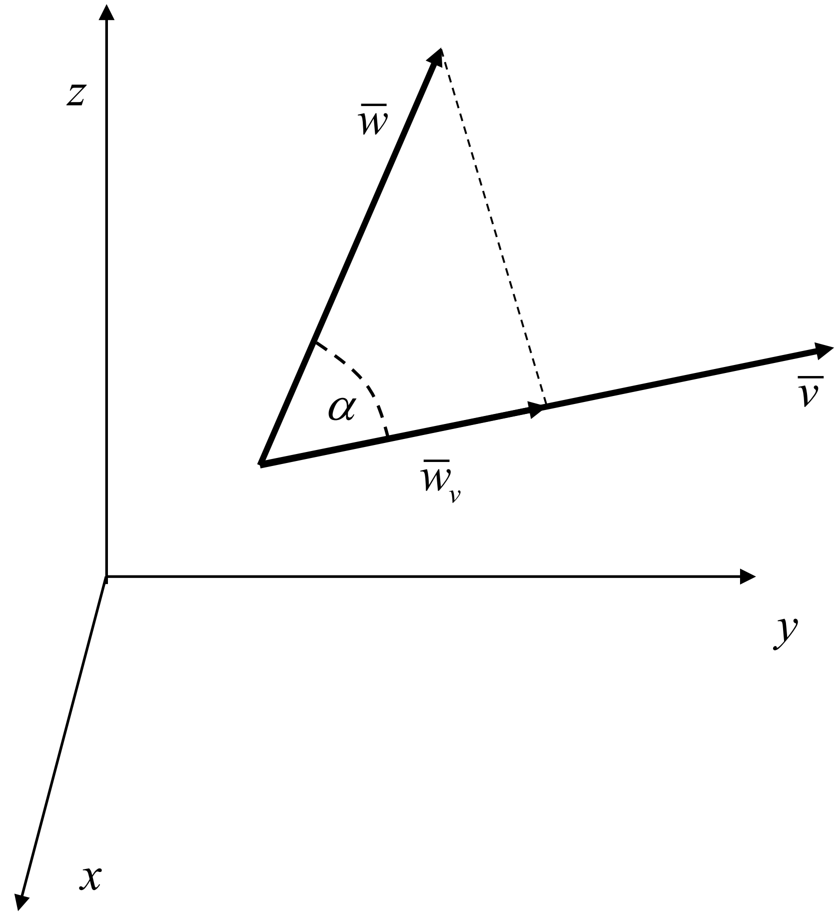


the direction is perpendicular to both $\overline{v}$ and $\overline{w}$ and the orientation is determined with the right hand rule

VECTORS: projections in the direction of another vector

Let's consider two vectors in Cartesian coordinates:

$$\overline{v} = v_x \hat{e}_x + v_y \hat{e}_y + v_z \hat{e}_z \qquad \overline{w} = w_x \hat{e}_x + w_y \hat{e}_y + w_z \hat{e}_z$$



The scalar projection of $\overline{w}$ in the direction of $\overline{v}$ is the scalar:

$$\boxed{w_v = |\overline{w}| \cos\alpha = \frac{\overline{w} \cdot \overline{v}}{|\overline{v}|}}$$

The vector projection of $\overline{w}$ in the direction of $\overline{v}$ is the vector:

$$\overline{w}_v = |\overline{w}| \cos\alpha \, \hat{e}_v$$

Exercise:

Prove that $\boxed{\overline{w}_v = \dfrac{\overline{w} \cdot \overline{v}}{|\overline{v}|^2} \overline{v}}$

You can use the expression above to prove that:

$$\boxed{\begin{aligned} a_x &= \overline{a} \cdot \hat{e}_x \\ a_y &= \overline{a} \cdot \hat{e}_y \\ a_z &= \overline{a} \cdot \hat{e}_z \end{aligned}}$$

VECTORS: distance between two points

Let's consider two position vectors $\overline{v}, \overline{w}$ that identify two points, P and Q.

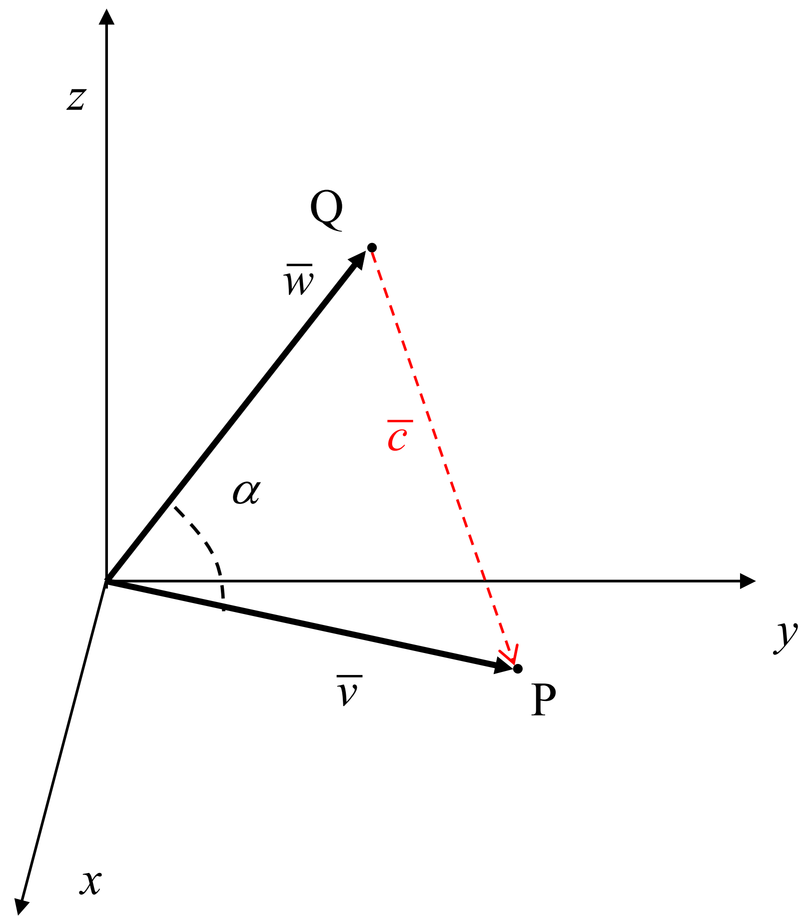
The distance between P and Q is the length L of the vector $\overline{c} = \overline{v} - \overline{w}$

$$L = |\overline{c}| = \sqrt{\overline{c} \cdot \overline{c}} = \sqrt{(\overline{v} - \overline{w}) \cdot (\overline{v} - \overline{w})} =$$

$$= \sqrt{\overline{v} \cdot \overline{v} - \overline{v} \cdot \overline{w} - \overline{w} \cdot \overline{v} + \overline{w} \cdot \overline{w}} =$$

$$= \sqrt{|\overline{v}|^2 + |\overline{w}|^2 - 2\overline{v} \cdot \overline{w}}$$

$$\boxed{L = \sqrt{|\overline{v}|^2 + |\overline{w}|^2 - 2\overline{v} \cdot \overline{w}}}$$



Warning: never write $\bar{v}^2$. It is not clear which product you are using.

CYLINDRICAL COORDINATE SYSTEMS

A point P can be identified by the coordinates:

x, y, z (Cartesian coordinates)

ρ, ϕ, z (cylindrical coordinates)

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases}$$

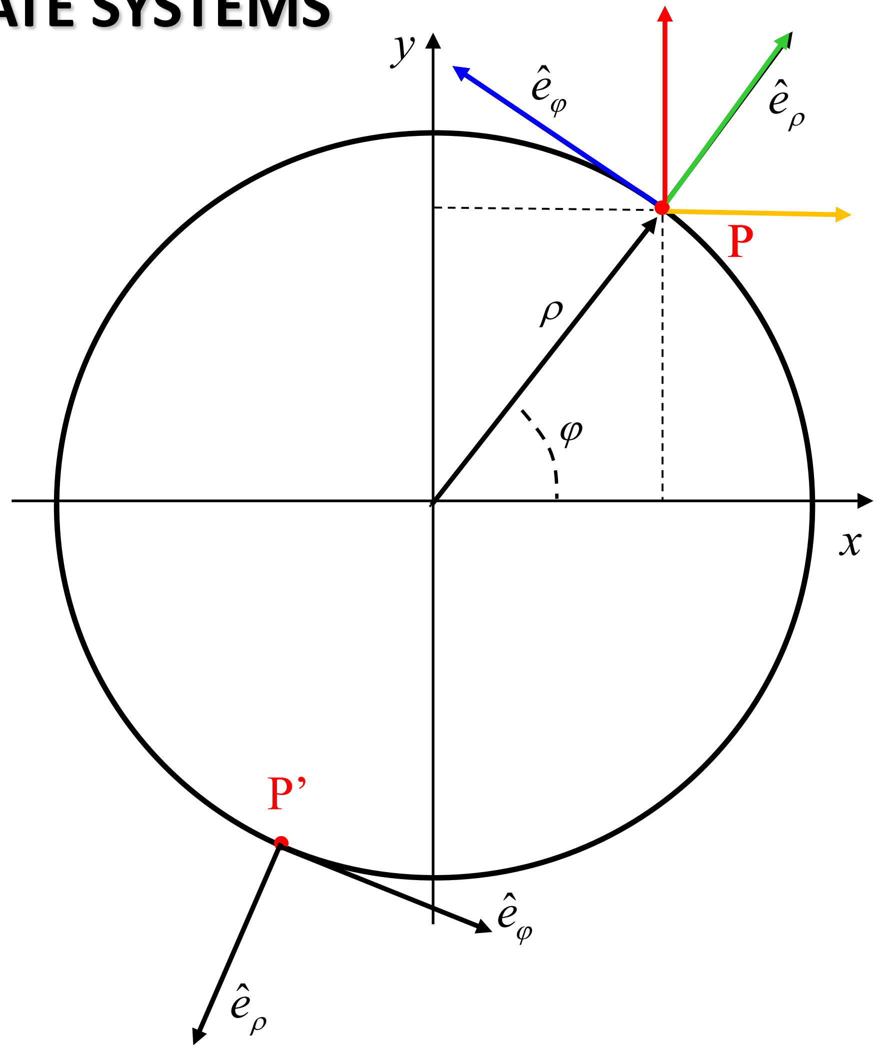
The basis vectors are:

$\hat{e}_x, \hat{e}_y, \hat{e}_z$ in the Cartesian coordinate system

$\hat{e}_\rho, \hat{e}_\varphi, \hat{e}_z$ in the cylindrical coordinate system

The direction of the basis vectors in a cylindrical coordinate system depends on the position.

$$\begin{cases} \hat{e}_\rho = \cos\varphi\hat{e}_x + \sin\varphi\hat{e}_y \\ \hat{e}_\varphi = -\sin\varphi\hat{e}_x + \cos\varphi\hat{e}_y \\ \hat{e}_z = \hat{e}_z \end{cases}$$



IMPORTANT: The basis vectors in a cylindrical coordinate system are orthonormal:

$$\begin{cases} \hat{e}_\rho\cdot\hat{e}_\varphi = (\cos\varphi\hat{e}_x + \sin\varphi\hat{e}_y)\cdot(-\sin\varphi\hat{e}_x + \cos\varphi\hat{e}_y) = -\sin\varphi\cos\varphi + \sin\varphi\cos\varphi = 0 \\ \hat{e}_\rho\cdot\hat{e}_z = (\cos\varphi\hat{e}_x + \sin\varphi\hat{e}_y)\cdot\hat{e}_z = 0 \\ \hat{e}_\varphi\cdot\hat{e}_z = (-\sin\varphi\hat{e}_x + \cos\varphi\hat{e}_y)\cdot\hat{e}_z = 0 \end{cases}$$

$$\begin{cases} \hat{e}_\rho\cdot\hat{e}_\rho = 1 \\ \hat{e}_\varphi\cdot\hat{e}_\varphi = 1 \\ \hat{e}_z\cdot\hat{e}_z = 1 \end{cases}$$

Example: THE MAGNETIC FIELD AROUND A STRAIGHT WIRE

The magnetic field $\overline{B}$ around a straight wire which carries an electric current I depends on the distance from the wire. The amplitude of the magnetic field is:

$$\left|\overline{B}\right| = \frac{\mu_0 I}{2\pi\rho}$$

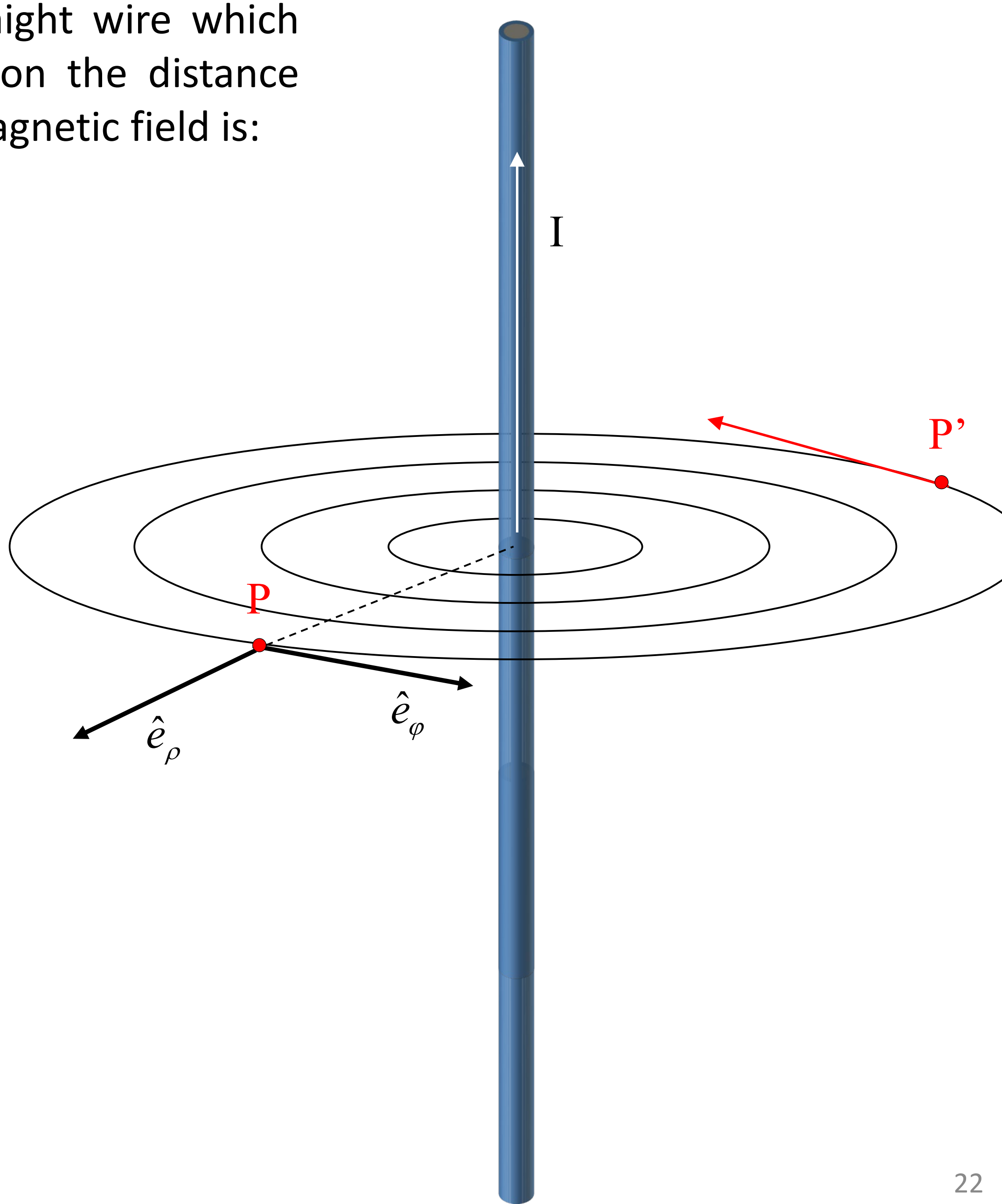
The direction is perpendicular to the wire, in the azimuthal direction. So, it is more convenient to express the filed using cylindrical coordinates:

$$\overline{B} = \frac{\mu_0 I}{2\pi\rho}\hat{e}_\varphi$$

Note that: $\hat{e}_\varphi$ **depends on the position.** The direction of $\overline{B}$ in P is different from the direction in P'.

In cartesian coordinates, the expression of the field looks more complicated:

$$\bar{B} = \frac{\mu_0 I}{2\pi}\left(\frac{-y\hat{e}_x + x\hat{e}_y}{x^2 + y^2}\right)$$



ADDITION OF VECTORS DEFINED IN DIFFERENT COORDINATE SYSTEMS

Consider two vectors:

$\overline{v} = (2,1,0)$ in the Cartesian coordinate system

$\overline{w} = (2,0,0)$ in the cylindrical coordinate system

Is this correct: $\cancel{\overline{v} + \overline{w} = (2,1,0) + (2,0,0) = (4,1,0)}$? **NO**

Let's rewrite the vectors using the basis of the coordinate systems:

$$\overline{v} = (2,1,0) = 2\hat{e}_x + \hat{e}_y$$

$$\overline{w} = (2,0,0) = 2\hat{e}_\rho$$

$$\overline{v} + \overline{w} = 2\hat{e}_x + \hat{e}_y + 2\hat{e}_\rho = 2\hat{e}_x + \hat{e}_y + 2(\cos\varphi\hat{e}_x + \sin\varphi\hat{e}_y) =$$

$$= (2 + 2\cos\varphi)\hat{e}_x + (1 + 2\sin\varphi)\hat{e}_y$$

It is always convenient to express a vector using the basis of the coordinate system.

It will avoid major errors!

CYLINDRICAL COORDINATE SYSTEMS: the position vector

The position vector $\overline{r}$ of a point P is a vector from the origin to the point P.
In general, the position vector in Cartesian coordinates x,y,z is expressed as:

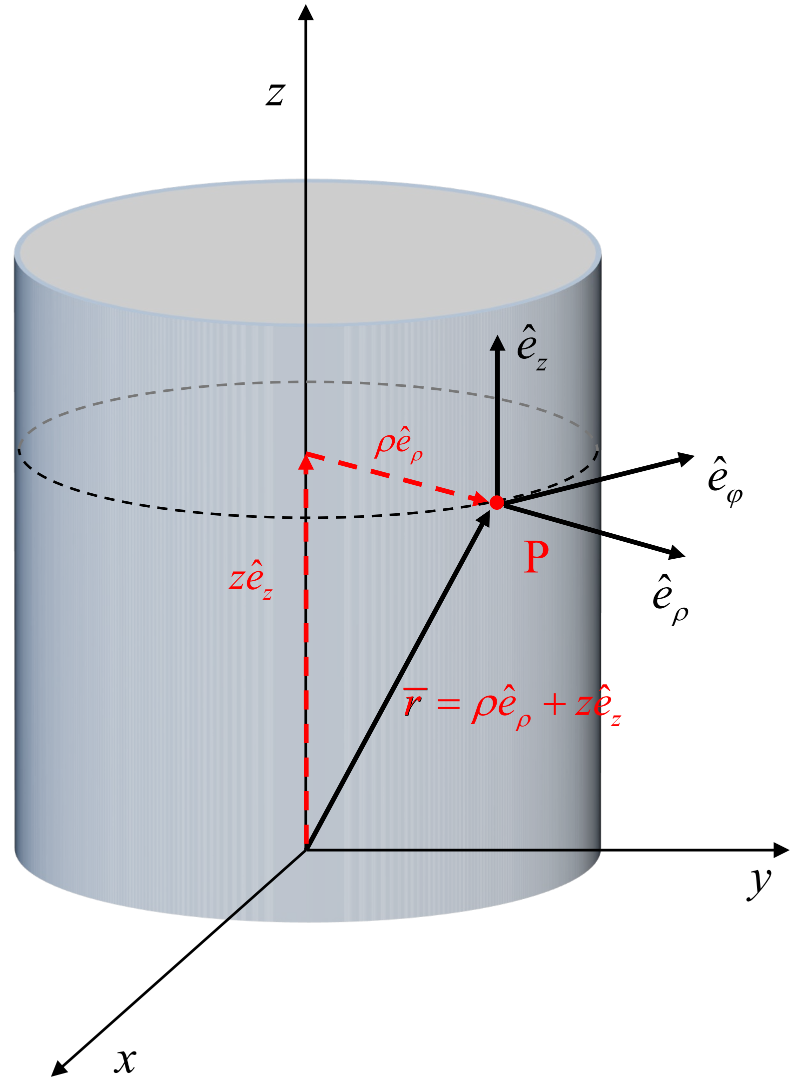
$$\overline{r} = (x, y, z) = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$$

Now, consider a cylindrical coordinate system ρ, φ, z.
Is it correct to say that the position vector in a cylindrical coordinate system can be expressed as:

$$\cancel{\overline{r} = (\rho, \varphi, z) = \rho\hat{e}_\rho + \varphi\hat{e}_\varphi + z\hat{e}_z} \text{ ?}$$

No!

The position vector in cylindrical coordinate is: $\boxed{\overline{r} = \rho\hat{e}_\rho + z\hat{e}_z}$



CYLINDRICAL COORDINATE SYSTEMS: differential elements

Assume that the radius of the cylinder is ρ_0, and the height z_0. The arc l defined by the angle φ on the circumference C has length: $l=\varphi\rho$.
The differential elements are:

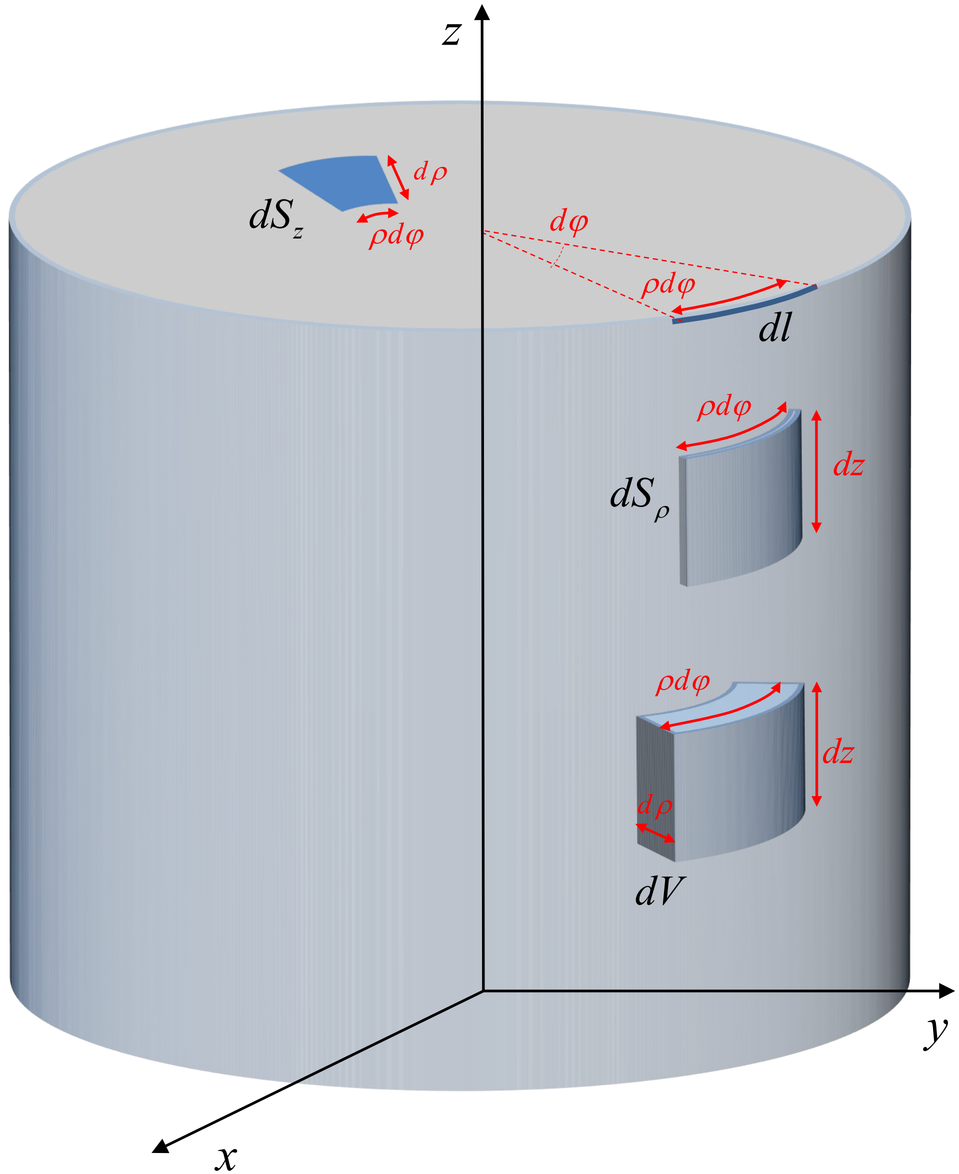
$$\boxed{\begin{aligned} dl &= \rho d\varphi \\ dS_z &= \rho d\varphi d\rho \\ dS_\rho &= \rho d\varphi dz \\ dV &= \rho d\varphi d\rho dz \end{aligned}}$$

$$C = \int dl = \int_0^{2\pi} \rho_0 d\varphi = 2\pi\rho_0$$

$$S_z = \int dS_z = \int_0^{\rho_0}\int_0^{2\pi} \rho d\varphi d\rho = \pi\rho_0^2$$

$$S_\rho = \int dS_\rho = \int_0^{z_0}\int_0^{2\pi} \rho_0 d\varphi dz = 2\pi\rho_0 z_0$$

$$V = \int dV = \int_0^{z_0}\int_0^{\rho_0}\int_0^{2\pi} \rho d\varphi d\rho dz = \pi z_0 \rho_0^2$$



SPHERICAL COORDINATE SYSTEMS

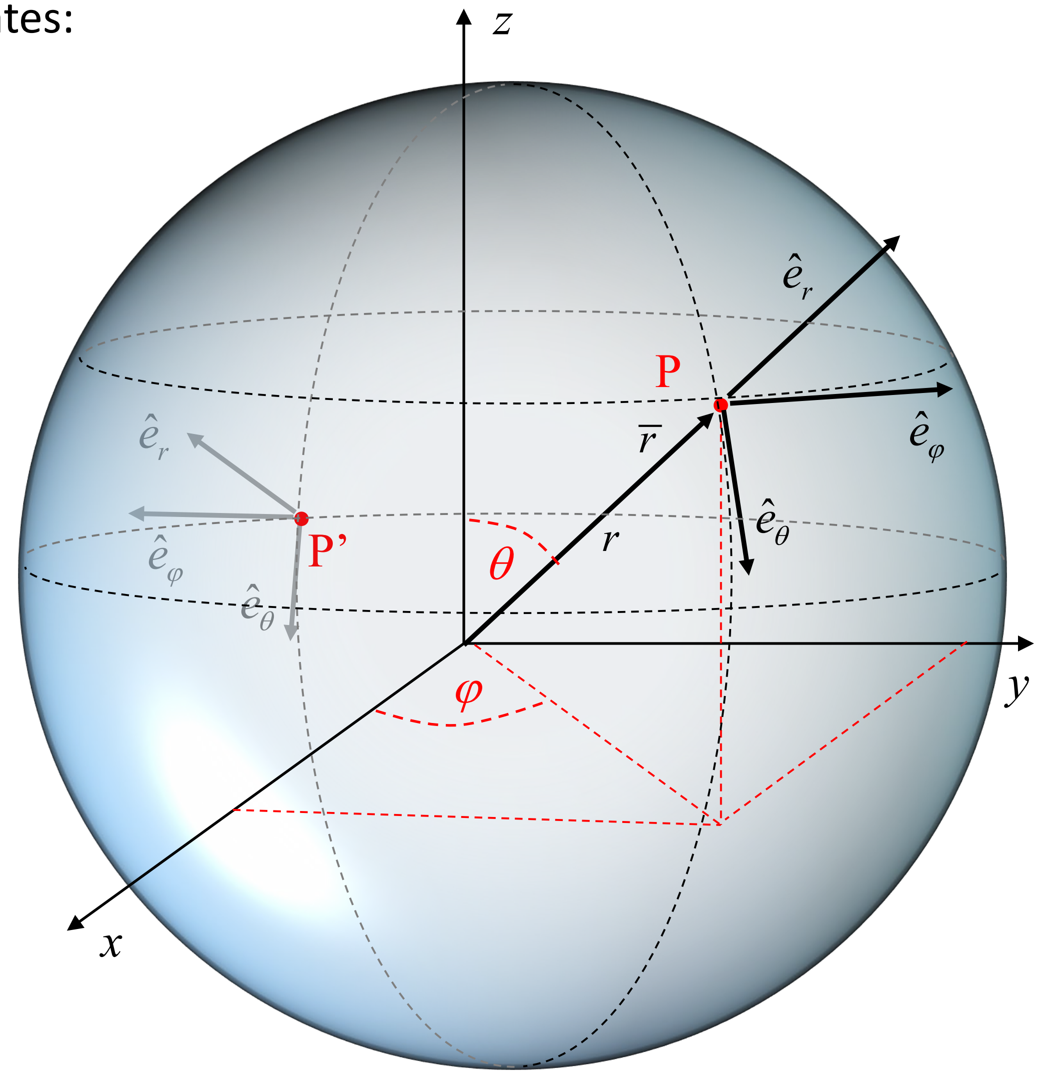
A point P can be identified by the coordinates:

x, y, z (Cartesian coordinate system)

r, θ, ϕ (spherical coordinate system)

$$0 \le r \le \infty$$
$$0 \le \theta \le \pi$$
$$0 \le \varphi \le 2\pi$$

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$



The basis vectors are:

$\hat{e}_x, \hat{e}_y, \hat{e}_z$ in the Cartesian coordinate system

$\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi$ in the spherical coordinate system

The direction of the basis vectors in a cylindrical coordinate system depends on the position.

$$\begin{cases} \hat{e}_r = \sin\theta\cos\varphi\hat{e}_x + \sin\theta\sin\varphi\hat{e}_y + \cos\theta\hat{e}_z \\ \hat{e}_\theta = \cos\theta\cos\varphi\hat{e}_x + \cos\theta\sin\varphi\hat{e}_y - \sin\theta\hat{e}_z \\ \hat{e}_\varphi = -\sin\varphi\hat{e}_x + \cos\varphi\hat{e}_y \end{cases}$$

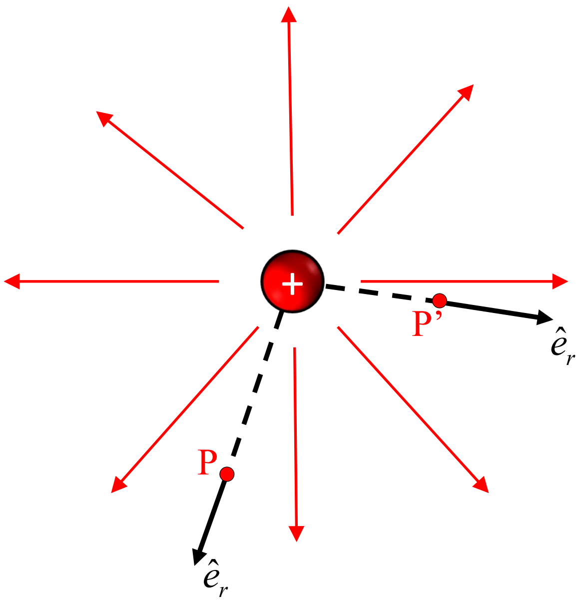
IMPORTANT. The basis vectors in a spherical coord. sys. are orthonormal:

$$\hat{e}_r \cdot \hat{e}_\theta = 0, \quad \hat{e}_r \cdot \hat{e}_\varphi = 0, \quad \hat{e}_\theta \cdot \hat{e}_\varphi = 0 \qquad \hat{e}_r \cdot \hat{e}_r = 1, \quad \hat{e}_\theta \cdot \hat{e}_\theta = 1, \quad \hat{e}_\varphi \cdot \hat{e}_\varphi = 1$$

Example: THE ELECTRIC FIELD PRODUCED BY A POINT CHARGE

The electric field $\overline{E}$ produced by a point charge with electric charge Q has amplitude:

$$\left|\overline{E}\right| = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$



and, if the charge is located in the origin, its direction is radial. So, it is convenient to use a spherical coordinate system to express the electric field:

$$\overline{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{e}_r$$

Note: $\hat{e}_r$ **depends on the position!** The direction of $\overline{E}$ in P is different from the direction in P'.

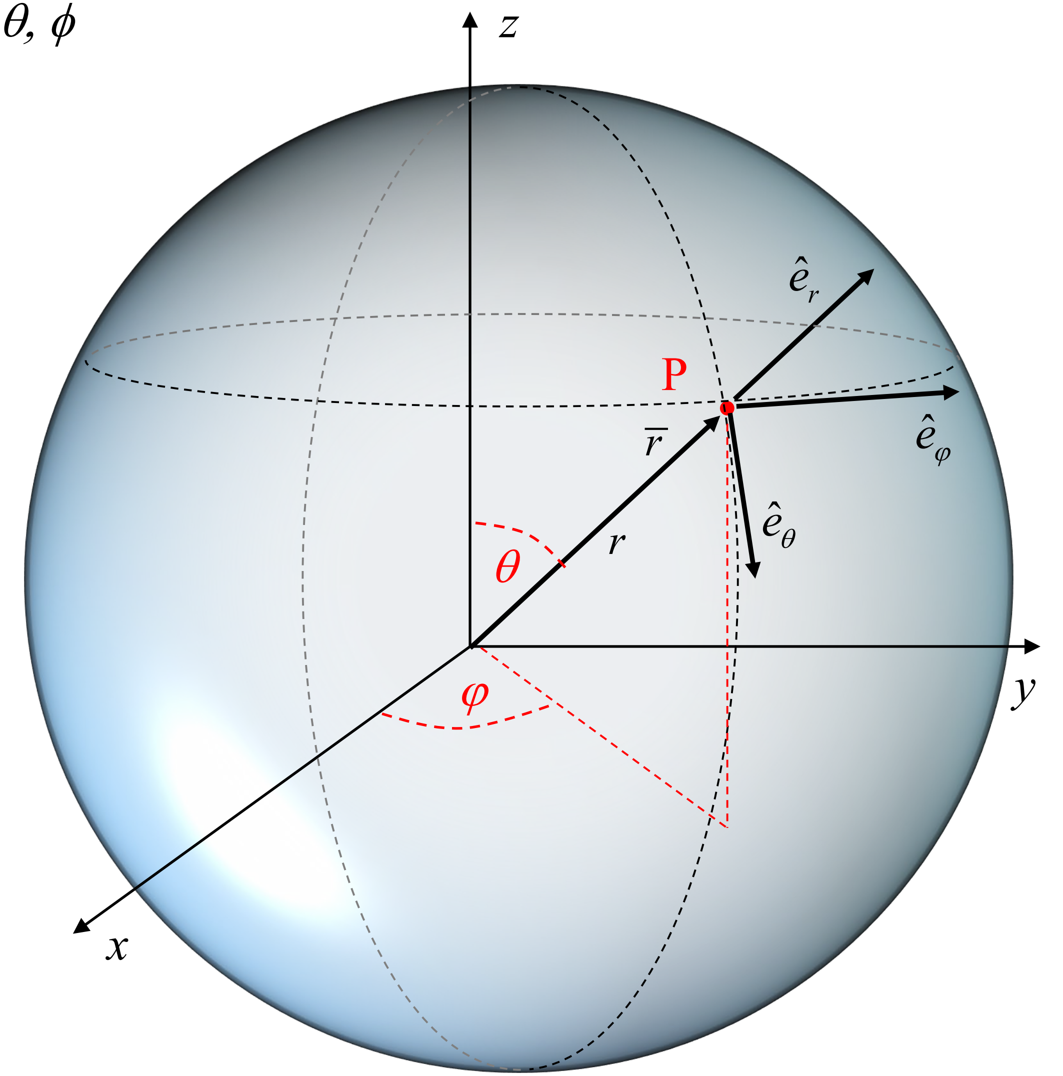
In cartesian coordinates, the expression of the electric field looks more complicated:

$$\overline{E} = \frac{Q}{4\pi\varepsilon_0}\frac{x\hat{e}_x + y\hat{e}_y + z\hat{e}_z}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$\Rightarrow$It is much more convenient to use spherical coordinates

SPHERICAL COORDINATE SYSTEMS: the position vector

Consider a spherical coordinate system $r,\ \theta,\ \phi$



Is the position vector in a spherical coordinate system:

$$\cancel{\overline{r} = (r,\theta,\varphi) = r\hat{e}_\rho + \theta\hat{e}_\theta + \varphi\hat{e}_\varphi}\ \ ?$$

No!

<u>Exercise:</u>
express the position vector in a spherical coordinate system.

The position vector in a spherical coordinate system is: $\boxed{\overline{r} = r\hat{e}_r}$

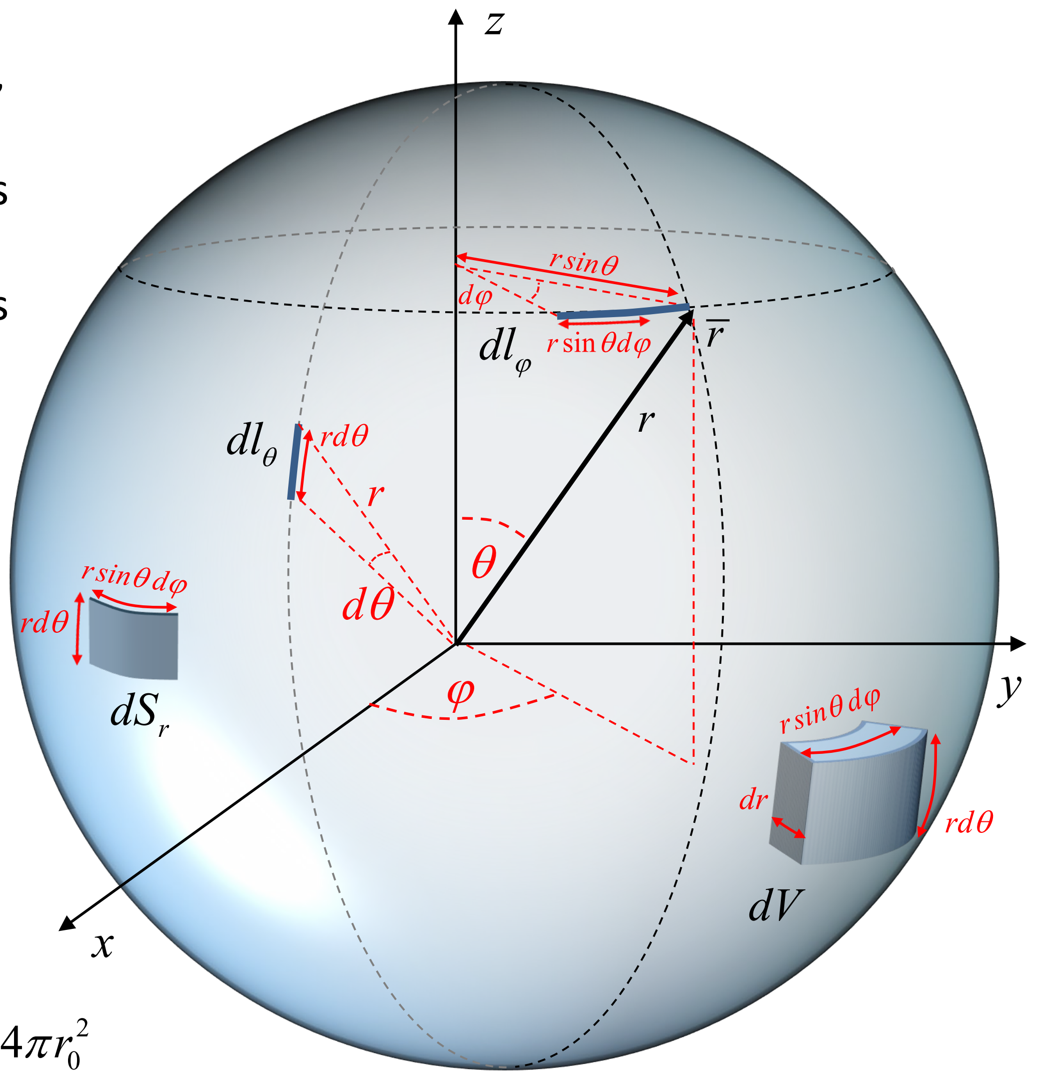
SPHERICAL COORDINATE SYSTEMS: differential elements

Assume that the radius of the sphere is r_0,

- The arc l_θ parallel to the x-z plane has length: $l_\theta = \theta r$.
- The arc l_φ parallel to the x-y plane has length: $l_\varphi = \varphi r \sin\theta$.

The differential elements are:

$$\boxed{\begin{aligned} dl_\theta &= r d\theta \\ dl_\varphi &= r\sin\theta d\varphi \\ dS_r &= r^2 \sin\theta d\varphi d\theta \\ dV &= r^2 \sin\theta d\varphi d\theta dr \end{aligned}}$$



$$S_r = \int dS_r = \int_0^\pi \int_0^{2\pi} r_0^2 \sin\theta d\theta d\varphi = 4\pi r_0^2$$

$$V = \int dV = \int_0^{r_0} \int_0^\pi \int_0^{2\pi} r_0^2 \sin\theta d\theta d\varphi dr = \frac{4}{3}\pi r_0^3$$

SCALAR PRODUCT IN
CYLINDRICAL AND SPHERICAL COORDINATE SYSTEMS

The scalar product in cylindrical and spherical coordinate systems can be calculated in a way similar to the Cartesian. This is because **the basis vectors are orthonormal**.

cartesian coordinate system:

$$\left.\begin{aligned} \overline{v} &= v_x\hat{e}_x + v_y\hat{e}_y + v_z\hat{e}_z \\ \overline{w} &= w_x\hat{e}_x + w_y\hat{e}_y + w_z\hat{e}_z \end{aligned}\right\} \Rightarrow \boxed{\overline{v}\cdot\overline{w} = v_x w_x + v_y w_y + v_z w_z}$$

cylindrical coordinate system:

$$\left.\begin{aligned} \overline{v} &= v_\rho\hat{e}_\rho + v_\varphi\hat{e}_\varphi + v_z\hat{e}_z \\ \overline{w} &= w_\rho\hat{e}_\rho + w_\varphi\hat{e}_\varphi + w_z\hat{e}_z \end{aligned}\right\} \Rightarrow \overline{v}\cdot\overline{w} = (v_\rho\hat{e}_\rho + v_\varphi\hat{e}_\varphi + v_z\hat{e}_z)\cdot(w_\rho\hat{e}_\rho + w_\varphi\hat{e}_\varphi + w_z\hat{e}_z) =$$

$$\begin{aligned} = {} & v_\rho\hat{e}_\rho\cdot w_\rho\hat{e}_\rho + v_\rho\hat{e}_\rho\cdot w_\varphi\hat{e}_\varphi + v_\rho\hat{e}_\rho\cdot w_z\hat{e}_z + \\ & + v_\varphi\hat{e}_\varphi\cdot w_\rho\hat{e}_\rho + v_\varphi\hat{e}_\varphi\cdot w_\varphi\hat{e}_\varphi + v_\varphi\hat{e}_\varphi\cdot w_z\hat{e}_z + \\ & + v_z\hat{e}_z\cdot w_\rho\hat{e}_\rho + v_z\hat{e}_z\cdot w_\varphi\hat{e}_\varphi + v_z\hat{e}_z\cdot w_z\hat{e}_z \end{aligned}$$

=0

=1

Remember that:

$$\begin{cases} \hat{e}_\rho\cdot\hat{e}_\varphi = 0 \\ \hat{e}_\rho\cdot\hat{e}_z = 0 \\ \hat{e}_\varphi\cdot\hat{e}_z = 0 \end{cases} \qquad \begin{cases} \hat{e}_\rho\cdot\hat{e}_\rho = 1 \\ \hat{e}_\varphi\cdot\hat{e}_\varphi = 1 \\ \hat{e}_z\cdot\hat{e}_z = 1 \end{cases}$$

$$\Rightarrow \boxed{\overline{v}\cdot\overline{w} = v_\rho w_\rho + v_\varphi w_\varphi + v_z w_z}$$

spherical coordinate system:

(in a way similar to the cylindrical, one can prove that:)

$$\left.\begin{aligned} \overline{v} &= v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_\varphi\hat{e}_\varphi \\ \overline{w} &= w_r\hat{e}_r + w_\theta\hat{e}_\theta + w_\varphi\hat{e}_\varphi \end{aligned}\right\} \Rightarrow \boxed{\overline{v}\cdot\overline{w} = v_r w_r + v_\theta w_\theta + v_\varphi w_\varphi}$$

CROSS PRODUCT IN
CYLINDRICAL AND SPHERICAL COORDINATE SYSTEMS

The cross product in cylindrical and spherical coordinate systems can be calculated in a way similar to the Cartesian. This is because **the basis vectors are orthonormal**.

cartesian coordinate system:

$$\left.\begin{aligned}\overline{v} &= v_x\hat{e}_x + v_y\hat{e}_y + v_z\hat{e}_z \\ \overline{w} &= w_x\hat{e}_x + w_y\hat{e}_y + w_z\hat{e}_z\end{aligned}\right\} \Rightarrow \boxed{\overline{v}\times\overline{w} = \begin{vmatrix}\hat{e}_x & \hat{e}_y & \hat{e}_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z\end{vmatrix} = (v_y w_z - v_z w_y)\hat{e}_x + (v_z w_x - v_x w_z)\hat{e}_y + (v_x w_y - v_y w_x)\hat{e}_z}$$

cylindrical coordinate system:

$$\left.\begin{aligned}\overline{v} &= v_\rho\hat{e}_\rho + v_\varphi\hat{e}_\varphi + v_z\hat{e}_z \\ \overline{w} &= w_\rho\hat{e}_\rho + w_\varphi\hat{e}_\varphi + w_z\hat{e}_z\end{aligned}\right\} \Rightarrow \boxed{\overline{v}\times\overline{w} = \begin{vmatrix}\hat{e}_\rho & \hat{e}_\varphi & \hat{e}_z \\ v_\rho & v_\varphi & v_z \\ w_\rho & w_\varphi & w_z\end{vmatrix} = (v_\varphi w_z - v_z w_\varphi)\hat{e}_\rho + (v_z w_\rho - v_\rho w_z)\hat{e}_\varphi + (v_\rho w_\varphi - v_\varphi w_\rho)\hat{e}_z}$$

spherical coordinate system:

$$\left.\begin{aligned}\overline{v} &= v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_\varphi\hat{e}_\varphi \\ \overline{w} &= w_r\hat{e}_r + w_\theta\hat{e}_\theta + w_\varphi\hat{e}_\varphi\end{aligned}\right\} \Rightarrow \boxed{\overline{v}\times\overline{w} = \begin{vmatrix}\hat{e}_r & \hat{e}_\theta & \hat{e}_\varphi \\ v_r & v_\theta & v_\varphi \\ w_r & w_\theta & w_\varphi\end{vmatrix} = (v_\theta w_\varphi - v_\varphi w_\theta)\hat{e}_r + (v_\varphi w_r - v_r w_\varphi)\hat{e}_\theta + (v_r w_\theta - v_\theta w_r)\hat{e}_\varphi}$$

INTEGRALS OF EXPRESSIONS CONTAINING VECTORS

In practical application, you will find often integrals of vectors.

- If the vector is expressed in a Cartesian coordinate system, this is not a problem
- If the vector is not expressed in a Cartesian coordinate system, we must be very carefull

(1) Vector expressed in a Cartesian coordinate system

- The basis of a Cartesian coordinate system, $\hat{e}_x, \hat{e}_y, \hat{e}_z$ are constant: they always point in the same direction and their absolute value is 1 → we can move them out of the integral.
- Example: $\int_0^2 \overline{v} dx \qquad \text{with} \qquad \overline{v} = zx\hat{e}_x + y\hat{e}_y + xy\hat{e}_z$

$$\int_0^2 \left(zx\hat{e}_x + y\hat{e}_y + xy\hat{e}_z\right)dx = \int_0^2 zx\hat{e}_x dx + \int_0^2 y\hat{e}_y dx + \int_0^2 xy\hat{e}_z dx = z\hat{e}_x \int_0^2 xdx + y\hat{e}_y \int_0^2 dx + y\hat{e}_z \int_0^2 xdx$$

$$= z\hat{e}_x \left[\frac{x^2}{2}\right]_0^2 + y\hat{e}_y \left[x\right]_0^2 + y\hat{e}_z \left[\frac{x^2}{2}\right]_0^2 = 2z\hat{e}_x + 2y\hat{e}_y + 2y\hat{e}_z$$

INTEGRALS OF EXPRESSIONS CONTAINING VECTORS

(2) Vector expressed in a non-Cartesian coordinate system

- The basis might not be constant in space: the direction could depend on the position.
  - we can NOT move them out of the integral.
  - we need to express the basis vectors using the Cartesian basis (that are constant)

- Example in a cylindrical coordinate system:

$$\int_{-\pi/2}^{\pi/2} \overline{v}\, d\varphi \qquad \text{with} \quad \overline{v} = \rho \hat{e}_\varphi$$

  - $\hat{e}_\varphi$ depends on the angle φ, so it depends on the variable of integration.
  - So, we cannot move the vector outside the integral.
  - We need to express the vectro in a Cartesian coordinate system:

$$\hat{e}_\varphi = -\sin\varphi \hat{e}_x + \cos\varphi \hat{e}_y$$

$$\int_{-\pi/2}^{\pi/2} \rho \hat{e}_\varphi d\varphi = \rho \int_{-\pi/2}^{\pi/2} \left(-\sin\varphi \hat{e}_x + \cos\varphi \hat{e}_y\right) d\varphi = \rho \int_{-\pi/2}^{\pi/2} \left(-\sin\varphi\right) \hat{e}_x d\varphi + \rho \int_{-\pi/2}^{\pi/2} \cos\varphi \hat{e}_y d\varphi =$$

$$= \rho \hat{e}_x \int_{-\pi/2}^{\pi/2} \left(-\sin\varphi\right) d\varphi + \rho \hat{e}_y \int_{-\pi/2}^{\pi/2} \cos\varphi d\varphi = \rho \hat{e}_x \left[\cos\varphi\right]_{-\pi/2}^{\pi/2} + \rho \hat{e}_y \left[\sin\varphi\right]_{-\pi/2}^{\pi/2} = 2\rho \hat{e}_y$$