DD2365/2022 – lecture 4 Adaptive methods

Johan Hoffman

Navier-Stokes equations

The *incompressible Navier-Stokes equations* then takes the form,

$$\dot{u} + (u \cdot \nabla)u + \nabla p - \nu \Delta u = f,$$
$$\nabla \cdot u = 0,$$

with the *kinematic viscosity* $\nu = \mu/\rho$

No slip boundary condition: $u = 0$

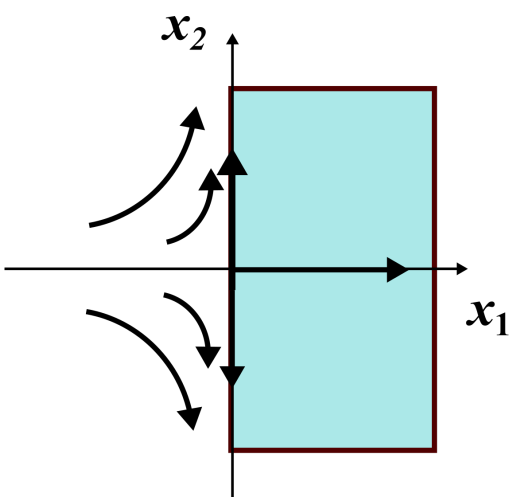
Slip boundary conditions: $u \cdot n = 0$

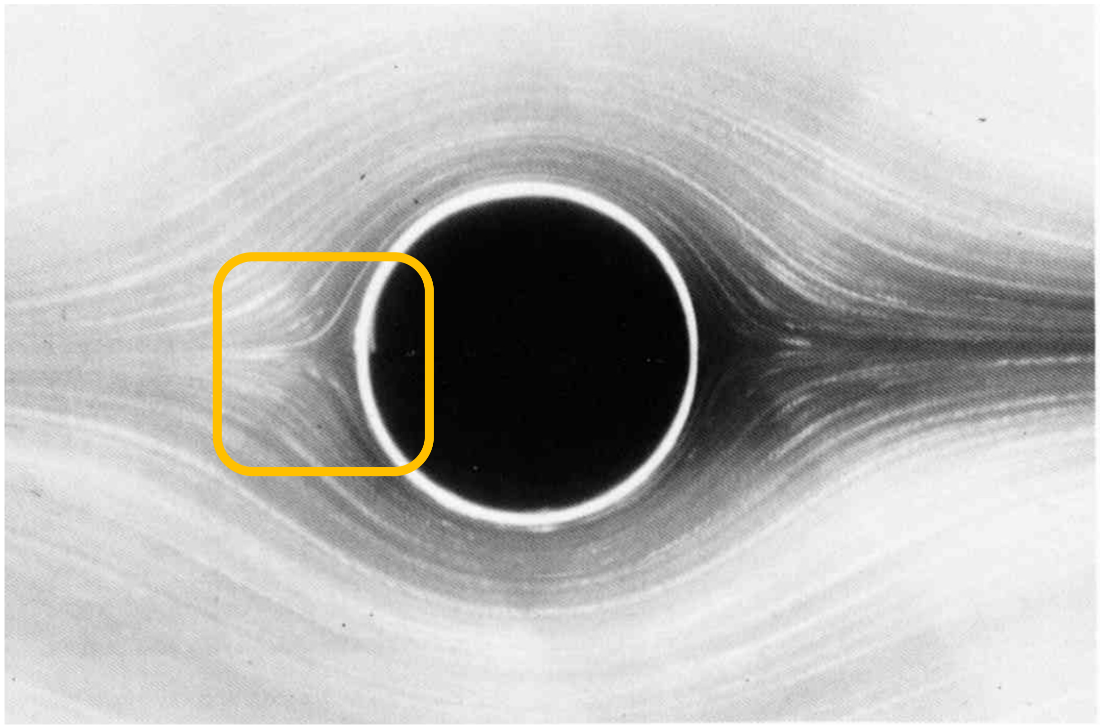
Friction boundary conditions: $n^T \sigma t_i = \beta u \cdot t_i$

Outflow boundary conditions: $n^T \sigma = 0$

Incompressible flow – attachment point

- $\nabla \cdot u = 0$
- $\dfrac{\partial u_2}{\partial x_2} = -\dfrac{\partial u_1}{\partial x_1}$

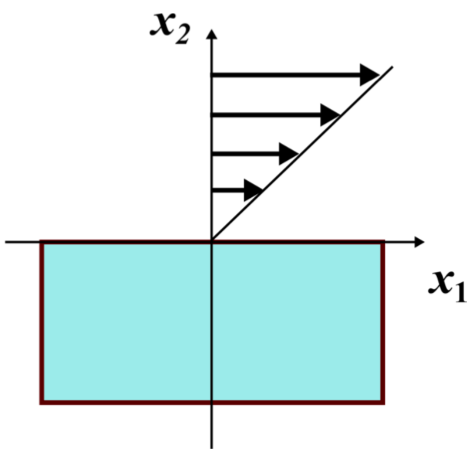


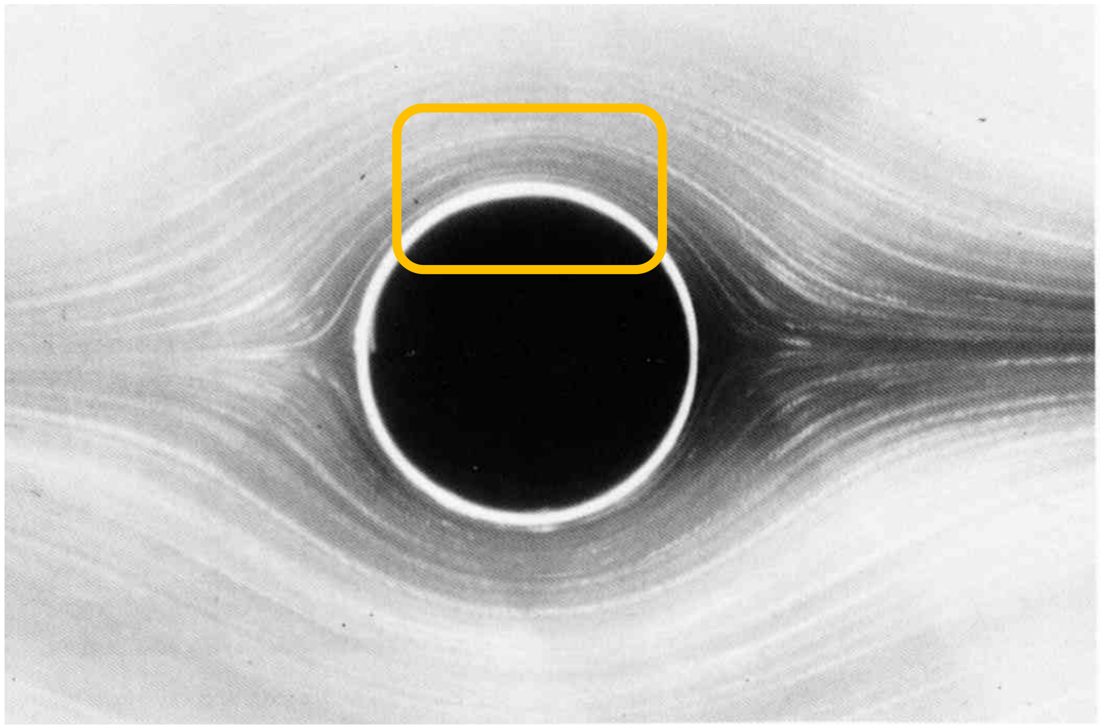


[Water and aluminum dust.]

Incompressible flow – boundary layer

- $\nabla \cdot u = 0$
- $u_1 = f(x_2)$
- $u_2 = 0$

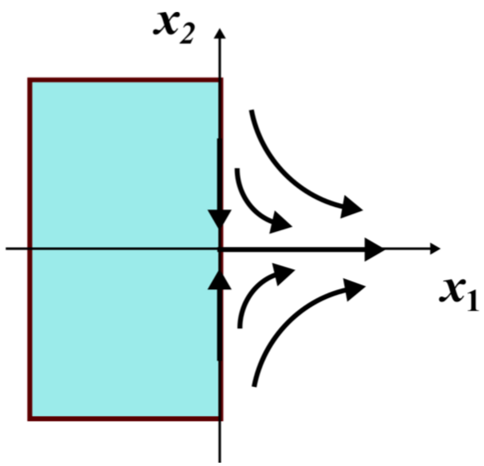


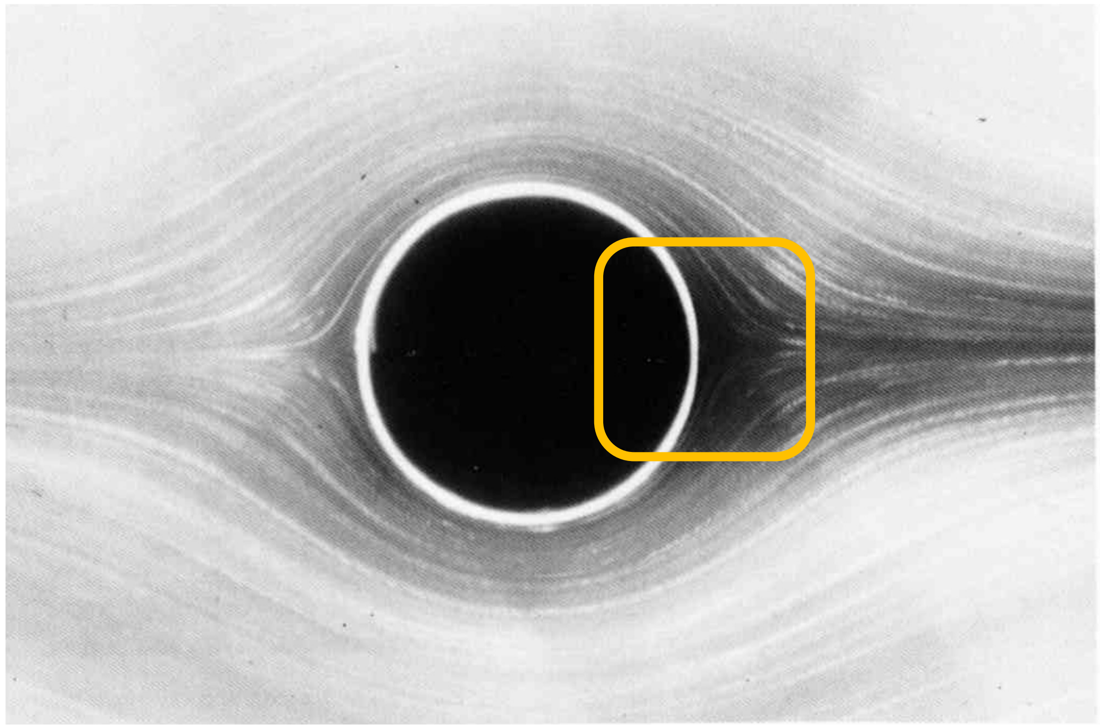


[Water and aluminum dust.]

Cylinder (Re = 0.16) – separation point

- $\nabla \cdot u = 0$
- $\frac{\partial u_1}{\partial x_1} = -\frac{\partial u_2}{\partial x_2}$

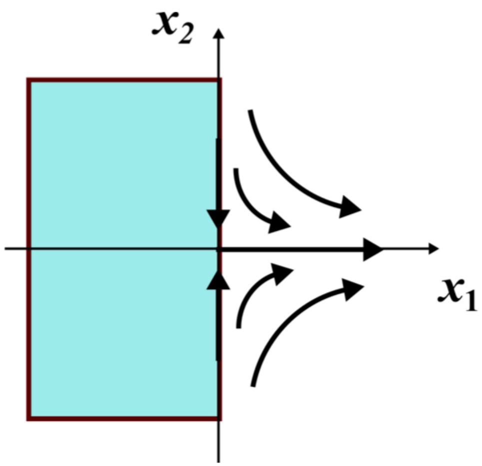


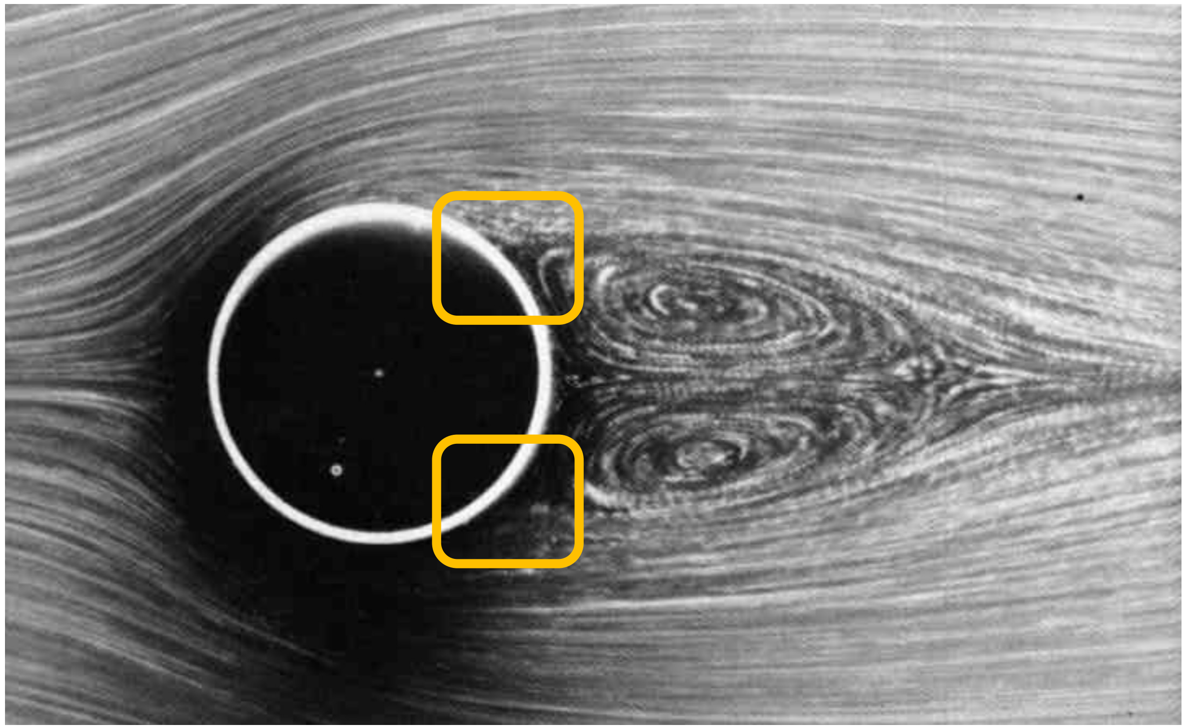


[Water and aluminum dust.]

Cylinder (Re = 26) – 2 separation points

- $\nabla \cdot u = 0$
- $\dfrac{\partial u_1}{\partial x_1} = -\dfrac{\partial u_2}{\partial x_2}$

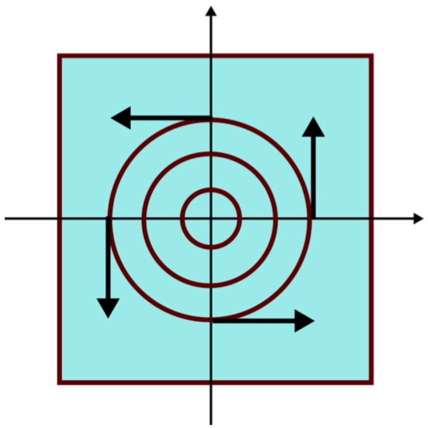


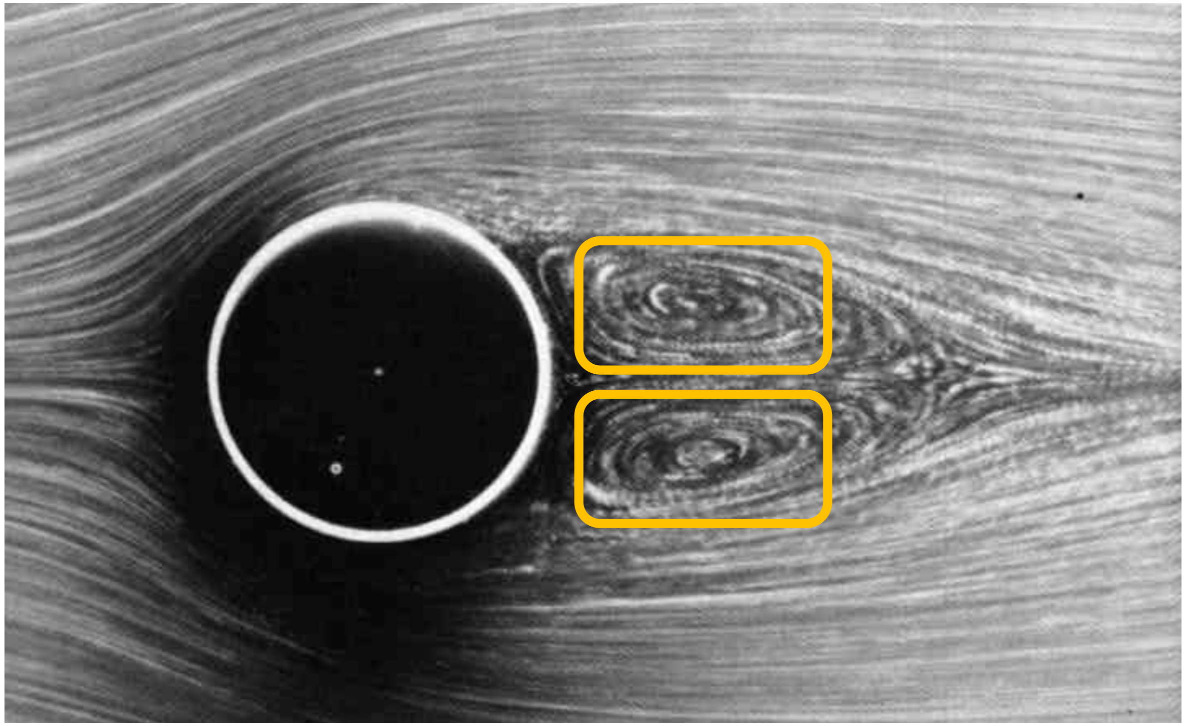


[Oil and magnesium.]

Cylinder (Re = 26) – 2 vortices

- $\nabla \cdot u = 0$
- $u_1 = f(x_2)$
- $u_2 = g(x_1)$



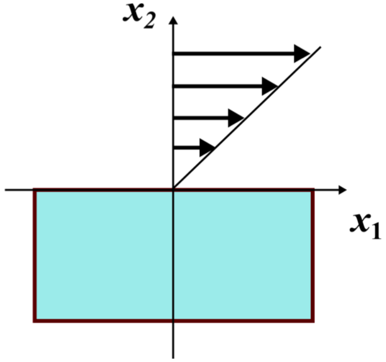


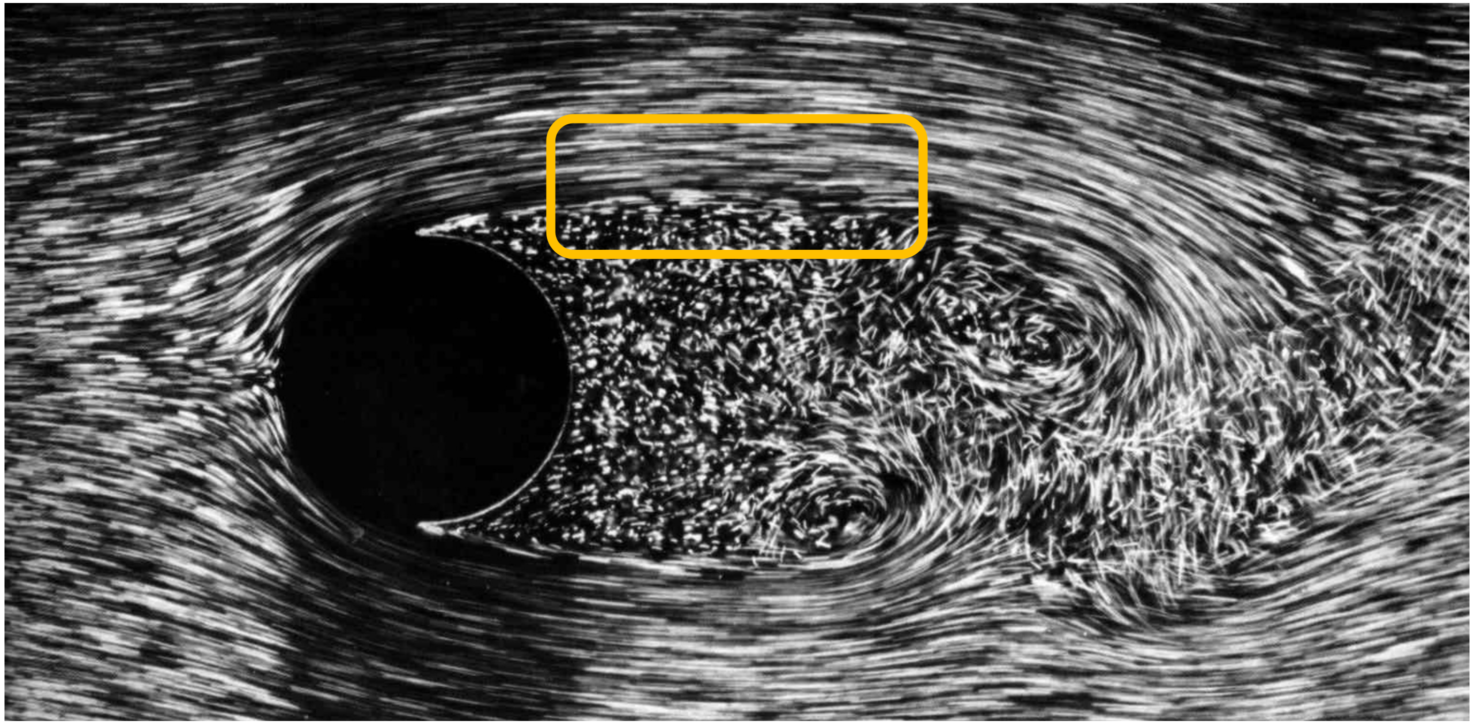
[Oil and magnesium.]

Cylinder (Re = 300) – Karman vortex street

[Wind and smoke.]

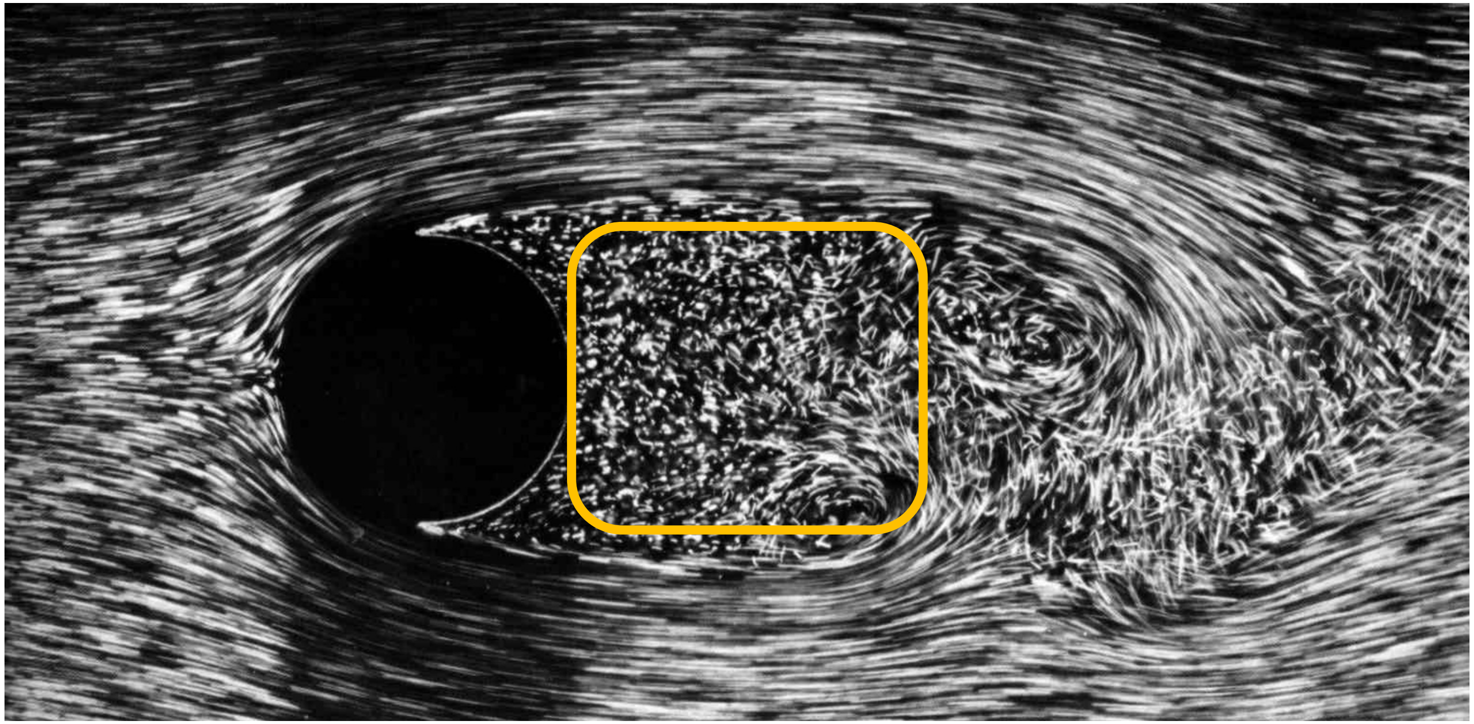
Cylinder (Re = 2000) – shear layer





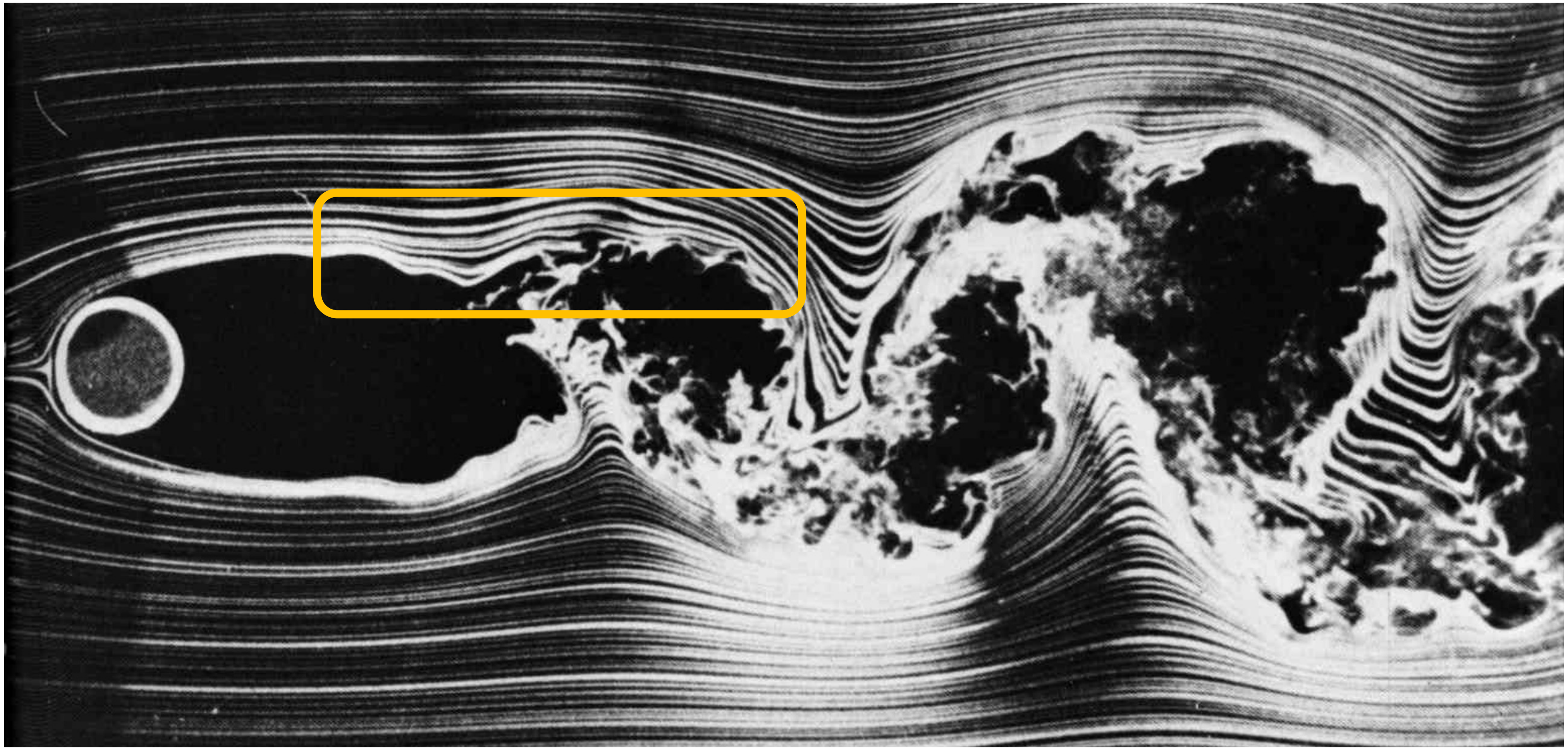
[Water and air bubbles.]

Cylinder (Re = 2000) – 3D turbulent wake



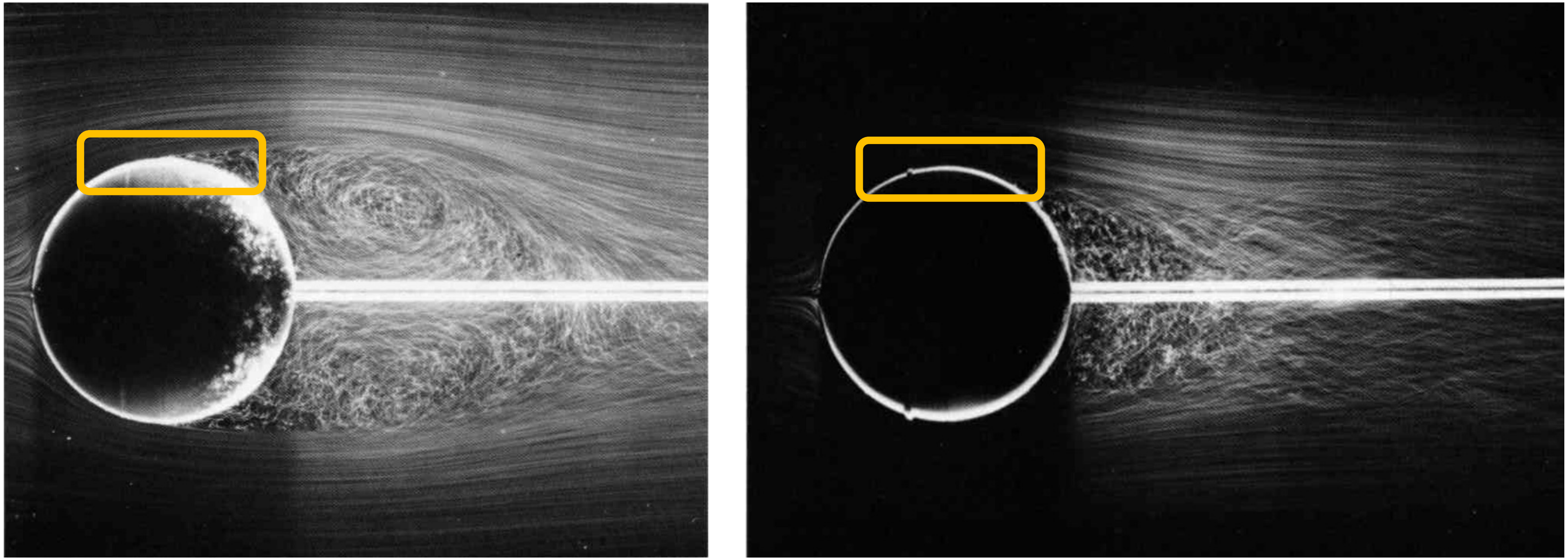
[Water and air bubbles.]

Re = 10 000 – turbulent shear layers

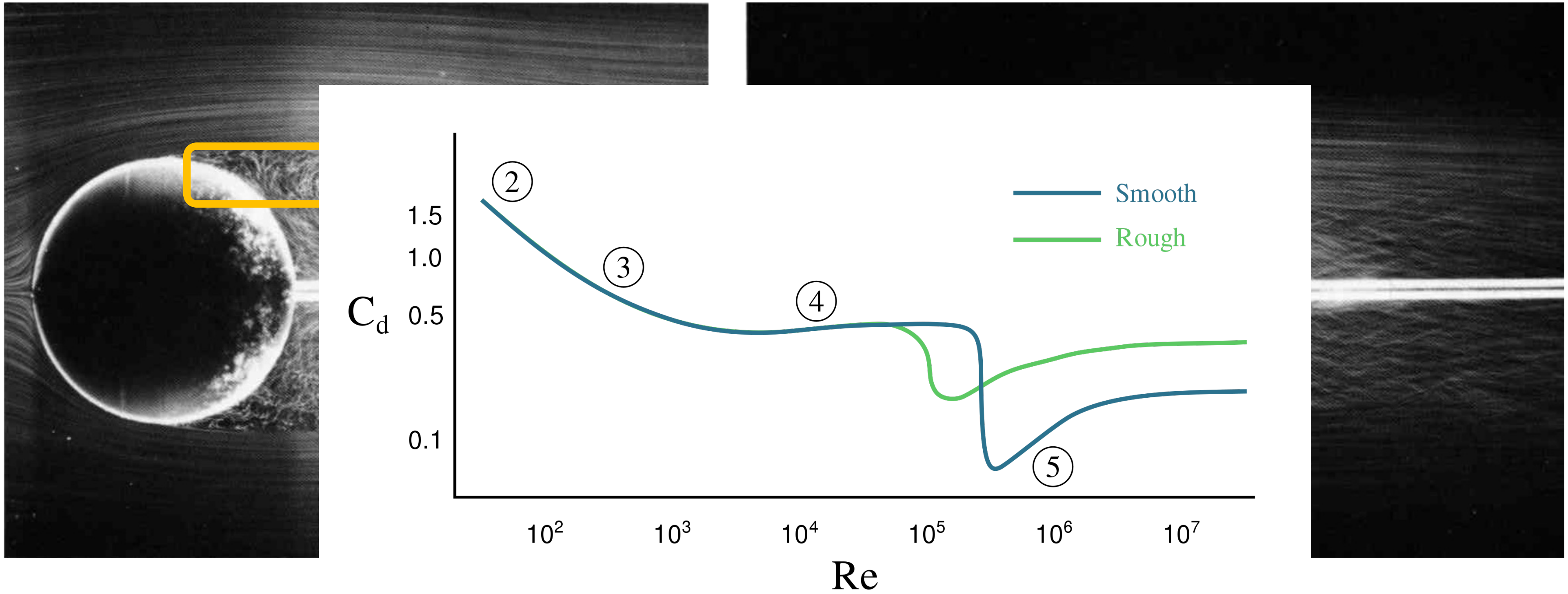


[Water and air bubbles.]

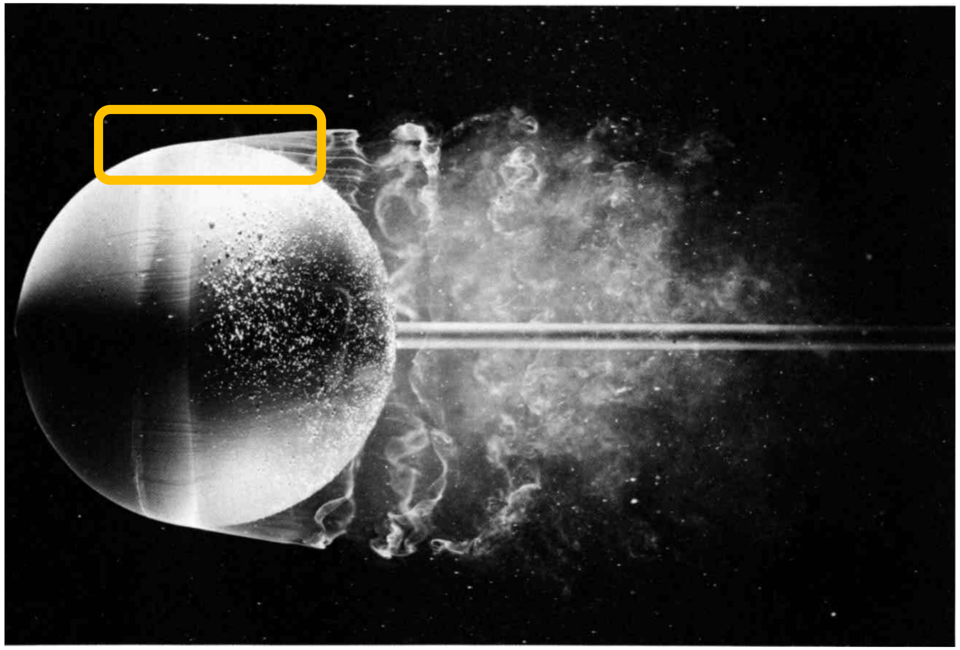
Sphere: Re = 15 000 vs 30 000
turbulent boundary layers (drag crisis)

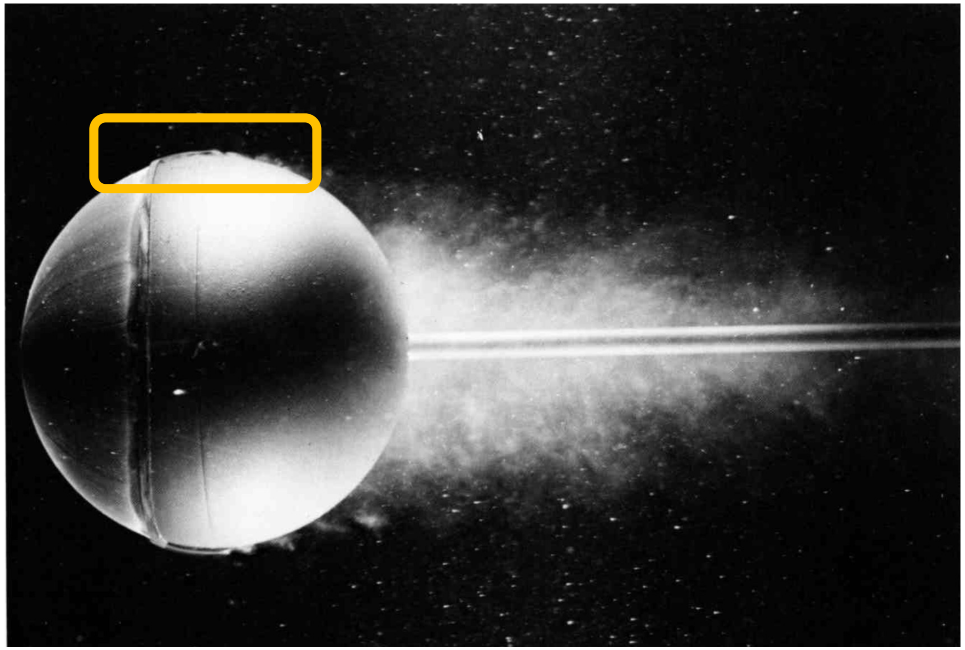


Sphere: Re = 15 000 vs 30 000
turbulent boundary layers (drag crisis)

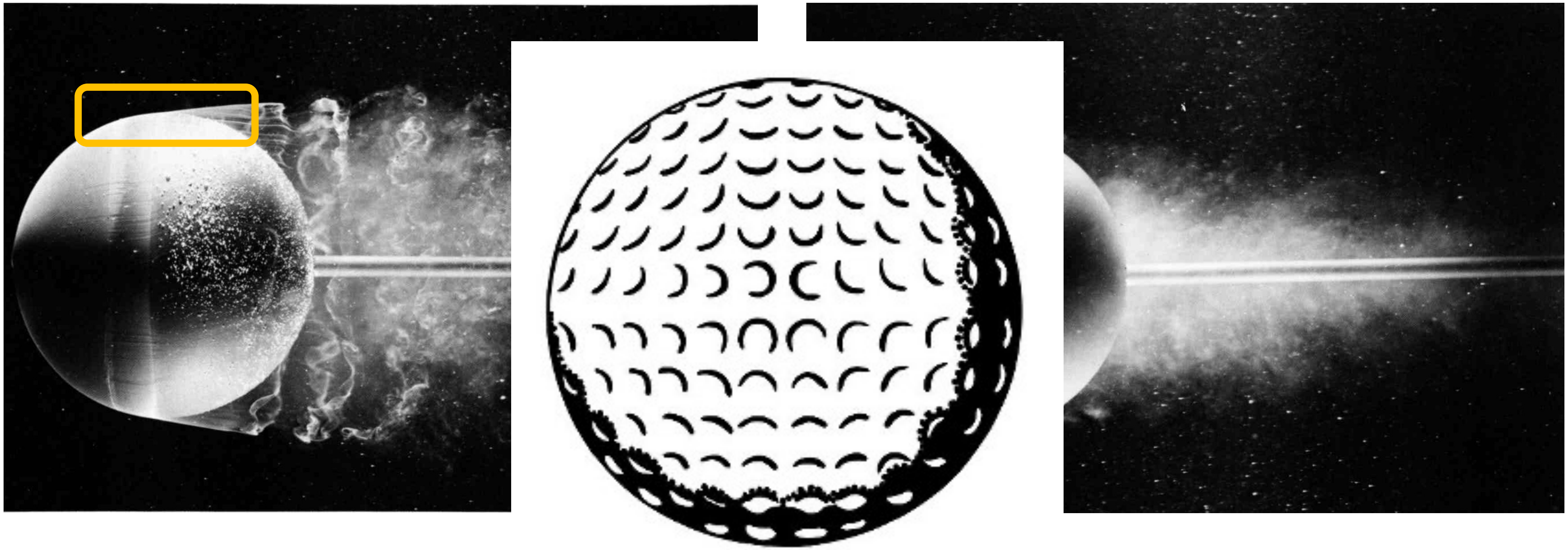


Sphere: Re = 15 000 vs 30 000
trip wire – to trigger turbulent boundary layer



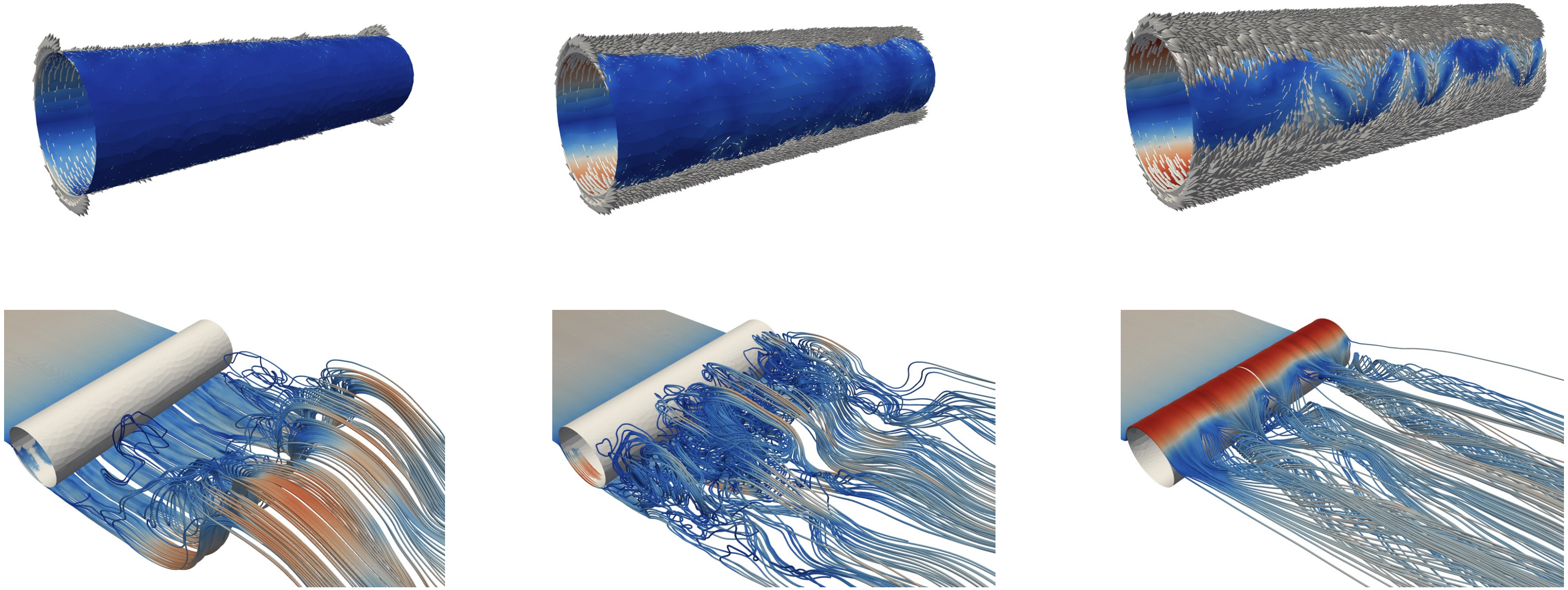


Sphere: Re = 15 000 vs 30 000
trip wire – to trigger turbulent boundary layer

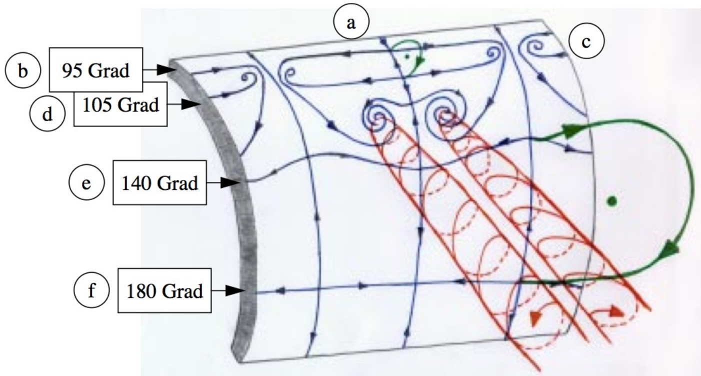


[https://en.wikipedia.org/wiki/Golf_Ball#/media/File:Golf_Ball.jpg]

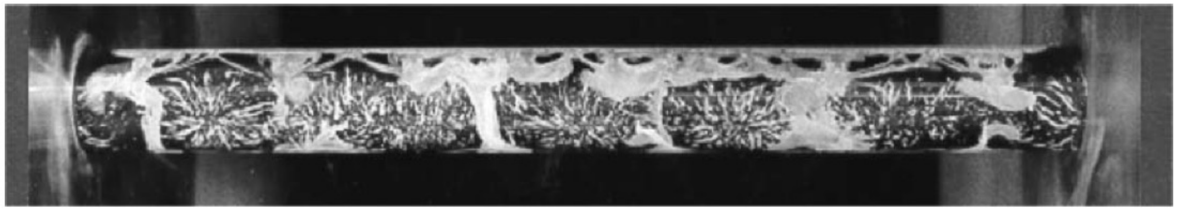
Simulation of drag crisis – slip/friction bc

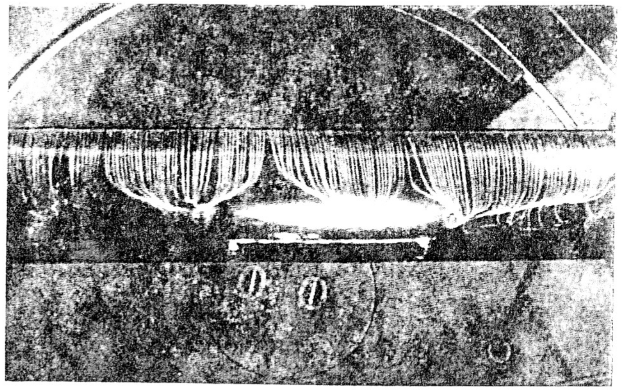


Streamwise vortex structures in the wake

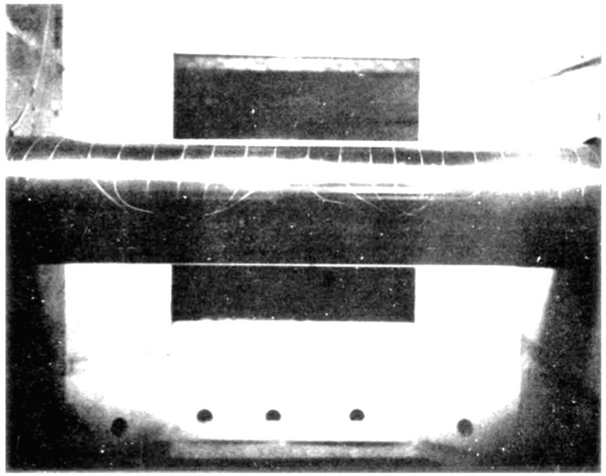


[Gölling DLR 2001]





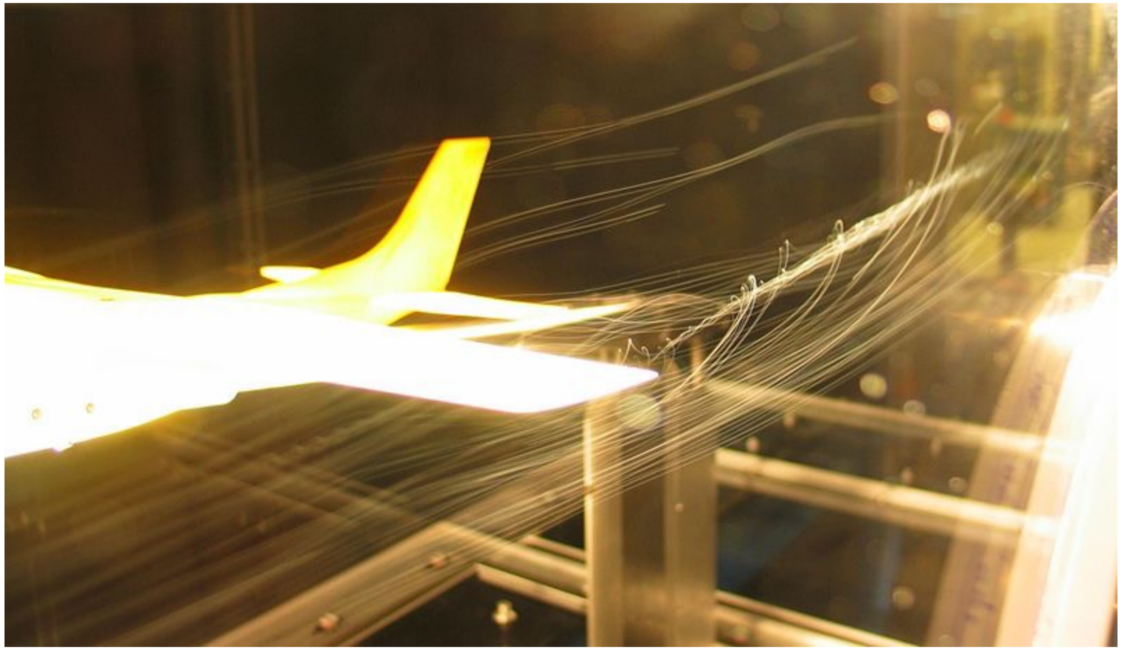
[Korotkin 1976]



[Humphreys JFM 1960]

Delayed separation and vortex structures

[https://en.wikipedia.org/wiki/File:Airplane_vortex_edit.jpg]



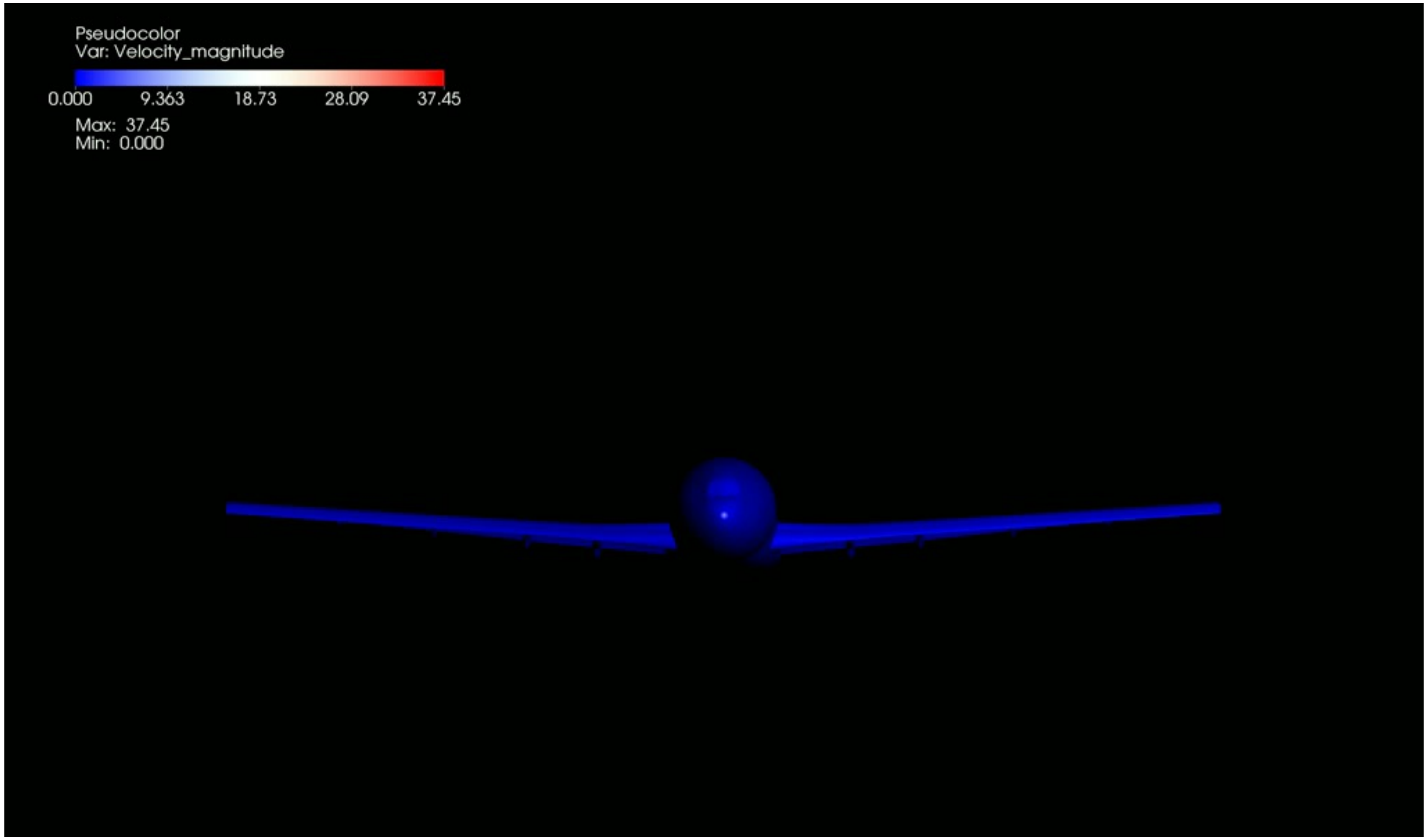
[https://sv.m.wikipedia.org/wiki/Fil:Cessna_182_model-wingtip-vortex.jpg]

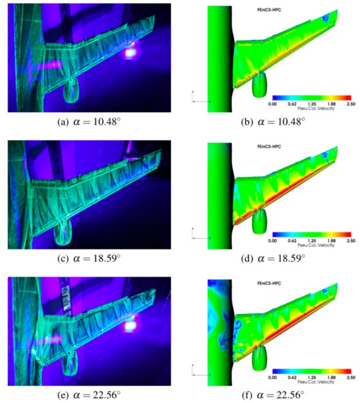
Delayed separation - downwash

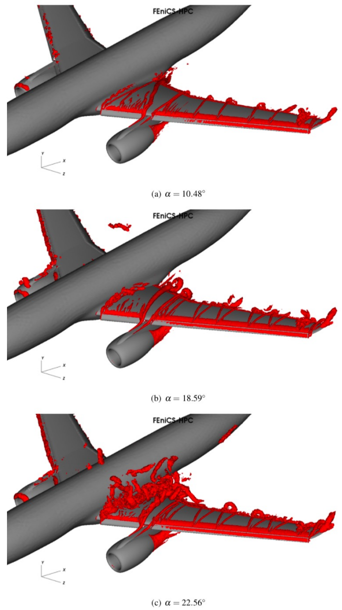
[https://i.redd.it/7yj9h0x2h9f61.jpg]

[https://www.grc.nasa.gov/www/k-12/airplane/downwash.html]

Simulation of airflow past airplane

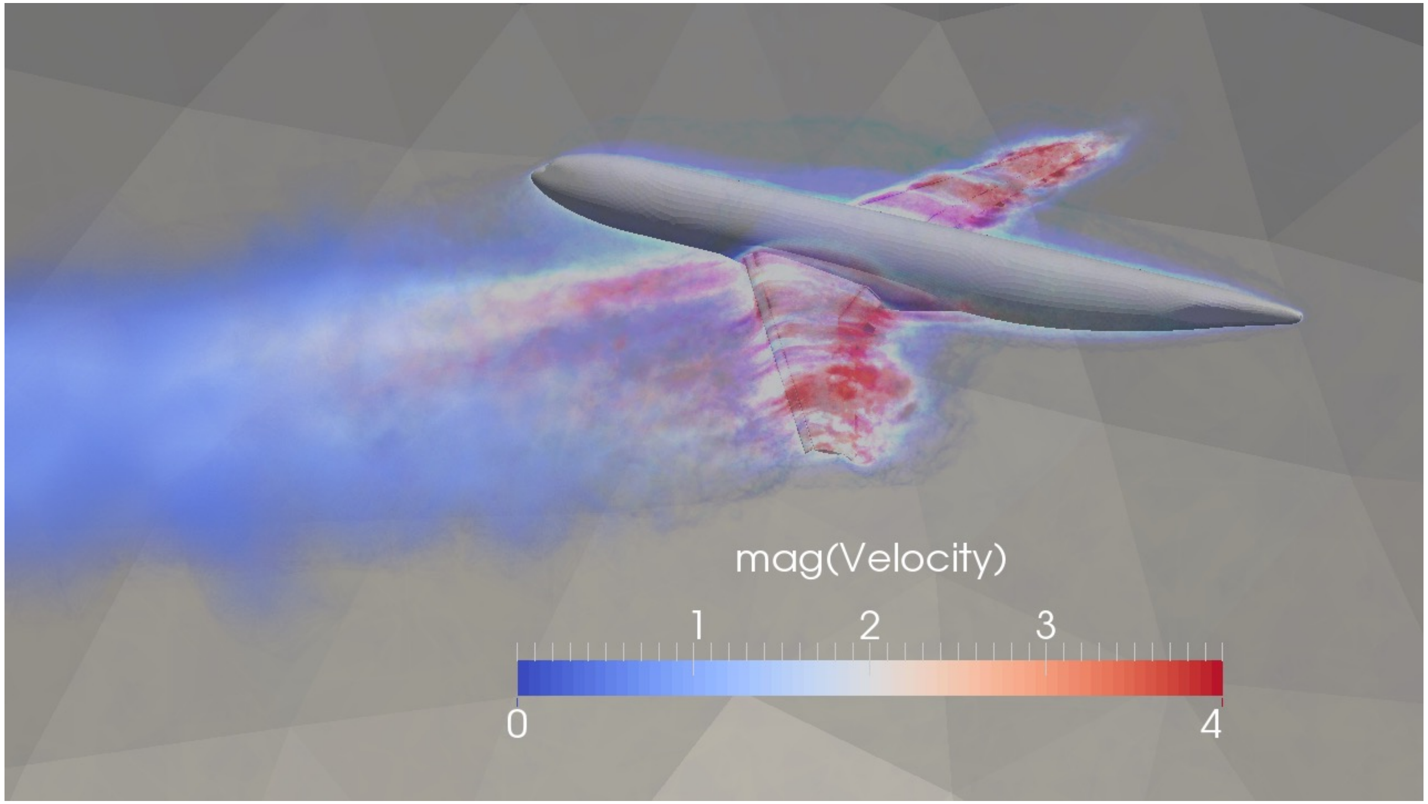


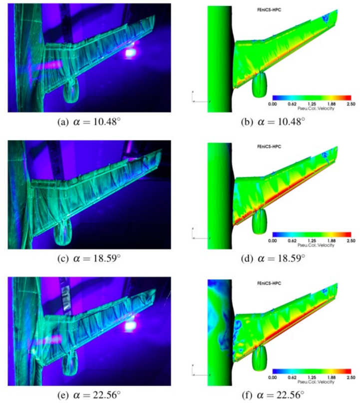


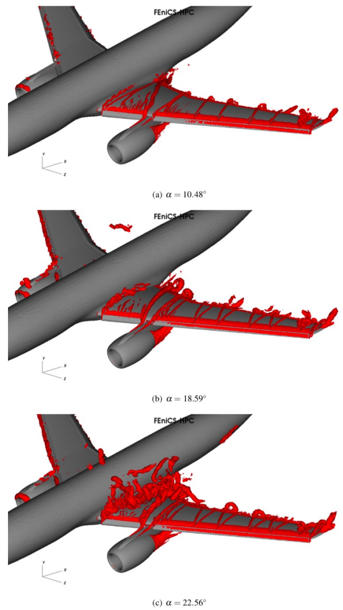


[Jansson et al., Springer, 2018]

Simulation of airflow past airplane

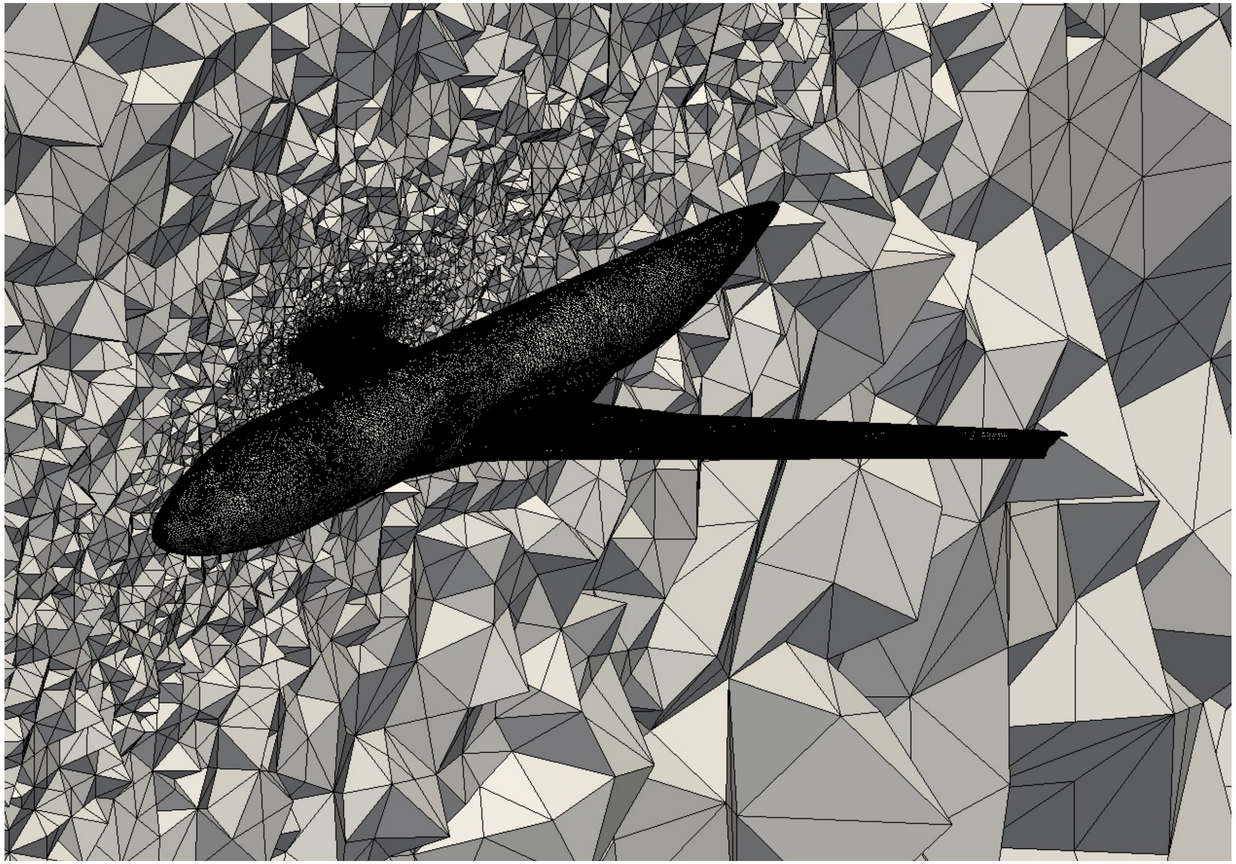


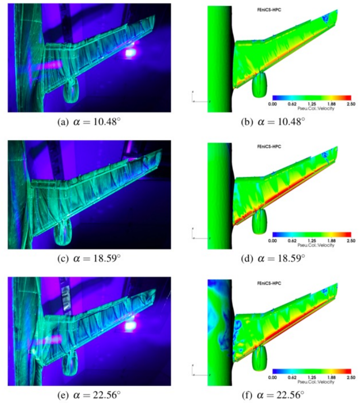


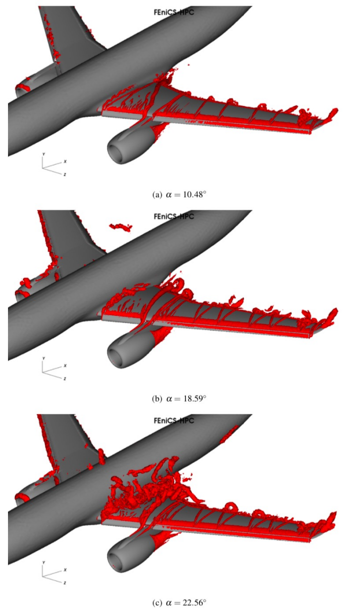


[Jansson et al., Springer, 2018]

Discretization by a mesh

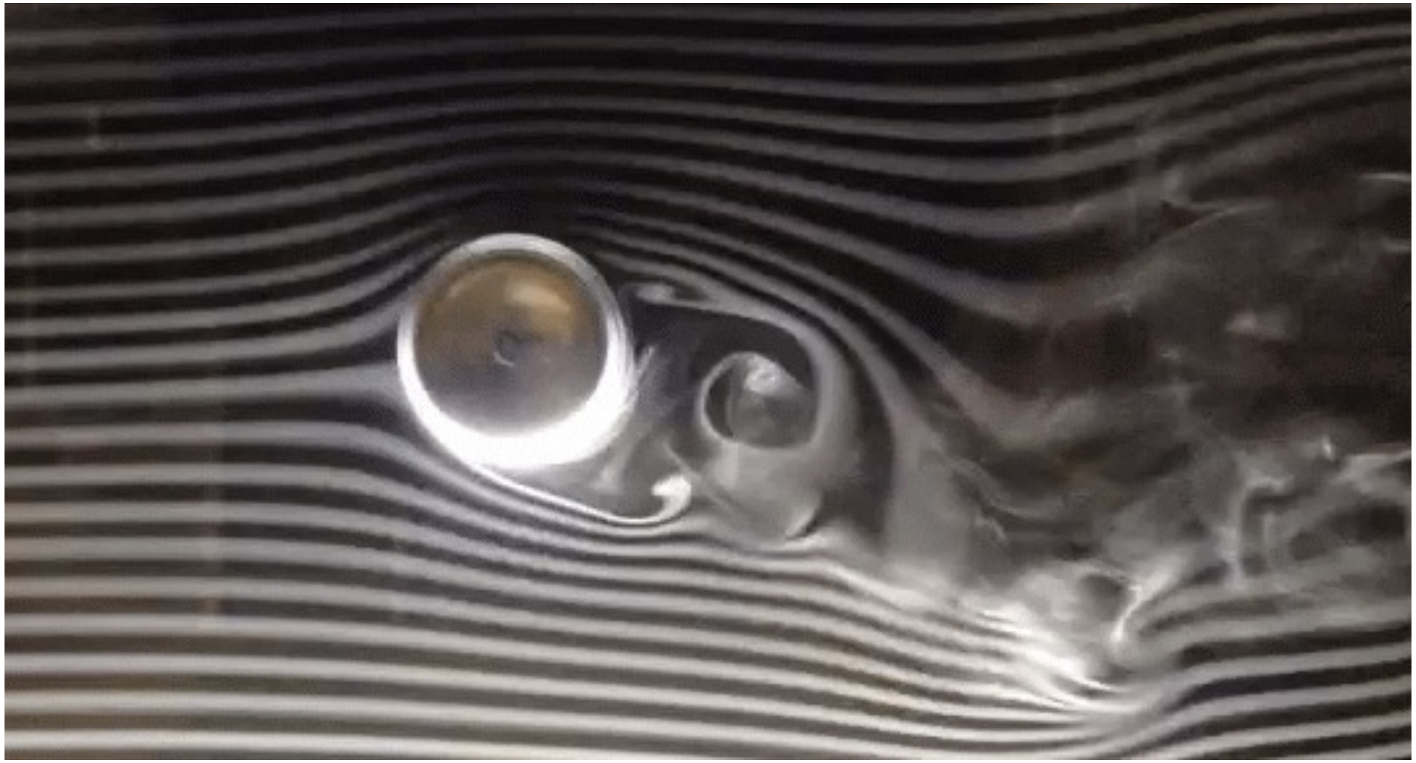






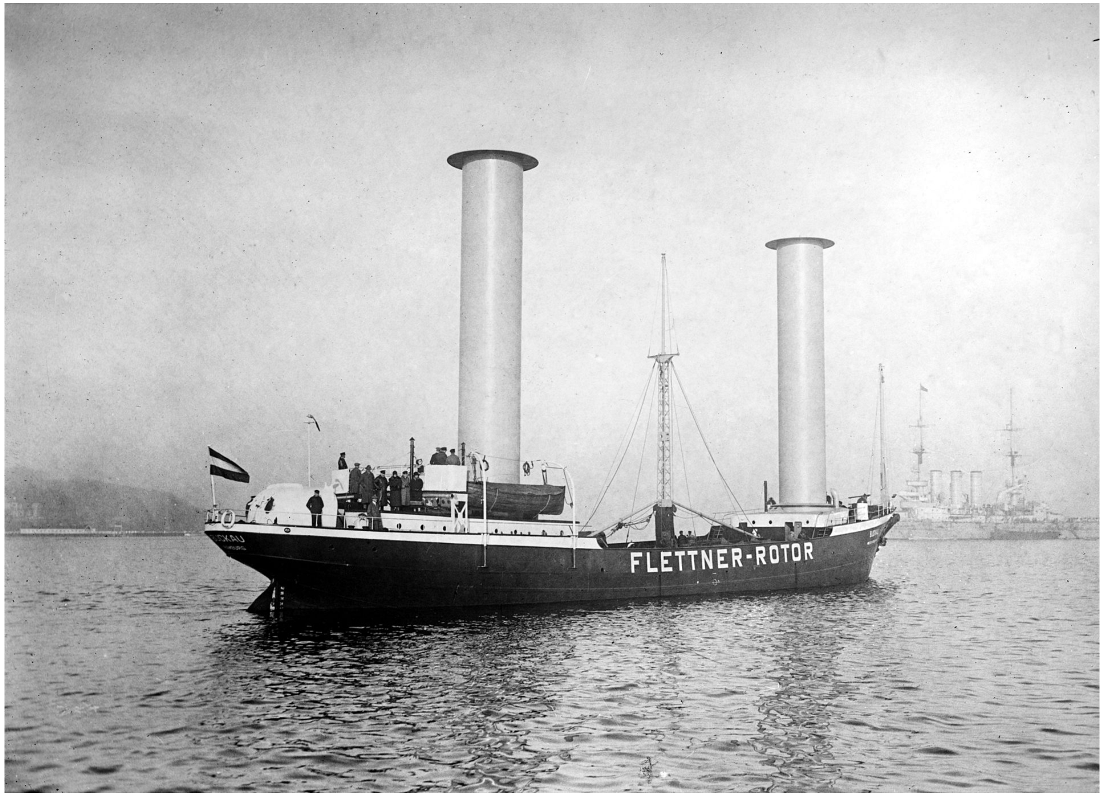
[Jansson et al., Springer, 2018]

Magnus effect – downwash through rotation



[https://en.wikipedia.org/wiki/Magnus_effect#/media/File:Magnus-anim-canette.gif]

Buckau - Flettner rotor ship 1924

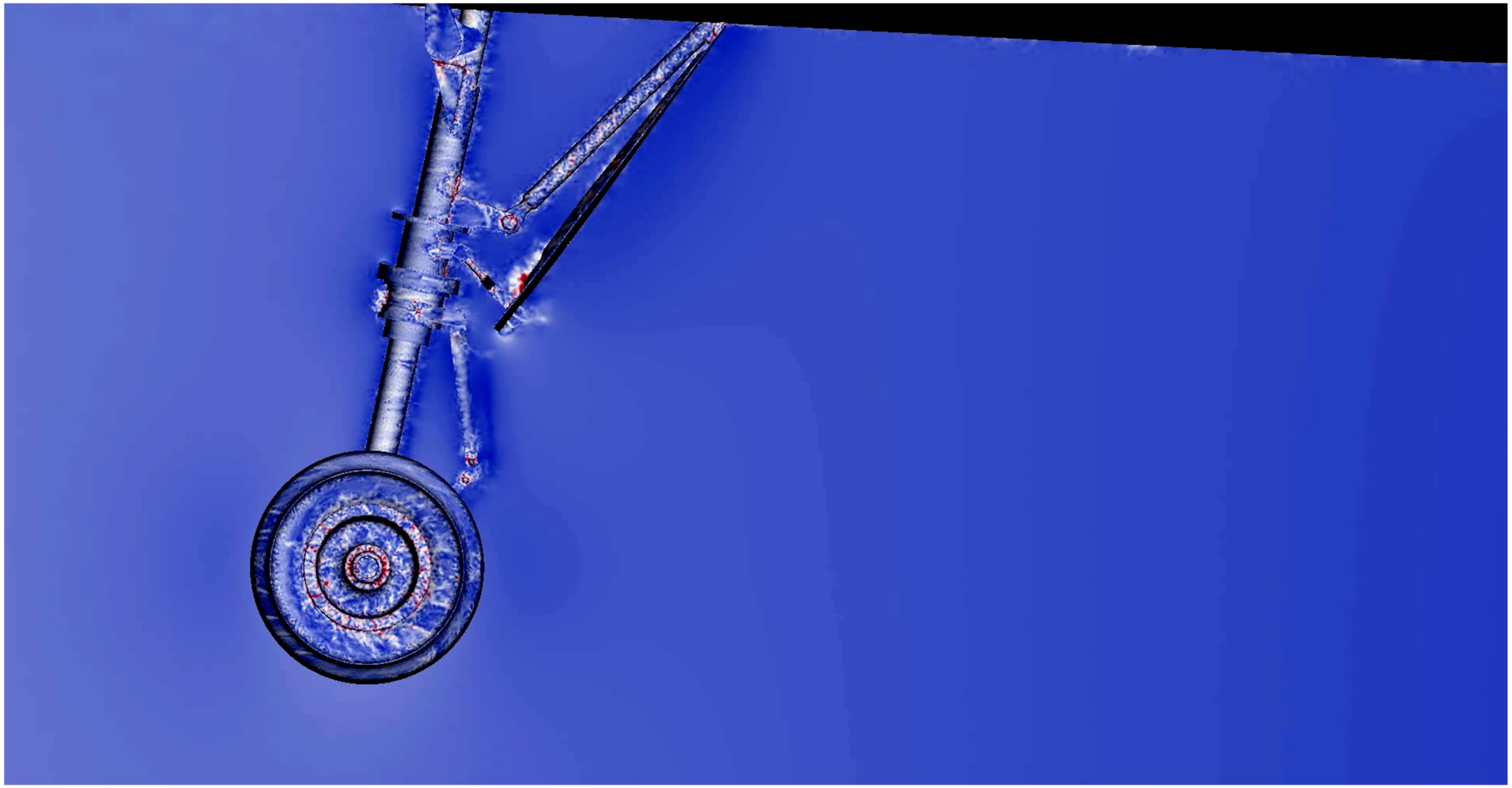


[https://en.wikipedia.org/wiki/Flettner_rotor#/media/File:Buckau_Flettner_Rotor_Ship_LOC_37764u.jpg]

Grace - Flettner rotor ship 2019

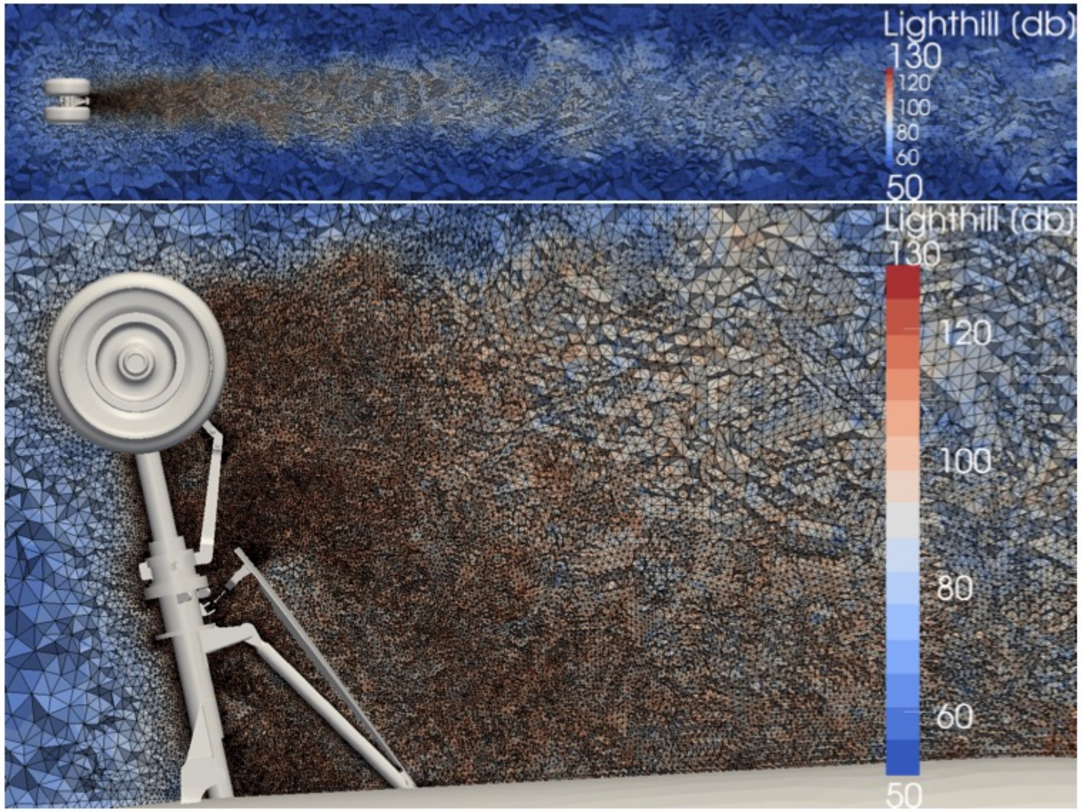
Grace - Flettner rotor ship 2019

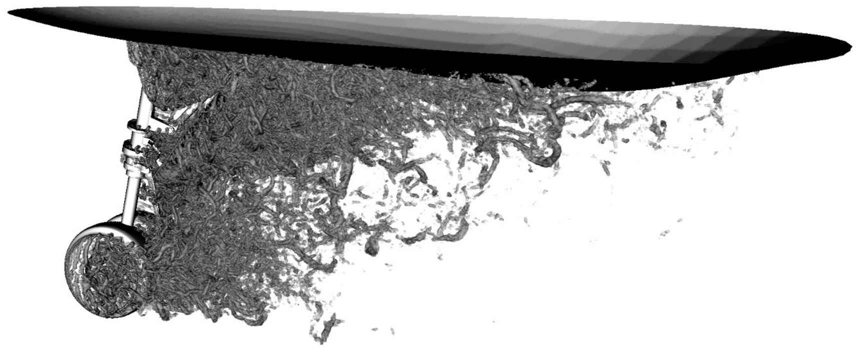
Simulation of airflow past landing gear



[De Abreu et al., Computers and Fluids, 2016]

Adaptively refined mesh





[De Abreu et al., Computers and Fluids, 2016]

Demo Lab 2

Elliptic PDEs: existence and uniqueness

For a Hilbert space V consisting of functions with finite norm $\|\cdot\|_V$, we formulate the corresponding variational problem: find $u \in V$, such that

$$a(u,v) = L(v), \quad \forall v \in V, \tag{3.16}$$

with $a : V \times V \to \mathbb{R}$ a bilinear form and $L : V \to \mathbb{R}$ a linear form.

Theorem 5 (Lax-Milgram theorem)**.** *The variational problem (3.16) has a unique solution $u \in V$, if the bilinear form is elliptic and bounded, and the linear form is bounded. That is, there exist constants $\alpha > 0$, $C_1, C_2 < \infty$, such that for $u, v \in V$,*

$$\begin{array}{ll} (i) & a(v,v) \geq \alpha \|v\|_V^2, \\ (ii) & a(u,v) \leq C_1 \|u\|_V \|v\|_V, \\ (iii) & L(v) \leq C_2 \|v\|_V. \end{array}$$

Energy norm and stability of solutions

Partial differential equations rarely admit closed form solutions, but we can still infer some characteristics of the solutions from the weak form (3.16). For an elliptic variational problem, a symmetric bilinear form defines an inner product $(\cdot,\cdot)_E = a(\cdot,\cdot)$ on the Hilbert space V, with an associated *energy norm*

$$\|\cdot\|_E = a(\cdot,\cdot)^{1/2},$$

For the energy norm we can derive the following stability estimate for the solution $u \in V$ to the variational problem (3.16),

$$\|u\|_E^2 = a(u,u) = L(u) \leq C_2\|u\|_V \leq (C_2/\alpha)\|u\|_E,$$

so that

$$\|u\|_E \leq (C_2/\alpha).$$

Optimality of Galerkin's method

In a Galerkin finite element method we seek an approximation $U \in V_h$,

$$a(U, v) = L(v), \quad \forall v \in V_h, \tag{3.19}$$

with $V_h \subset V$ a finite dimensional subspace, which in the case of a finite element method is a piecewise polynomial space. For an elliptic problem, existence and uniqueness of a solution follows from Lax-Milgram's theorem.

Since $V_h \subset V$, the weak form (3.16) is satisfied also for $v \in V_h$, and by subtracting (3.19) from (3.16) we obtain the Galerkin orthogonality property,

$$a(u - U, v) = 0, \quad \forall v \in V_h.$$

Optimality of Galerkin's method

For an elliptic problem with symmetric bilinear form we can show that the Galerkin approximation is optimal in the energy norm, since

$$\begin{aligned}\|u-U\|_E^2 &= a(u-U,u-U) = a(u-U,u-v) + a(u-U,v-U)\\ &= a(u-U,u-v) \leq \|u-U\|_E\|u-v\|_E,\end{aligned}$$

and hence

$$\|u-U\|_E \leq \|u-v\|_E, \quad \forall v \in V_h.$$

For an elliptic non-symmetric bilinear form, we can prove *Cea's lemma*,

$$\|u-U\|_V \leq \frac{C_1}{\alpha}\|u-v\|_V, \quad \forall v \in V, \tag{3.20}$$

which follows from

$$\begin{aligned}\|u-U\|_V^2 &\leq (1/\alpha)a(u-U,u-U) = (1/\alpha)a(u-U,u-v)\\ &\leq (C_1/\alpha)\|u-U\|_V\|u-v\|_V.\end{aligned}$$

A priori error estimation

For a Galerkin finite element method the approximation space V_h consists of piecewise polynomial functions defined over a mesh that approximates the domain $\Omega \subset \mathbb{R}^n$.

Cea's lemma (3.20) provides an estimate of the Galerkin error in terms of an arbitrary function $v \in V_h$, which we can choose to be an interpolant of the exact solution $v = \mathcal{I}^h u$, with

$$\mathcal{I}^h : V \to V_h$$

an interpolation operator, from which we obtain the *a priori* error estimate

$$\|u - U\|_V \leq (C_1/\alpha)\|u - \mathcal{I}^h u\|_V,$$

only in terms of the exact solution to the variational problem.

A priori error estimation

$$\left(\sum_K \|v - \mathcal{I}^h v\|^p_{W^{s,p}(K)}\right)^{1/p} \leq Ch^{k-s}|v|_{W^{k,p}(\Omega)}, \quad \forall v \in W^{s,p}(\Omega),$$

where

$$|v|_{W^{k,p}(\Omega)} = \sum_{|\alpha|=k} \|D^\alpha u\|^p_{L^p(\Omega)},$$

A posteriori error estimation

In contrast to an *a priori* error estimate which is expressed in terms of the unknown exact solution $u \in V$, an *a posteriori* error estimate is bounded in terms of a computed approximate solution $U \in V_h$. We define a bounded linear functional

$$M(\cdot) = (\cdot, \psi),$$

with ψ the Riesz representer of the functional $M \in V'$, guaranteed to exist by the Riesz representation theorem. To estimate the error with respect to $M(\cdot)$, we introduce an adjoint problem: find $\varphi \in V$, such that

$$a(v, \varphi) = M(v), \quad \forall v \in V. \tag{3.21}$$

A posteriori error estimation

An *a posteriori* error representation then follows from (3.16) and (3.21),

$$M(u) - M(U) = a(u, \varphi) - a(U, \varphi) = L(\varphi) - a(U, \varphi) = r(U, \varphi), \quad (3.22)$$

with the weak residual functional $r(U, \cdot) = L(\cdot) - a(U, \cdot) \in V'$, acting on the adjoint solution $\varphi \in V$,

$$r(U, \varphi) = L(\varphi) - a(U, \varphi).$$

Adaptive methods

With $U \in V_h$ a finite element approximation computed over a mesh $\mathcal{T}^h$, we can split the integral over the elements K in $\mathcal{T}^h$, so that the a posteriori error representation (3.22) is expressed as

$$M(u) - M(U) = r(U, \varphi) = \sum_{K \in \mathcal{T}^h} r(U, \varphi)|_K = \sum_{K \in \mathcal{T}^h} \mathcal{E}_K,$$

with the local error indicator

$$\mathcal{E}_K = r(U, \varphi)|_K,$$

defined for each element K. To approximate the error indicator we can compute an approximation $\Phi \approx \varphi$ to the adjoint problem (3.21), so that

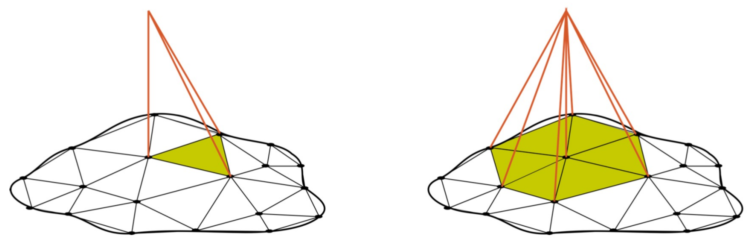
$$\mathcal{E}_K \approx r(U, \Phi)|_K.$$

Finite element method - mesh

For a simplicial mesh $\mathcal{T}^h$, the global approximation space of continuous piecewise polynomial functions V_h is spanned by the global nodal basis $\{\phi_j\}$, where each basis function ϕ_j is associated to a global vertex N_j. Hence with Dirichlet boundary conditions the finite element approximation $U \in V_h$ can be expressed as

$$U(x) = \sum_{N_j \in \mathcal{N}_I} U(N_j)\phi_j(x) + \sum_{N_j \in \mathcal{N}_D} U(N_j)\phi_j(x),$$

with $\mathcal{N}_I$ all internal vertices in the mesh and $\mathcal{N}_D$ all vertices on the Dirichlet boundary, and where $U(N_j)$ is the node which corresponds to function evaluation at the vertex N_j.



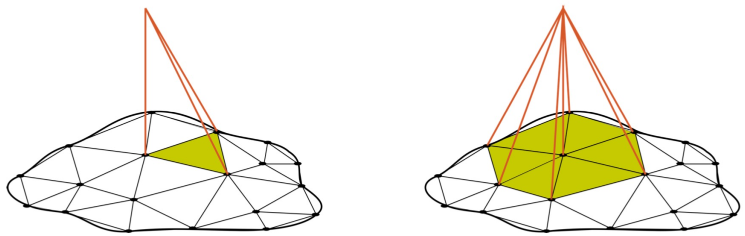
Finite element method - mesh

The finite element method takes the form of a matrix problem

$$Ax = b, \tag{3.24}$$

where $a_{ij} = a(\phi_j, \phi_i)$, $x_j = U(N_j)$ and $b_i = L(\phi_i)$. To compute the Galerkin finite element approximation, we thus have to construct the matrix A and vector b, and then solve the resulting matrix problem (3.24) to obtain the nodal values $U(N_j)$.

$$U(x) = \sum_{N_j \in \mathcal{N}_I} U(N_j)\phi_j(x) + \sum_{N_j \in \mathcal{N}_D} U(N_j)\phi_j(x)$$



Conforming triangular mesh

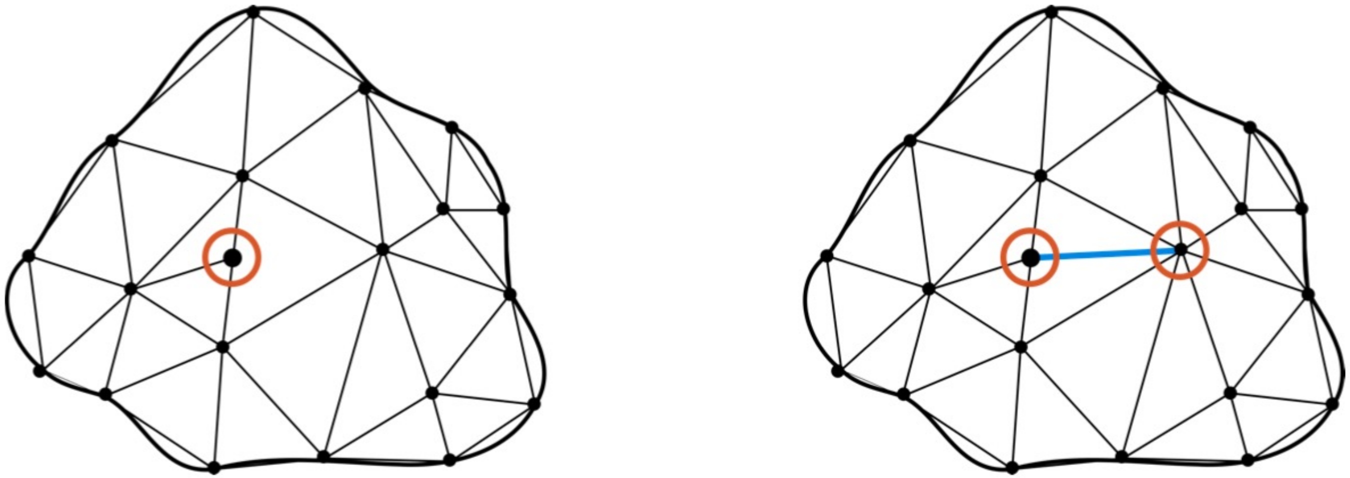


Figure 10.5. *Illustration of a non-conforming triangular mesh with a hanging node (left), which can be made into a conforming mesh by adding a new edge that eliminates the hanging node (right).*

Mesh generation

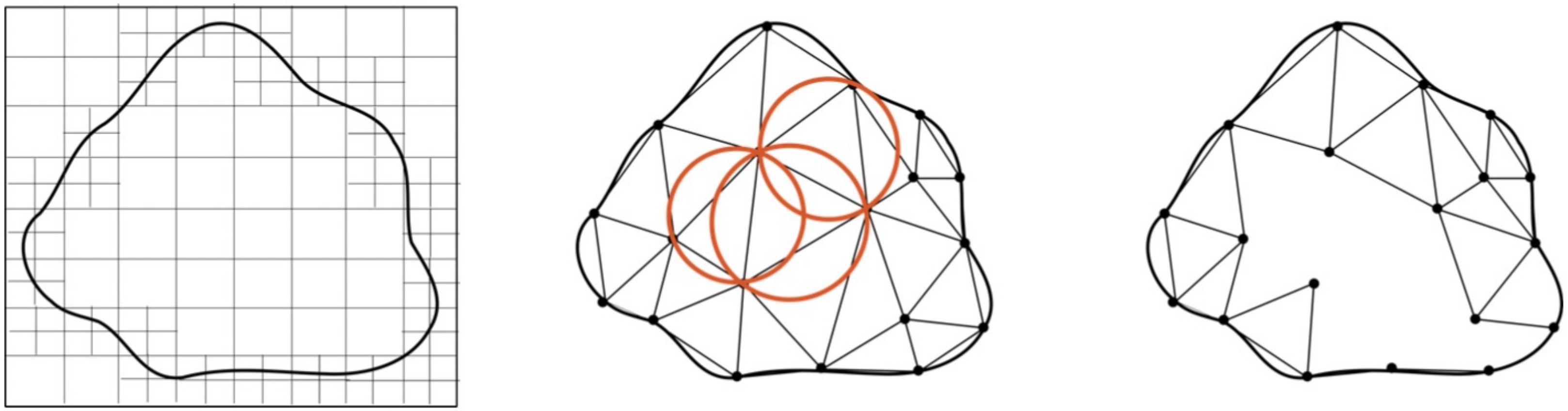
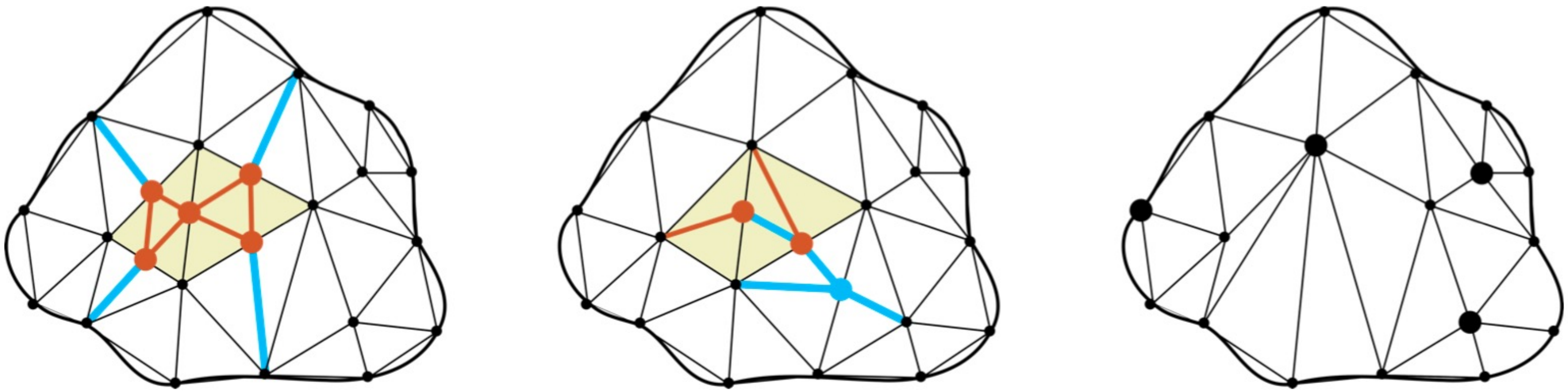


Figure 10.6. *Mesh generation by a quadtree algorithm (left), by the Delaunay condition (center), and by advancing front mesh generation (right).*

Mesh refinement and coarsening



Stokes equations

The Stokes equations for a domain $\Omega \subset \mathbb{R}^n$ with boundary $\nabla\Omega = \Gamma_D \cup \Gamma_N$, and associated normal n, takes the form

$$\begin{aligned} -\Delta u + \nabla p = f, &\qquad x \in \Omega, \\ \nabla \cdot u = 0, &\qquad x \in \Omega, \\ u = g_D, &\qquad x \in \Gamma_D, \\ -\nabla u \cdot n + pn = g_N, &\qquad x \in \Gamma_N. \end{aligned}$$

First assume that $\partial\Omega = \Gamma_D$ and $g_D = 0$, that is, homogeneous Dirichlet boundary conditions for the velocity. We then seek a weak solution to the Stokes equations in the following spaces,

$$V = H_0^1(\Omega) \times \ldots \times H_0^1(\Omega) = [H_0^1(\Omega)]^n,$$
$$Q = \{q \in L^2(\Omega) : \int_\Omega q\, dx = 0\},$$

Stokes equations – variational form

We derive the variational formulation by taking the inner product of the momentum equation with a test function $v \in V$, and the inner product of the continuity equation with a test function $q \in Q$. By Green's formula and the homogeneous Dirichlet boundary condition, we obtain the variational formulation as: find $(u, p) \in V \times Q$, such that

$$\begin{aligned} a(u,v) + b(v,p) &= (f,v), \quad \forall v \in V, && (5.6 \\ -b(u,q) &= 0, \quad \forall q \in Q, && (5.7 \end{aligned}$$

$$a(v,w) = (\nabla v, \nabla w) = \int_\Omega \nabla v : \nabla w \, dx, \qquad \nabla v : \nabla w = \sum_{i,j=1}^{3} \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j}$$

$$b(v,q) = -(\nabla \cdot v, q) = -\int_\Omega (\nabla \cdot v) q \, dx,$$

Stokes equations – finite element method

We seek an approximation $(U, P) \in V_h \times Q_h$, such that,

$$a(U, v) + b(v, P) = (f, v), \tag{5.11}$$

$$-b(U, q) = 0, \tag{5.12}$$

for all $(v, q) \in V_h \times Q_h$, where V_h and Q_h are finite element approximation spaces. There exists a unique solution to (5.11)-(5.12), under similar conditions as for the continuous variational problem.

A posteriori error estimation

The Stokes equations take the form

$$\nabla p - \Delta u = f, \quad \nabla \cdot u = 0,$$

together with boundary conditions for $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_F$

$$u = g_D, \quad x \in \Gamma_D$$

$$u = 0, \quad x \in \Gamma_F$$

$$-\nabla u \cdot n + pn = 0, \quad x \in \Gamma_N$$

Here Γ_D is the part of the boundary where we prescribe Dirichlet boundary conditions, Γ_N a part of the boundary where we apply a homogeneous Neumann boundary condition, and Γ_F the part of the boundary over which we want to compute the force.

We seek a finite element approximation $(U, P) \in V_h \times Q_h$ such that

$$-(P, \nabla \cdot v) + (\nabla U, \nabla v) + (\nabla \cdot U, q) = (f, v)$$

for all test functions $(v, q) \in \hat{V}_h \times Q_h$, where $\hat{V}_h$ are the test functions v such that $v = 0$ for $x \in \Gamma_D$. Here $V_h \subset V, Q_h \subset Q, \hat{V}_h \subset \hat{V}$ are finite dimensional subspaces defined over the computational mesh by finite element basis functions.

A posteriori error estimation

We consider the linear functional $F : V \times Q \to \mathbb{R}$,

$F(v, q) = (v, \psi_1)_\Omega + (q, \psi_2)_\Omega + \langle \nabla v \cdot n - pn, \psi_3 \rangle_{\Gamma_F}$

corresponding to weighted mean values of v and q, and the force on the surface $\Gamma_F \subset \partial\Omega$, which generates the adjoint Stokes equations

$-\nabla\theta - \Delta\varphi = \psi_1, \quad -\nabla \cdot \varphi = \psi_2,$

together with boundary conditions that reflect the primal equations and the chosen functional.

$\varphi = 0, \quad x \in \Gamma_D$

$\varphi = \psi_3, \quad x \in \Gamma_F$

$-\nabla\varphi \cdot n - \theta n = 0, \quad x \in \Gamma_N$

The weak form of the adjoint Stokes equations take the form: find $(\varphi, \theta) \in \hat{V} \times Q$ such that

$-(q, \nabla \cdot \varphi) + (\nabla v, \nabla\varphi) + (\nabla \cdot v, \theta) = (v, \psi_1)_\Omega + (q, \psi_2)_\Omega - \langle \nabla v \cdot n - qn, \psi_3 \rangle_{\Gamma_F} = F(v, q)$

for all test functions $(v, q) \in V \times Q$.

A posteriori error estimation

Since the Stokes equations are linear we can express the error in the linear functional with respect to an approximation $(u, p) \approx (U, P)$ as

$$F(u,p) - F(U,P) = (f,\varphi) + (P, \nabla\cdot\varphi) - (\nabla U, \nabla\varphi) - (\nabla\cdot U, \theta) = r(U,P;\varphi,\theta) = \sum_K \mathcal{E}_K$$

where we used that $F(u,p) = (f,\varphi)$ since $\varphi \in \hat{V}$, with the error indicator

$$\mathcal{E}_K = r(U,P;\varphi,\theta)|_K,$$

which is the local residual on weak form with the solution to the adjoint equation as test function. The error indicator $\mathcal{E}_K$ can be used as an indicator for where to refine the mesh to reduce the global error as efficiently as possible.

A posteriori error estimation

Note however that since $(U, P) \in V \times Q$ is the solution of a Galerkin finite element method, if we use the approximation $(\varphi, \theta) \approx (\varphi_h, \theta_h) \in \hat{V}_h \times Q_h$, the error indicators sum to zero. Hence, this sum cannot be used as a stopping criterion for an adaptive algorithm. Instead we may use error estimates of the type

$$\mathcal{E}_K \leq C h_K (\|\nabla \varphi_h\|_K + \|\nabla \theta_h\|_K) \|R(U, P)\|_K$$

where $R(U, P) = (R_1(U, P), R_2(U))$ is the residual of the equations in strong form, with

$$R_1(U, P) = f + \Delta U - \nabla P$$

$$R_2(U) = \nabla \cdot U$$

$$\begin{aligned} |F(u) - F(U)| &= r(U, \varphi) = \sum_{i=1}^{n+1} \int_{I_i} (f\varphi - U'\varphi')\, dx, \\ |F(u) - F(U)| &= (R(U), \varphi) = (R(U), \varphi - \pi_h \varphi) \\ &= \sum_{i=1}^{n+1} \int_{I_i} R(U)(\varphi - \pi_h \varphi)\, dx \leq \sum_{i=1}^{n+1} C_i \|h_i^2 R(U)\| \|\varphi''\|. \end{aligned}$$

Demo Lab 3