DD2365/2022 – lecture 2
Viscous flow

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Finite element method: Poisson equation

We now consider the *Poisson equation* for a function $u : \mathbb{R}^n \to \mathbb{R}$,

$$-\Delta u = f, \quad \text{in } \Omega, \tag{3.1}$$

with the domain $\Omega \subset \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}$ given data. The Poisson equation

For the equation to have a unique solution we need to specify boundary conditions. We may prescribe *Dirichlet boundary conditions*,

$$u|_{\partial\Omega} = g_D, \tag{3.2}$$

Neumann boundary conditions,

$$\nabla u \cdot n|_{\partial\Omega} = g_N, \tag{3.3}$$

or a linear combination of the two, referred to as a *Robin boundary condition*,

$$\nabla u \cdot n|_{\partial\Omega} = \alpha(u|_{\partial\Omega} - g_D) + g_N, \tag{3.4}$$

with $\alpha(x)$ a given weight function.

Homogeneous Dirichlet bc

With homogeneous Dirichlet boundary conditions, we have the problem

$$\begin{aligned} -\Delta u &= f, \qquad \text{in } \Omega, \\ u &= 0, \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.5}$$

To make the problem statement precise, let the trial and test functions belong to a certain function space V. With $V = H_0^1(\Omega)$ we obtain the following variational formulation: find $u \in V$, such that

$$(\nabla u, \nabla v) = (f, v), \quad \forall v \in V, \tag{3.6}$$

since the boundary term vanishes as the test function is an element of the vector space $H_0^1(\Omega)$.

Homogeneous Neumann bc

$$
\begin{aligned}
-\Delta u &= f, && \text{in } \Omega, \\
\nabla u \cdot n &= 0, && \text{on } \partial\Omega.
\end{aligned}
\tag{3.7}
$$

With $V = H^1(\Omega)$ we have the following variational formulation: find $u \in V$, such that

$$
(\nabla u, \nabla v) = (f, v), \quad \forall v \in V, \tag{3.8}
$$

However, it turns out that the variational problem (3.8) has no unique solution, since for any solution $u \in V$, also $u + C$ is a solution, with $C \in \mathbb{R}$ any constant. To ensure a unique solution, we need an extra condition for the solution which determines the arbitrary constant, for example, we may change the trial space to

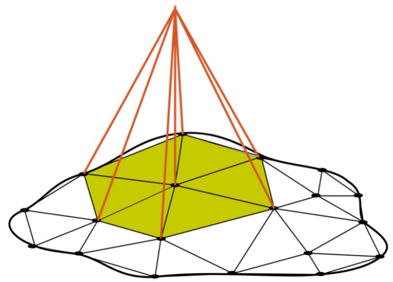
$$
V = \{u \in H^1(\Omega) : \int_\Omega u(x)\, dx = 0\}. \tag{3.9}
$$

Galerkin Finite Element Method

To formulate a Galerkin method for the Poisson equation we replace the Hilbert space V by a finite dimensional subspace $V_h \subset V$ in the variational formulation of the equation. We hence seek $U \in V_h$, such that

$$(\nabla U, \nabla v) = (f, v), \quad \forall v \in V_h. \tag{3.13}$$

For a simplicial mesh $\mathcal{T}^h$, the global approximation space of continuous piecewise polynomial functions V_h is spanned by the global nodal basis $\{\phi_j\}$, where each basis function ϕ_j is associated to a global vertex N_j. Hence with Dirichlet boundary conditions the finite element approximation $U \in V_h$ can be expressed as



$$U(x) = \sum_{N_j \in \mathcal{N}_I} U(N_j)\phi_j(x) + \sum_{N_j \in \mathcal{N}_D} U(N_j)\phi_j(x),$$

with $\mathcal{N}_I$ all internal vertices in the mesh and $\mathcal{N}_D$ all vertices on the Dirichlet boundary, and where $U(N_j)$ is the node which corresponds to function evaluation at the vertex N_j.

FEM for general variational problem

For a Hilbert space V consisting of functions with finite norm $\|\cdot\|_V$, we formulate the corresponding variational problem: find $u \in V$, such that

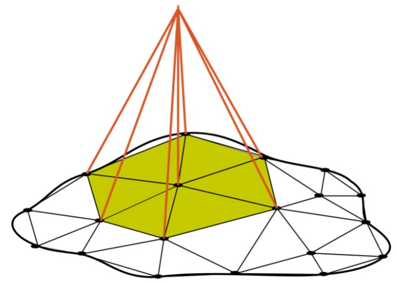
$$a(u,v) = L(v), \quad \forall v \in V, \tag{3.16}$$

with $a : V \times V \to \mathbb{R}$ a bilinear form and $L : V \to \mathbb{R}$ a linear form.

The finite element method takes the form of a matrix problem

$$Ax = b, \tag{3.24}$$

where $a_{ij} = a(\phi_j, \phi_i)$, $x_j = U(N_j)$ and $b_i = L(\phi_i)$. To compute the Galerkin finite element approximation, we thus have to construct the matrix A and vector b, and then solve the resulting matrix problem (3.24) to obtain the nodal values $U(N_j)$.



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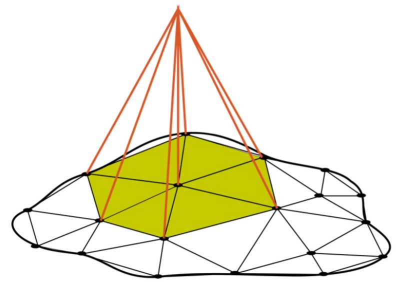
In the case of Dirichlet boundary conditions, the rows in the matrix corresponding to boundary nodes $N_j \in \mathcal{N}_D$ are replaced by a row with one on the diagonal and with all other components zero. To enforce the Dirichlet boundary condition, each corresponding vector component is then set to the interpolated Dirichlet boundary value $b_j = \mathcal{I}^h g_D(N_j)$.

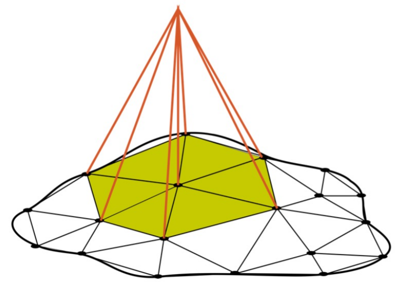


FEM for general variational problem

The boundary nodal values are given by the interpolated Dirichlet boundary condition $U(N_j) = \mathcal{I}^h g_D(N_j)$, for all $N_j \in \mathcal{N}_D$. Hence, if we order the nodes so that the boundary nodes have the highest indices, the matrix problem (3.24) has a block structure,

$$\left[\begin{array}{c|c} A_{II} & A_{ID} \\ \hline 0_{DI} & I_{DD} \end{array}\right] \left[\begin{array}{c} x_I \\ \hline x_D \end{array}\right] = \left[\begin{array}{c} b_I \\ \hline b_D \end{array}\right],$$

where A_{II} is a square $n_I \times n_I$ matrix, with n_I the number of internal nodes, A_{ID} an $n_I \times n_D$ matrix, with n_D the number of boundary nodes, I_{DD} an $n_D \times n_D$ identity matrix, 0_{DI} an $n_D \times n_I$ zero matrix, and b_D is an n_D vector with components $(b_D)_j = \mathcal{I}^h g_D(N_j)$.



Assembly algorithm

The matrix and vector are constructed by an assembly algorithm, which loops over all elements K in the mesh to compute the local element matrices $A^K = (a_{ij}^K)$, with

$$a_{i,j}^K = a(\lambda_j, \lambda_i)|_K,$$

and the local element vector

$$b_i^K = L(\lambda_i)|_K,$$

with $a(\cdot,\cdot)|_K$ and $L(\cdot)|_K$ the bilinear and linear forms restricted to element K, and with $\{\lambda_i\}_{i=1}^{n_q-1}$ the element shape functions, for example, local Lagrange basis functions over K. The integrals are often approximated by quadrature over a reference element $\hat{K}$, based on a map $F_K : \hat{K} \to K$.

Algorithm 1: Assembly of matrix $A = (a_{i,j})$ and vector $b = (b_i)$

for $K \in \mathcal{T}^h$ **do**
- **for** $i = 0, 1..., n_q - 1$ **do**
  - $b_i^K = L(\lambda_i)|_K$ ▷ `compute element vector`
  - $b_{loc2glob(i)}$ += b_i^K ▷ `add to global vector`
- **end**
- **for** $i = 0, 1..., n_q - 1$ **do**
  - **for** $j = 0, 1, ..., n_q - 1$ **do**
    - $a_{i,j}^K = a(\lambda_j, \lambda_i)|_K$ ▷ `compute element matrix`
    - $a_{loc2glob(i),loc2glob(j)}$ += $a_{i,j}^K$ ▷ `add to global matrix`
  - **end**
- **end**

end

Assembly algorithm

To add the local element matrix and element vector to the global matrix and vector, we use an index map

$$loc2glob : i_K \to i_A,$$

which maps the index of each local degree of freedom $i \in i_K$, to the corresponding index in the global matrix $loc2glob(i) \in i_A$.

Algorithm 1: Assembly of matrix $A = (a_{i,j})$ and vector $b = (b_i)$

for $K \in \mathcal{T}^h$ **do**
  for $i = 0, 1..., n_q - 1$ **do**
    $b_i^K = L(\lambda_i)|_K$ ▷ `compute element vector`
    $b_{loc2glob(i)} \mathrel{+}= b_i^K$ ▷ `add to global vector`
  end
  for $i = 0, 1..., n_q - 1$ **do**
    for $j = 0, 1, ..., n_q - 1$ **do**
      $a_{i,j}^K = a(\lambda_j, \lambda_i)|_K$ ▷ `compute element matrix`
      $a_{loc2glob(i),loc2glob(j)} \mathrel{+}= a_{i,j}^K$ ▷ `add to global matrix`
    end
  end
end

Assembly algorithm

Algorithm 1: Assembly of matrix $A = (a_{i,j})$ and vector $b = (b_i)$

```
for K ∈ T^h do
    for i = 0, 1..., n_q − 1 do
        b_i^K = L(λ_i)|_K                                  ▷ compute element vector
        b_loc2glob(i) += b_i^K                             ▷ add to global vector
    end
    for i = 0, 1..., n_q − 1 do
        for j = 0, 1, ..., n_q − 1 do
            a_{i,j}^K = a(λ_j, λ_i)|_K                     ▷ compute element matrix
            a_loc2glob(i),loc2glob(j) += a_{i,j}^K         ▷ add to global matrix
        end
    end
end
```

Existence and uniqueness

We can express a general linear partial differential equation as the abstract problem,

$$A(u) = f, \quad \text{in } \Omega, \tag{3.14}$$

with boundary conditions,

$$B(u) = g, \quad \text{on } \partial\Omega. \tag{3.15}$$

For a Hilbert space V consisting of functions with finite norm $\|\cdot\|_V$, we formulate the corresponding variational problem: find $u \in V$, such that

$$a(u, v) = L(v), \quad \forall v \in V, \tag{3.16}$$

with $a : V \times V \to \mathbb{R}$ a bilinear form and $L : V \to \mathbb{R}$ a linear form.

Existence and uniqueness

Theorem 5 (Lax-Milgram theorem)**.** *The variational problem (3.16) has a unique solution $u \in V$, if the bilinear form is elliptic and bounded, and the linear form is bounded. That is, there exist constants $\alpha > 0$, $C_1, C_2 < \infty$, such that for $u, v \in V$,*

$$\begin{aligned} (i) &\qquad a(v,v) \geq \alpha \|v\|_V^2, \\ (ii) &\qquad a(u,v) \leq C_1 \|u\|_V \|v\|_V, \\ (iii) &\qquad L(v) \leq C_2 \|v\|_V. \end{aligned}$$

Ex: linear reaction-diffusion equation

$$
\begin{aligned}
-\Delta u + u = f, &\qquad \text{in } \Omega, \\
u = 0, &\qquad \text{on } \partial\Omega.
\end{aligned}
$$

The corresponding bilinear form,

$$a(u, v) = (\nabla u, \nabla v) + (u, v),$$

is elliptic in $H_0^1(\Omega)$ with $\alpha = 1$, since $\|v\|_1^2 = a(v, v)$, and continuous with $C_1 = 1$ by Cauchy-Schwarz inequality,

$$a(u, v) = (\nabla u, \nabla v) + (u, v) \leq \|\nabla u\| \|\nabla v\| + \|u\| \|v\| \leq \|u\|_1 \|v\|_1.$$

With $f \in H^{-1}(\Omega)$, the linear form is continuous with $C_2 = \|f\|_{-1}$, and hence the variational problem has a unique solution.

Energy norm and stability of solutions

Partial differential equations rarely admit closed form solutions, but we can still infer some characteristics of the solutions from the weak form (3.16). For an elliptic variational problem, a symmetric bilinear form defines an inner product $(\cdot,\cdot)_E = a(\cdot,\cdot)$ on the Hilbert space V, with an associated *energy norm*

$$\|\cdot\|_E = a(\cdot,\cdot)^{1/2},$$

which is equivalent to the norm $(\cdot,\cdot)_V$, since

$$\alpha\|\cdot\|_V^2 \leq (\cdot,\cdot)_E \leq C_1\|\cdot\|_V^2.$$

Energy norm and stability of solutions

For the energy norm we can derive the following stability estimate for the solution $u \in V$ to the variational problem (3.16),

$$\|u\|_E^2 = a(u,u) = L(u) \leq C_2 \|u\|_V \leq (C_2/\alpha)\|u\|_E,$$

so that

$$\|u\|_E \leq (C_2/\alpha).$$

Stability of Poisson's equation

For the Poisson problem (3.6), $\|\nabla u\| \leq C\|f\|$, which follows from

$$\|\nabla u\|^2 = (f,u) \leq \|f\|\|u\| \leq C\|f\|\|\nabla u\| \leq \frac{C^2}{2}\|f\|^2 + \frac{1}{2}\|\nabla u\|^2,$$

where we used Cauchy-Schwarz inequality, Poincaré-Friedrich inequality, and the following version of Young's inequality.

Theorem 6 (Young's inequality). *For $a, b \geq 0$ and $\epsilon > 0$,*

$$ab \leq \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2.$$

Theorem 4 (Poincaré-Friedrich's inequality). *For all $u \in H^1(\Omega)$ there exists a constant $C > 0$, such that*

$$\|u\|^2_{L^2(\Omega)} \leq C(\|u\|^2_{L^2(\partial\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}).$$

Optimality of Galerkin's method

In a Galerkin finite element method we seek an approximation $U \in V_h$,

$$a(U, v) = L(v), \quad \forall v \in V_h, \tag{3.19}$$

with $V_h \subset V$ a finite dimensional subspace, which in the case of a finite element method is a piecewise polynomial space. For an elliptic problem, existence and uniqueness of a solution follows from Lax-Milgram's theorem.

Since $V_h \subset V$, the weak form (3.16) is satisfied also for $v \in V_h$, and by subtracting (3.19) from (3.16) we obtain the Galerkin orthogonality property,

$$a(u - U, v) = 0, \quad \forall v \in V_h.$$

Optimality of Galerkin's method

For an elliptic problem with symmetric bilinear form we can show that the Galerkin approximation is optimal in the energy norm, since

$$\begin{aligned}\|u-U\|_E^2 &= a(u-U,u-U) = a(u-U,u-v) + a(u-U,v-U)\\ &= a(u-U,u-v) \le \|u-U\|_E\|u-v\|_E,\end{aligned}$$

and hence

$$\|u-U\|_E \le \|u-v\|_E, \quad \forall v \in V_h.$$

For an elliptic non-symmetric bilinear form, we can prove *Cea's lemma*,

$$\|u-U\|_V \le \frac{C_1}{\alpha}\|u-v\|_V, \quad \forall v \in V, \tag{3.20}$$

which follows from

$$\begin{aligned}\|u-U\|_V^2 &\le (1/\alpha)a(u-U,u-U) = (1/\alpha)a(u-U,u-v)\\ &\le (C_1/\alpha)\|u-U\|_V\|u-v\|_V.\end{aligned}$$

Poisson’s equation: diffusion

- Examples: Jupyter notebook?

Non-dimensionalization – Reynolds number

The *incompressible Navier-Stokes equations* then takes the form

$$\dot{u} + (u \cdot \nabla)u + \nabla p - \nu \Delta u = f,$$
$$\nabla \cdot u = 0,$$

with the *kinematic viscosity* $\nu = \mu/\rho$, and the kinematic pressure p

$$\dot{u} + (u \cdot \nabla)u + \nabla p - Re^{-1} \Delta u = f,$$
$$\nabla \cdot u = 0.$$

$$Re = \frac{UL}{\nu}$$

Limit cases: Euler and Stokes equations

Formally, in the limit $Re \to \infty$, the viscous term vanishes and we are left with the inviscid *Euler equations*,

$$\dot{u} + (u \cdot \nabla)u + \nabla p = f,$$
$$\nabla \cdot u = 0,$$

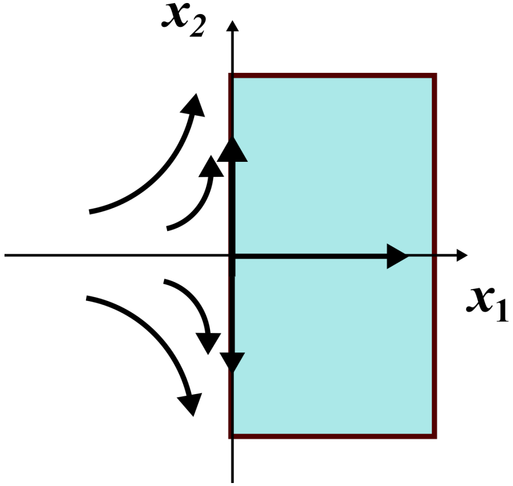
traditionally seen as a model for flow at high Reynolds numbers.

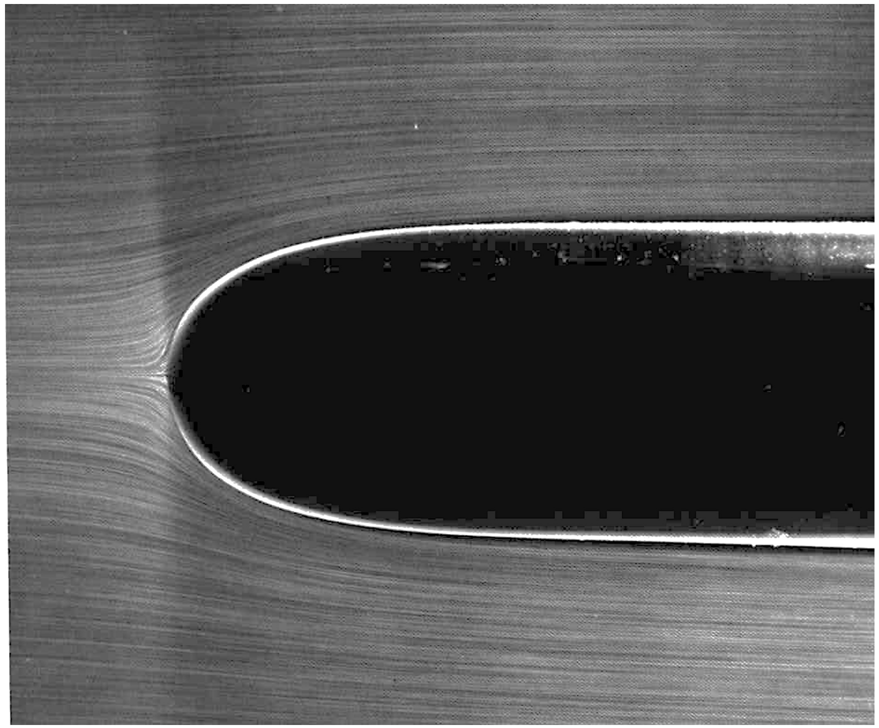
In the limit $Re \to 0$, we obtain the *Stokes equations* as a model of viscous flow,

$$\begin{aligned} -\Delta u + \nabla p &= f, \\ \nabla \cdot u &= 0 \end{aligned}$$

Incompressible flow

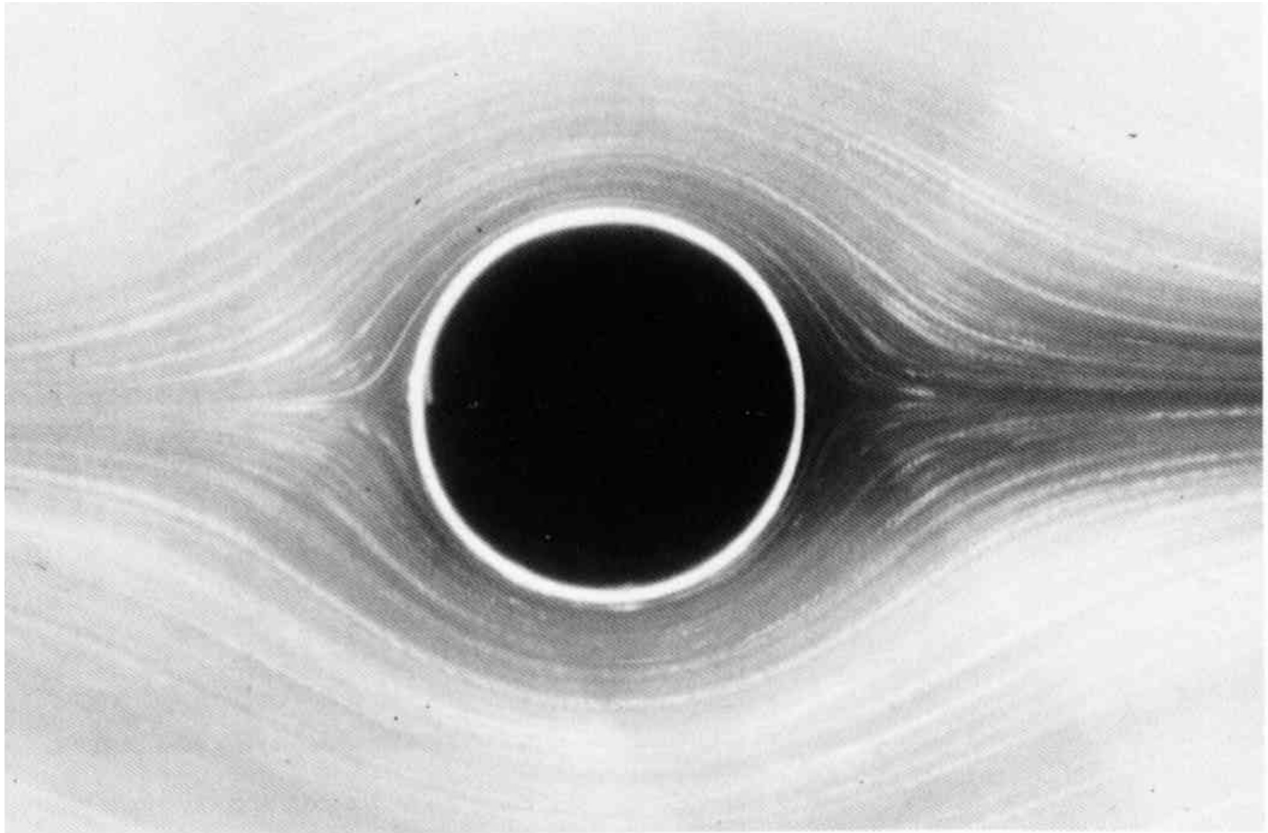
- Approximate small M by M = 0.
- Density constant
- Velocity divergence free: $\nabla \cdot u = 0$
- $\frac{\partial u_2}{\partial x_2} = -\frac{\partial u_1}{\partial x_1}$





[Water and air bubbles.]

Reynolds number Re = 0.16



[Water and aluminum dust.]

Stokes equations

The Stokes equations for a domain $\Omega \subset \mathbb{R}^n$ with boundary $\nabla\Omega = \Gamma_D \cup \Gamma_N$, and associated normal n, takes the form

$$\begin{aligned} -\Delta u + \nabla p &= f, && x \in \Omega, \\ \nabla \cdot u &= 0, && x \in \Omega, \\ u &= g_D, && x \in \Gamma_D, \\ -\nabla u \cdot n + pn &= g_N, && x \in \Gamma_N. \end{aligned}$$

Stokes equations

First assume that $\partial\Omega = \Gamma_D$ and $g_D = 0$, that is, homogeneous Dirichlet boundary conditions for the velocity. We then seek a weak solution to the Stokes equations in the following spaces,

$$V = H_0^1(\Omega) \times ... \times H_0^1(\Omega) = [H_0^1(\Omega)]^n,$$
$$Q = \{q \in L^2(\Omega) : \int_\Omega q \, dx = 0\},$$

where the extra condition in the vector space Q is needed to assure uniqueness of the pressure, which otherwise is undetermined up to a constant.

Stokes equations

We derive the variational formulation by taking the inner product of the momentum equation with a test function $v \in V$, and the inner product of the continuity equation with a test function $q \in Q$. By Green's formula and the homogeneous Dirichlet boundary condition, we obtain the variational formulation as: find $(u,p) \in V \times Q$, such that

$$\begin{aligned} a(u,v) + b(v,p) &= (f,v), \quad \forall v \in V, &(5.6)\\ -b(u,q) &= 0, \quad \forall q \in Q, &(5.7)\end{aligned}$$

$$a(v,w) = (\nabla v, \nabla w) = \int_\Omega \nabla v : \nabla w \, dx, \tag{5.8}$$

$$b(v,q) = -(\nabla \cdot v, q) = -\int_\Omega (\nabla \cdot v) q \, dx, \tag{5.9}$$

Stokes equations

Theorem 7. *The variational problem (5.6)-(5.7) has a unique weak solution* $(u,p) \in V \times Q$*, which satisfies the following stability inequality,*

$$\|u\|_V + \|q\|_Q \leq C\|f\|_{-1},$$

if the following conditions hold,

(i) $a(\cdot,\cdot)$ *is bounded and coercive, i.e. that exists a constant* $\alpha > 0$,

$$a(v,v) \geq \alpha\|v\|_V^2,$$

for all $v \in Z = \{v \in V : b(v,q) = 0, \forall q \in Q\}$,

(ii) $b(\cdot,\cdot)$ *is bounded and satisfies the inf-sup condition, i.e. there exists a constant* $\beta > 0$,

$$\inf_{q \in Q} \sup_{v \in V} \frac{b(v,q)}{\|v\|_V \|q\|_Q} \geq \beta.$$

Stokes equations

We now formulate a finite element method for solving Stokes equations. Since we use different approximation spaces for the velocity and the pressure, we refer to the method as a mixed finite element method.

We seek an approximation $(U, P) \in V_h \times Q_h$, such that,

$$\begin{aligned} a(U,v) + b(v,P) &= (f,v), && (5.11) \\ -b(U,q) &= 0, && (5.12) \end{aligned}$$

for all $(v, q) \in V_h \times Q_h$, where V_h and Q_h are finite element approximation spaces. There exists a unique solution to (5.11)-(5.12), under similar conditions as for the continuous variational problem.

Stokes equations

Theorem 8. *The mixed finite element problem (5.11)-(5.12) has a unique solution* $(U, P) \in V_h \times Q_h$*, if*

(i) $a(\cdot,\cdot)$ *is coercive, i.e. that exists a constant* $\alpha_h > 0$*, such that*

$$a(v, v) \geq \alpha_h \|v\|_V,$$

for all $v \in Z_h = \{v \in V_h : b(v, q) = 0, \forall q \in Q_h\}$*,*

(ii) $b(\cdot,\cdot)$ *satisfies the inf-sup condition, i.e. there exists a constant* $\beta_h > 0$*,*

$$\inf_{q \in Q_h} \sup_{v \in V_h} \frac{b(v, q)}{\|v\|_V \|q\|_Q} \geq \beta_h,$$

and this unique solution satisfies the following error estimate,

$$\|u - U\|_V + \|p - P\|_Q \leq C \left(\inf_{v \in V_h} \|u - v\| + \inf_{q \in Q_h} \|p - q\| \right),$$

for a constant $C > 0$*.*

Stokes equations

The pair of approximation spaces must be chosen to satisfy the inf-sup condition, with the velocity space sufficiently rich compared to the pressure space. For example, continuous piecewise quadratic approximation of the velocity and continuous piecewise linear approximation of the pressure, referred to as the Taylor-Hood elements. On the other hand, continuous piecewise linear approximation of both velocity and pressure is not inf-sup stable.

Stokes equations

We seek finite element approximations in the following spaces,

$$V_h = \{v = (v_1, v_2, v_3) : v_k(x) = \sum_{j=1}^{N} v_k^j \phi_j(x), k = 1, 2, 3\}$$

$$Q_h = \{q : q(x) = \sum_{j=1}^{M} q^j \psi_j(x)\},$$

which leads to a discrete system in matrix form,

$$\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix},$$

with u and p vectors holding the coordinates of U and P in the respective bases of V_h and Q_h.

Stokes equations

The matrix A is symmetric positive definite and thus invertible, so we can express

$$u = A^{-1}(f - Bp),$$

and since $B^T u = 0$,

$$B^T A^{-1} B p = B^T A^{-1} f,$$

which is the *Schur complement* equation. If $\mathrm{null}(B) = \{0\}$, then the matrix $S = B^T A^{-1} B$ is symmetric positive definite and can also be inverted.

Stokes equations

Schur complement methods take the form

$$p_k = p_{k-1} - C^{-1}(B^T A^{-1} B p_{k-1} - B^T A^{-1} f),$$

where C^{-1} is a preconditioner for $S = B^T A^{-1} B$. The Usawa algorithm is based on C^{-1} as a scaled identity matrix, which gives

1. Solve $Au_k = f - Bp_{k-1}$,
2. Set $p_k = p_{k-1} + \alpha B^T u_k$.

Stokes equations

Approximation spaces of equal order is possible, by stabilization of the standard Galerkin finite element method: find $(U, P) \in V_h \times Q_h$, such that,

$$\begin{aligned} a(U,v) + b(v,P) &= (f,v), \\ -b(U,q) + s(P,q) &= 0, \end{aligned}$$

for all $(v, q) \in V_h \times Q_h$, where $s(P, q)$ is a pressure stabilization term. The resulting discrete system takes the form,

$$\begin{bmatrix} A & B \\ B^T & S \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix},$$

where the stabilization term is chosen so that the matrix S is invertible.

Stokes equations

For example, the *Brezzi-Pitkäranta* stabilization takes the form,

$$s(P, q) = C \int_\Omega h^2 \nabla P \cdot \nabla q \, dx,$$

with $C > 0$ a constant.

Stokes equations: Lab1

- Jupyter notebook