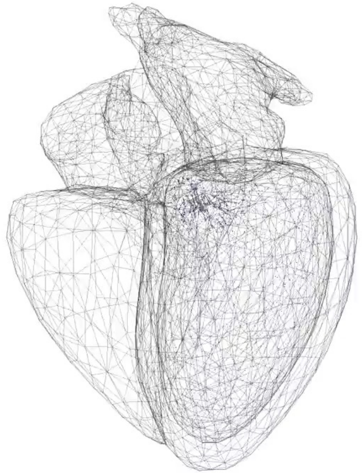
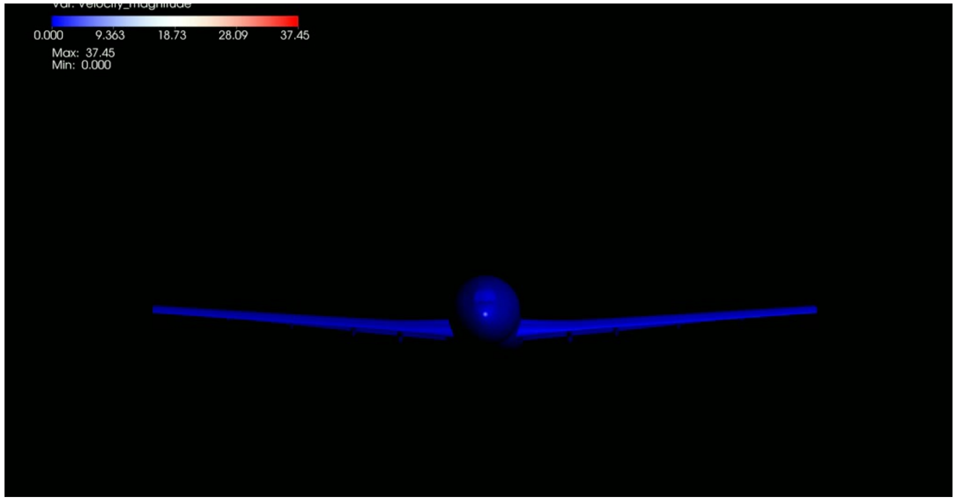
DD2365/2022 – lecture 1 Introduction

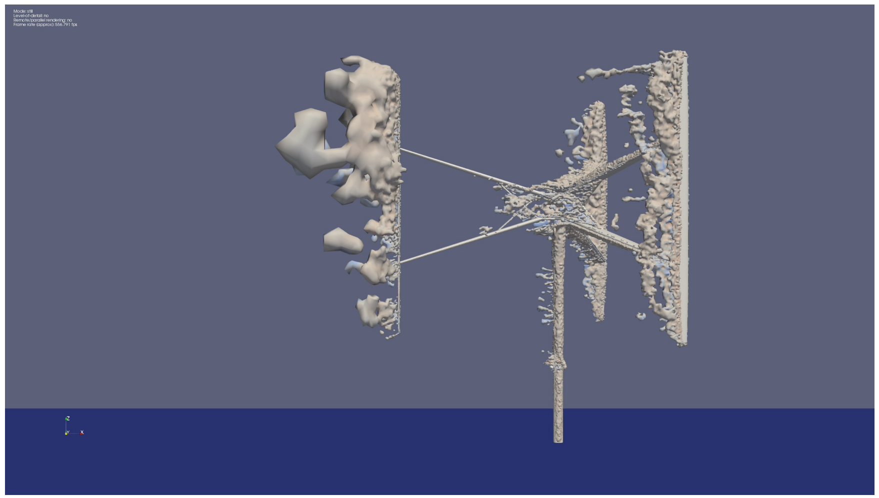
Johan Hoffman

Who am I?

- Professor of numerical analysis
- Research: fluid dynamics, medicine, renewable energy,…
- https://www.kth.se/profile/jhoffman







This course

Theory and labs (weeks 1-5)

- Mathematical model of fluid flow: Navier-Stokes equations
- Numerical approximation of Navier-Stokes equations
- Computer simulation of fluid flow

Project work (weeks 6-11)

- Research question
- Simulation experiments
- Analysis and conclusion

This course

Computer simulation of fluid flow

- Mathematical model: Navier-Stokes equations
- Numerical approximation: Finite Element Method (FEM)
- Computational platform: FEniCS Jupyter notebooks & Google Colab

- Course (open) GitHub repository:

  https://github.com/johanhoffman/DD2365_VT22

This course

Labs

1. Stokes equations (viscous flow)
2. Navier-Stokes equations
3. Adaptive finite element methods
4. Fluid-structure interaction

Today

- Vector calculus
- Function spaces
- Conservation laws
- Finite element method

[Lecture notes, chapters 1-3]

Function spaces: continuous functions

For $\Omega \subset \mathbb{R}^n$, we define the set of functions with k continuous derivatives,

$$C^k(\Omega) = \{\phi : D^\alpha \phi \in C(\Omega),\ |\alpha| \le k\},$$

with $C(\Omega) = C^0(\Omega)$ and $C^\infty(\Omega) = \cap_{k=0}^\infty C^k(\Omega)$. The subset $C_0^k(\Omega)$ consists of the functions $\phi \in C^k(\Omega)$ that have *compact support* in Ω, that is, the support

$$\operatorname{supp}(\phi) = \{x \in \Omega : \phi(x) \neq 0\},$$

is closed and bounded.

$D_j = \partial/\partial x_j$ is the differential operator of partial differentiation

Differential operators in R^n

The *gradient* of a scalar function $f \in C^1(\Omega)$ is denoted by

$$\text{grad}\, f = \nabla f = (D_1 f, ..., D_n f)^T,$$

or in index notation $D_i f$, with the *nabla* operator

$$\nabla = (D_1, ..., D_n)^T.$$

Further, the *directional derivative* $\nabla_v f$, of f in the direction of the vector field $v : \mathbb{R}^n \to \mathbb{R}^n$, is defined as

$$\nabla_v f = (v \cdot \nabla) f = v_j D_j f.$$

Differential operators in R^n

For the $C^1(\Omega)$ vector field $F : \mathbb{R}^n \to \mathbb{R}^m$, we define the *Jacobian* J,

$$J = F' = \nabla F = \begin{bmatrix} D_1F_1 & \cdots & D_nF_1 \\ \vdots & \ddots & \vdots \\ D_1F_m & \cdots & D_nF_m \end{bmatrix} = \begin{bmatrix} (\nabla F_1)^T \\ \vdots \\ (\nabla F_m)^T \end{bmatrix} = D_jF_i,$$

with directional derivative

$$\nabla_v F = (v \cdot \nabla)F = Jv = v_jD_jF_i.$$

Differential operators in R^n

For a scalar function $f \in C^2(\mathbb{R}^n)$, we define the *Laplacian*

$$\Delta f = \nabla^2 f = \nabla^T \nabla f = \nabla \cdot \nabla f = D_1^2 f + ... + D_n^2 f = D_i^2 f,$$

The *vector Laplacian* of a $C^2(\Omega)$ vector field $F : \mathbb{R}^n \to \mathbb{R}^n$, is defined as

$$\Delta F = \nabla^2 F = (\Delta F_1, ..., \Delta F_n)^T,$$

Gauss theorem

For a $C^1(\Omega)$ vector field $F : \mathbb{R}^n \to \mathbb{R}^n$, we define the *divergence*

$$\operatorname{div} F = \nabla \cdot F = D_1 F_1 + ... + D_n F_n = \frac{\partial F_i}{\partial x_i},$$

The divergence can be interpreted in terms of *Gauss theorem*, which states that the volume integral of the divergence of F in $\Omega \subset \mathbb{R}^n$, is equal to a surface integral over $\partial\Omega$ of F projected in the direction of the unit outward normal n of $\partial\Omega$,

$$\int_\Omega \nabla \cdot F \, dx = \int_{\partial\Omega} F \cdot n \, ds,$$

with the surface integral defined by a suitable parameterization of $\partial\Omega$.

Function spaces: integrable functions

For $\Omega \subset \mathbb{R}^n$ an open set and p a positive real number, we denote by $L^p(\Omega)$ the class of all Lebesgue measurable functions u defined on Ω, such that

$$\int_\Omega |u(x)|^p \, dx < \infty,$$

where we identify functions that are equal almost everywhere in Ω.

$L^p(\Omega)$ is a Banach space for $1 \leq p < \infty$, with the norm

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p}.$$

In the case of a vector valued function $u : \mathbb{R}^n \to \mathbb{R}^m$, we replace the integrand in the definitions by the l^p norm, and for a matrix function $u : \mathbb{R}^n \to \mathbb{R}^{m \times k}$, a generalized Frobenius norm

$$\sum_i^m \sum_j^k |u_{ij}(x)|^p.$$

Function spaces: integrable functions

Theorem 3 (Hölder's inequality for $L^p(\Omega)$)**.** *Let* $1 \leq p, q \leq \infty$ *and* $1/p + 1/q = 1$*. If* $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$ *then* $fg \in L^1(\Omega)$*, and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

$L^2(\Omega)$ is a Hilbert space with the inner product

$$(u, v) = (u, v)_{L^2(\Omega)} = \int_\Omega u(x)v(x)\, dx, \tag{2.1}$$

which induces the $L^2(\Omega)$ norm. For vector valued functions the integrand is replaced by the l_2 inner product, and for matrix functions by the Frobenius inner product. In what follows, we let $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$.

Function spaces: Sobolev spaces

To construct appropriate vector spaces for partial differential equations, we extend the L^p spaces with derivatives. We first define the *Sobolev norms*,

$$\|u\|_{k,p} = \left(\sum_{0 \leq |\alpha| \leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p},$$

for $1 \leq p < \infty$, and

$$\|u\|_{k,\infty} = \max_{0 \leq |\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)},$$

where $D^\alpha u$ refers to weak derivatives. Equipped with the Sobolev norms, we then define the *Sobolev spaces*,

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega),\, 0 \leq |\alpha| \leq k\},$$

for each positive integer k and $1 \leq p \leq \infty$, with $W^{0,p}(\Omega) = L^p(\Omega)$.

Function spaces: Sobolev spaces

The Sobolev spaces $H^k(\Omega) = W^{k,2}(\Omega)$ and $H_0^k(\Omega) = W_0^{k,2}(\Omega)$ are Hilbert spaces with the inner product and associated norm

$$(u,v)_k = \sum_{0\le\alpha\le k} (D^\alpha u, D^\alpha v), \quad \|u\|_k = (u,u)_k^{1/2},$$

for which Cauchy-Schwarz inequality is satisfied,

$$|(u,v)_k| \le \|u\|_k \|v\|_k.$$

We denote by $H^{-k}(\Omega)$ the dual space of $H_0^k(\Omega)$, with the norm

$$\|u\|_{-k} = \sup_{v\in H_0^k(\Omega)} \frac{|(u,v)|}{\|v\|_k} = \sup_{v\in H_0^k(\Omega):\|v\|_k=1} |(u,v)|,$$

satisfying a generalized Hölder inequality for $u \in H^{-k}(\Omega)$ and $v \in H_0^k(\Omega)$,

$$|(u,v)| \le \|u\|_{-k} \|v\|_k.$$

Partial integration/Green's theorem

For a scalar function $f : \mathbb{R}^n \to \mathbb{R}$, and a vector function $F : \mathbb{R}^n \to \mathbb{R}^n$, we have the following generalization of partial integration over a domain $\Omega \subset \mathbb{R}^n$, referred to as *Green's theorem*,

$$(\nabla f, F) = -(f, \nabla \cdot F) + \langle f, F \cdot n \rangle,$$

with n the unit outward normal vector for the boundary $\partial\Omega$, and where we use the notation

$$\langle v, w \rangle = (v, w)_{L^2(\partial\Omega)} \tag{2.2}$$

for the boundary integral. With $F = \nabla g$ and $g : \mathbb{R}^n \to \mathbb{R}$ a scalar function,

$$(\nabla f, \nabla g) = -(f, \Delta g) + \langle f, \nabla g \cdot n \rangle.$$

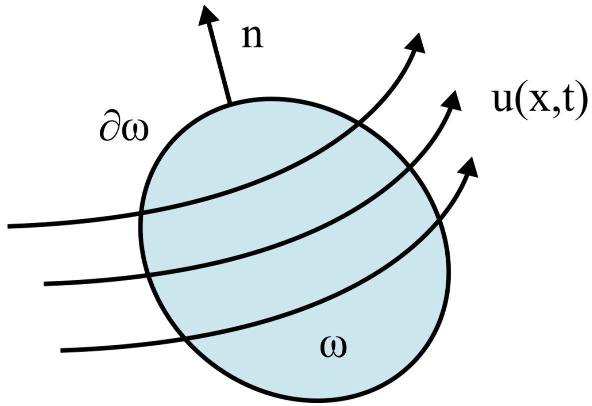
Conservation laws

Consider an arbitrary open subdomain $\omega \subset \mathbb{R}^n$. For a time $t > 0$, the total flow of a quantity with density $\phi(x,t)$ through the boundary $\partial\omega$ is given by

$$\int_{\partial\omega} \phi u \cdot n \, ds,$$

where n is the outward unit normal of $\partial\omega$, and $u = u(x,t)$ is the velocity of the flow. The change of the total quantity ϕ in ω is equal to the volume source or sink $s = s(x,t)$, minus the total flow of the quantity through the boundary $\partial\omega$,

$$\frac{d}{dt}\int_{\omega} \phi(x,t)\, dx = -\int_{\partial\omega} \phi u \cdot n \, ds + \int_{\omega} s(x,t)\, dx$$



Conservation laws

$$\frac{d}{dt}\int_\omega \phi(x,t)\,dx = -\int_{\partial\omega} \phi u \cdot n\,ds + \int_\omega s(x,t)\,dx,$$

Gauss' theorem,

$$\int_\omega \left(\frac{\partial}{\partial t}\phi(x,t) + \nabla\cdot(\phi u) - s\right)\,dx = 0,$$

and assuming the integrand is continuous in ω, we are lead to the general conservation equation

$$\dot{\phi} + \nabla\cdot(\phi u) - s = 0, \tag{5.1}$$

for $t > 0$ and $x \in \omega$, with $\omega \subset \mathbb{R}^n$ any open domain for which the equation is sufficiently regular.

Conservation of mass

Now consider the flow of a continuum with $\rho = \rho(x, t)$ the mass density of the continuum. The general continuity equation (5.1) with $\phi = \rho$ and zero source $s = 0$, gives the equation for conservation of mass

$$\dot{\rho} + \nabla \cdot (\rho u) = 0.$$

We say that a flow is *incompressible* if

$$\nabla \cdot u = 0,$$

or equivalently, if the *material derivative* is zero,

$$\frac{D\rho}{Dt} = \dot{\rho} + u \cdot \nabla \rho = 0,$$

since

$$0 = \dot{\rho} + \nabla \cdot (\rho u) = \frac{D\rho}{Dt} + \rho \nabla \cdot u.$$

Conservation of momentum

Newton's 2nd Law states that the change of *momentum* ρu over an arbitrary open subdomain $\omega \subset \mathbb{R}^n$, is equal to the sum of all forces, including *volume forces*,

$$\int_\omega \rho f \, dx,$$

for a force density $f = f(x,t) = (f_1(x,t), ..., f_n(x,t))$, and *surface forces*,

$$\int_{\partial\omega} n \cdot \sigma \, ds,$$

with the *Cauchy stress tensor* $\sigma = \sigma(x,t) = (\sigma_{ij}(x,t))$, and where we define $n \cdot \sigma = n^T \sigma = (\sigma_{ji} n_j)$. Gauss' theorem gives the total force as

$$\int_\omega \rho f \, dx + \int_{\partial\omega} n \cdot \sigma \, ds = \int_\omega (\rho f + \nabla \cdot \sigma) \, dx.$$

Conservation of momentum

The general continuity equation with $\phi = \rho u$, and the source given by the sum of all forces, leads to the equation for conservation of momentum

$$\frac{\partial}{\partial t}(\rho u) + \nabla \cdot (\rho u \otimes u) = \rho f + \nabla \cdot \sigma, \tag{5.2}$$

with $u \otimes u = uu^T$, the tensor product of the velocity vector field u. With the help of conservation of mass, we can rewrite the left hand side as

$$\frac{\partial}{\partial t}(\rho u) + \nabla \cdot (\rho u \otimes u) = u(\dot{\rho} + \nabla \cdot (\rho u)) + \rho(\dot{u} + (u \cdot \nabla)u) = \rho(\dot{u} + (u \cdot \nabla)u),$$

so that

$$\rho(\dot{u} + (u \cdot \nabla)u) = \rho f + \nabla \cdot \sigma. \tag{5.3}$$

We say that (5.2) is an equation on *conservation form*, whereas (5.3) is on *non-conservation form*.

Conservation of momentum

We define the *mechanical pressure* as the mean normal stress,

$$p_{mech} = -\frac{1}{3}\,\mathrm{tr}(\sigma) = -\frac{1}{3}I_1,$$

and the *deviatoric stress tensor* $\tau = \sigma + p_{mech}I$, with $\mathrm{tr}(\tau) = 0$, such that

$$\sigma = -p_{mech}I + \tau,$$

and we can write conservation of momentum as

$$\rho(\dot{u} + (u \cdot \nabla)u) = \rho f - \nabla p_{mech} + \nabla \cdot \tau.$$

Newtonian flow

To determine the deviatoric stress we need a constitutive model of the fluid. For a Newtonian fluid, the deviatoric stress depends linearly on the *strain rate tensor*

$$\epsilon = \frac{1}{2}(\nabla u + (\nabla u)^T) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

with $\tau = 2\mu\epsilon$, where μ is the *dynamic viscosity*, which we here assume to be constant.

The *incompressible Navier-Stokes equations* then takes the form,

$$\dot{u} + (u \cdot \nabla)u + \nabla p - \nu \Delta u = f, \tag{5.4}$$

$$\nabla \cdot u = 0, \tag{5.5}$$

with the *kinematic viscosity* $\nu = \mu/\rho$, and the kinematic pressure $p = p_{mech}/\rho$.

Non-dimensionalization

$$u = Uu_*, \quad p = Pp_*, \quad x = Lx_*, \quad f = Ff_*, \quad t = Tt_*,$$

where U, P, L, T are characteristic scales of the velocity, pressure, force, length and time, respectively. The resulting non-dimensionalized differential operators are scaled as,

$$\frac{\partial}{\partial t} = \frac{1}{T}\frac{\partial}{\partial t_*}, \quad \nabla = \frac{1}{L}\nabla_*, \quad \Delta = \frac{1}{L^2}\Delta_*,$$

which gives

$$\begin{aligned}
&\frac{U}{T}\frac{\partial}{\partial t_*}u_* + \frac{U^2}{L}(u_* \cdot \nabla_*)u_* + \frac{P}{L}\nabla_* p_* - \frac{\nu U}{L^2}\Delta_* u_* = Ff_*, \\
&\frac{U}{L}\nabla \cdot u_* = 0,
\end{aligned}$$

Non-dimensionalization

$$\dot{u} + (u \cdot \nabla)u + \nabla p - Re^{-1}\Delta u = f,$$
$$\nabla \cdot u = 0.$$

Here we have dropped the non-dimensional notation for simplicity, with

$$T = L/U, \quad P = U^2, \quad F = \frac{U^2}{L}, \quad Re = \frac{UL}{\nu},$$

where the *Reynolds number* Re determines the balance between inertial and viscous characteristics in the flow. For low Re linear viscous effects dominate, whereas for high Re we have a flow dominated by nonlinear inertial effect, and turbulence for sufficiently high Reynolds numbers.

Limit cases: Euler and Stokes equations

Formally, in the limit $Re \to \infty$, the viscous term vanishes and we are left with the inviscid *Euler equations*,

$$\dot{u} + (u \cdot \nabla)u + \nabla p = f,$$
$$\nabla \cdot u = 0,$$

traditionally seen as a model for flow at high Reynolds numbers. Alt

In the limit $Re \to 0$, we obtain the *Stokes equations* as a model of viscous flow, now with $P = \nu U/L$ and $F = \nu U/L^2$,

$$\begin{aligned} -\Delta u + \nabla p &= f, \\ \nabla \cdot u &= 0, \end{aligned}$$

Anatomy of fluid flow

- Density ρ
- Velocity u
- Pressure p
- Viscosity (dynamic viscosity μ, kinematic viscosity $\upsilon = \frac{\mu}{\rho}$)
- Gravity g
- Surface tension σ
- Speed of sound c
- …

Anatomy of fluid flow

- Mach number $M = \frac{u}{c}$
- Reynolds number Re = $\frac{\rho U L}{\mu} = \frac{U L}{\upsilon}$
- …

Compressibility – Shock waves

- Shock waves appear for M>1
- Flow is compressible for M>0.2
- Flow is incompressible for M<0.2

[https://en.wikipedia.org/wiki/Mach_number#/media/File:FA-18_Hornet_breaking_sound_barrier_(7_July_1999).jpg]

Compressibility – Shock waves

- Shock waves appear for M>1
- Flow is compressible for M>0.2
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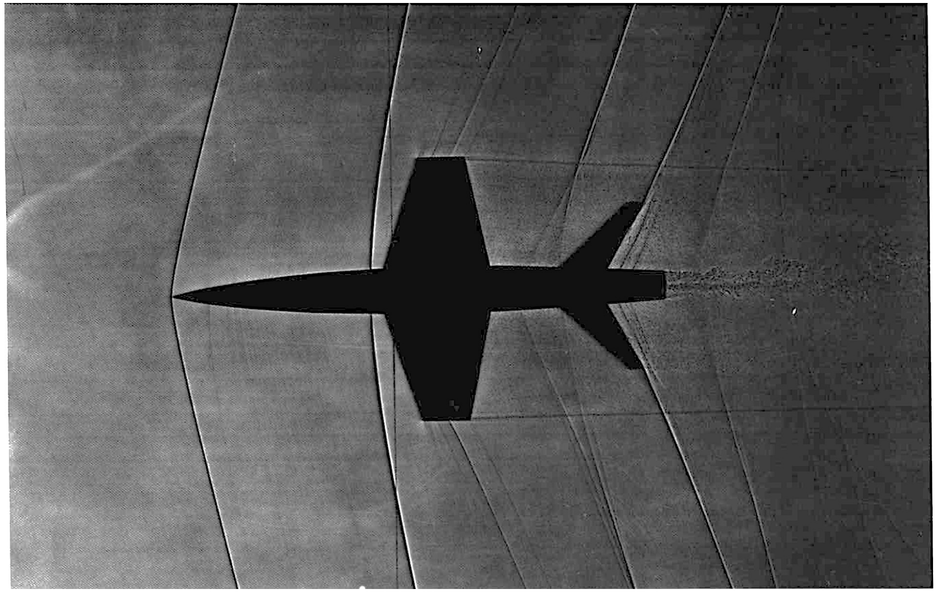
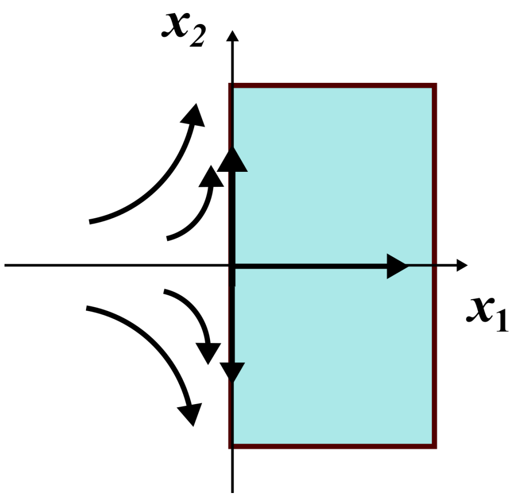
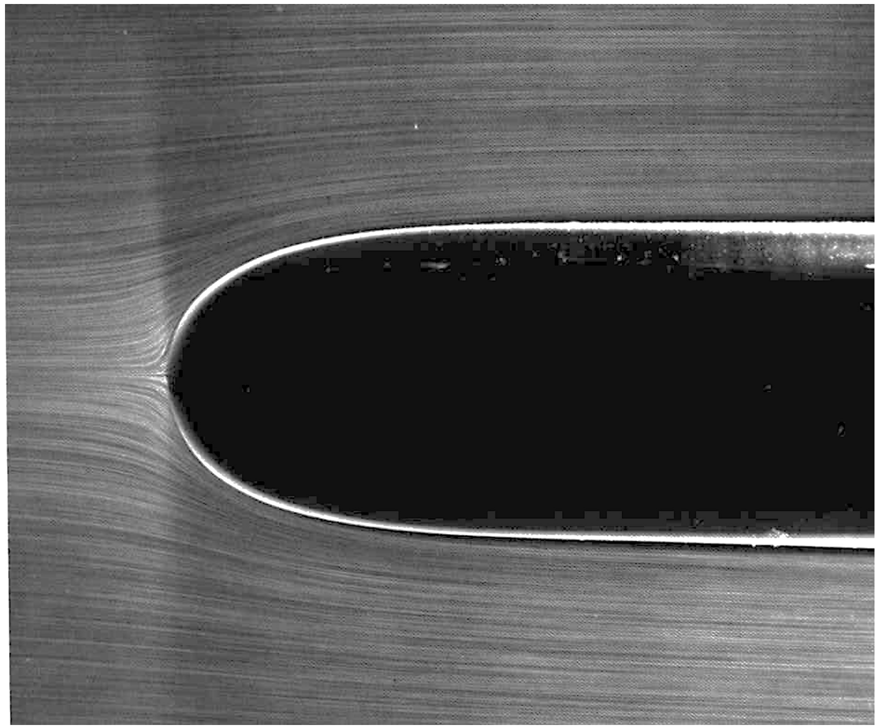


[Shadow graph]

Incompressible flow

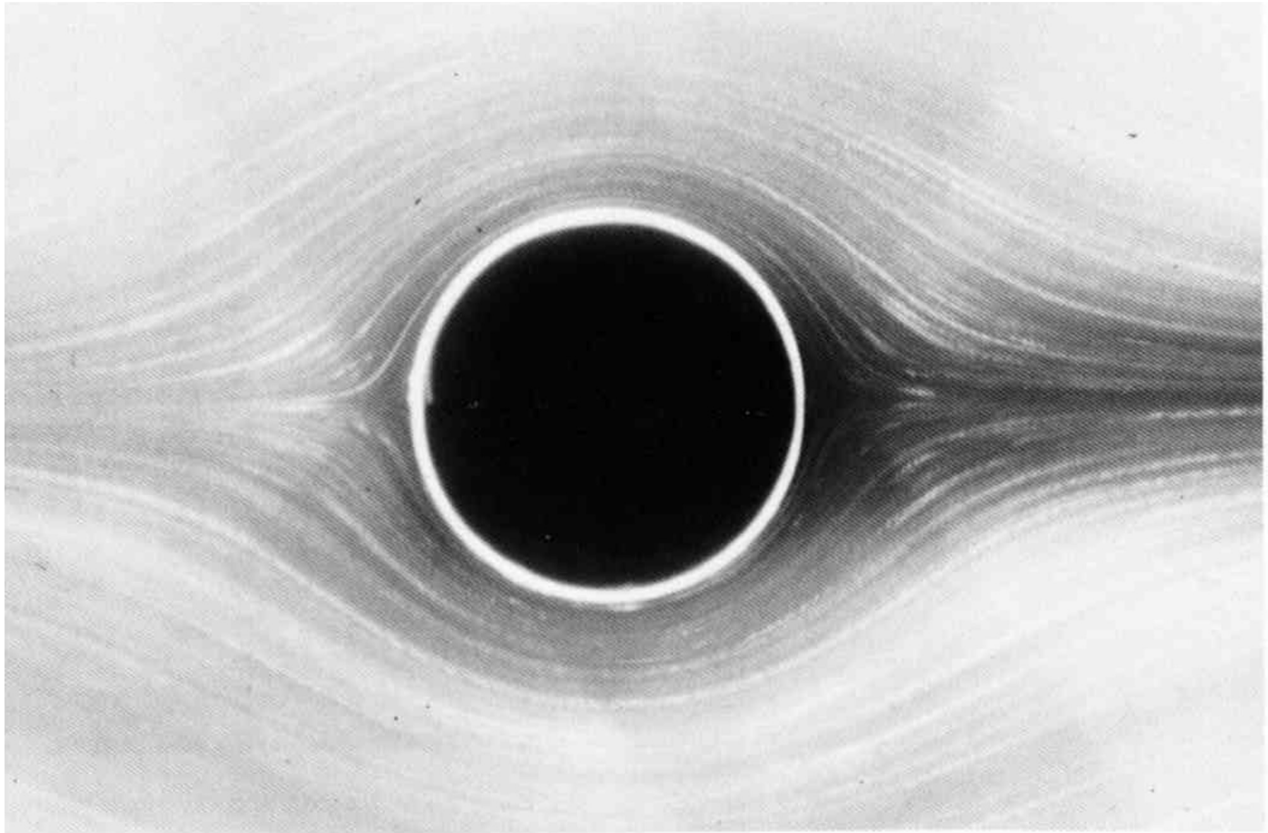
- Approximate small M by M = 0.
- Constant density
- Velocity divergence free: $\nabla \cdot u = 0$
- $\frac{\partial u_2}{\partial x_2} = -\frac{\partial u_1}{\partial x_1}$





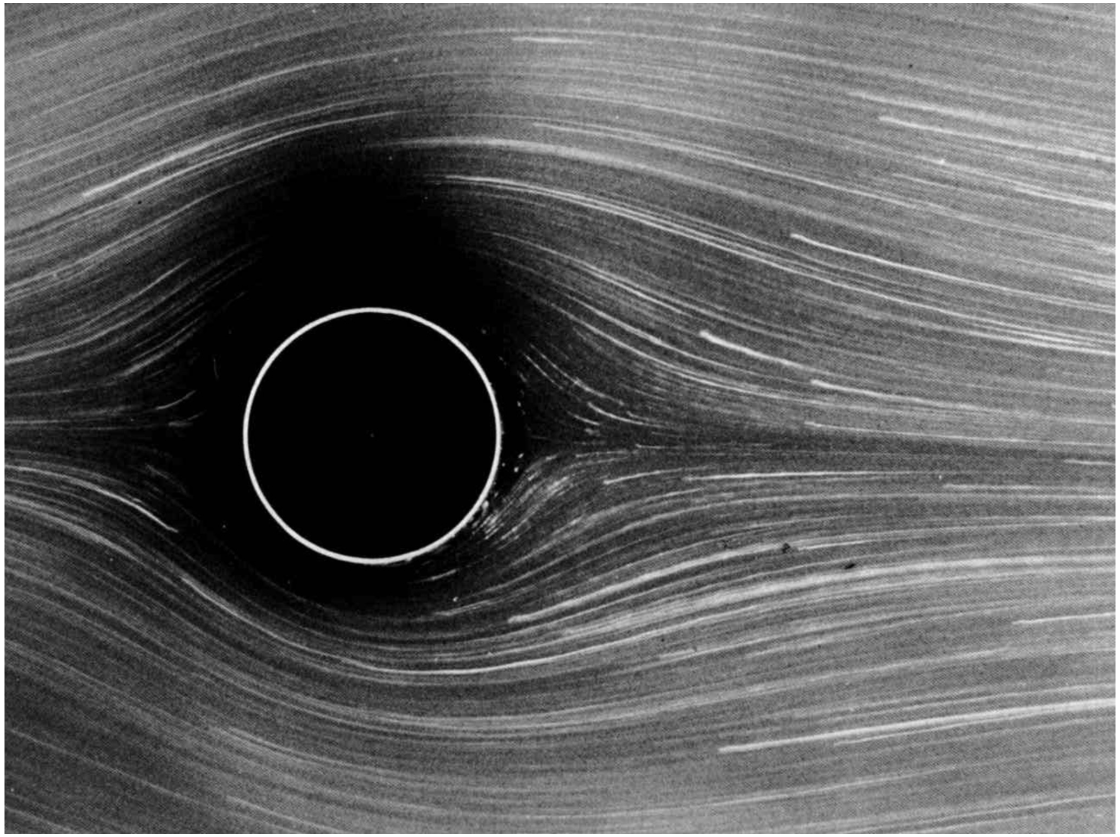
[Water and air bubbles.]

Reynolds number Re = 0.16



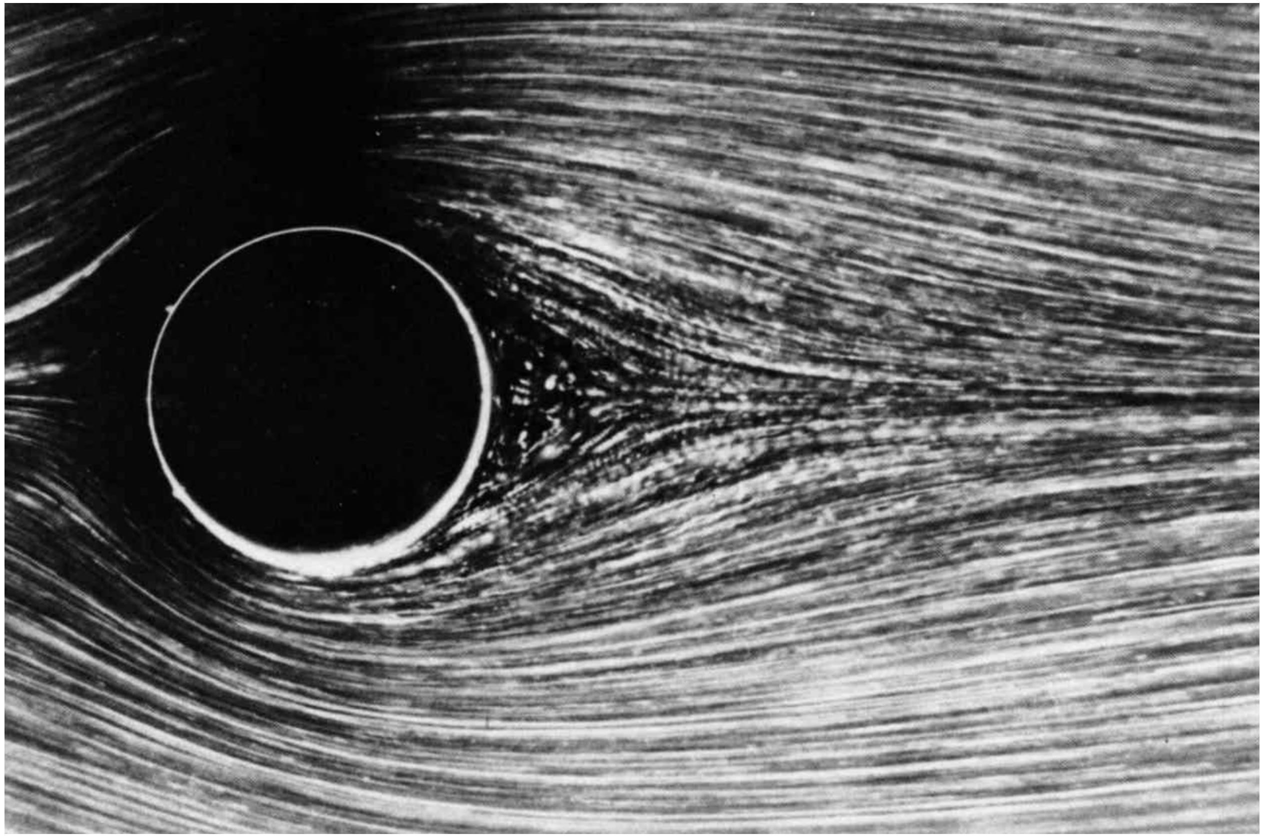
[Water and aluminum dust.]

Reynolds number Re = 1.54



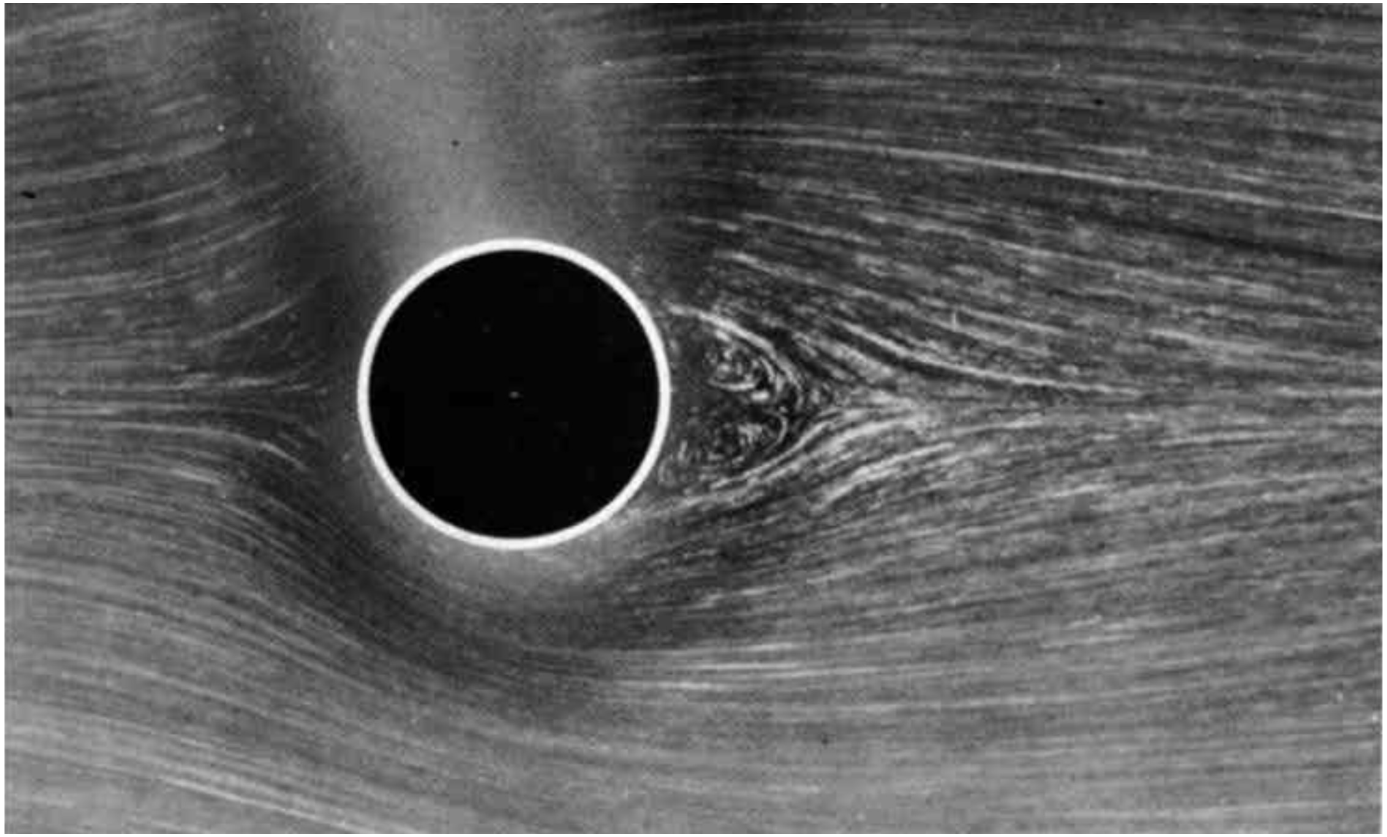
[Water and aluminum dust.]

Reynolds number Re = 9.6



[Water and aluminum dust.]

Reynolds number Re = 9.6

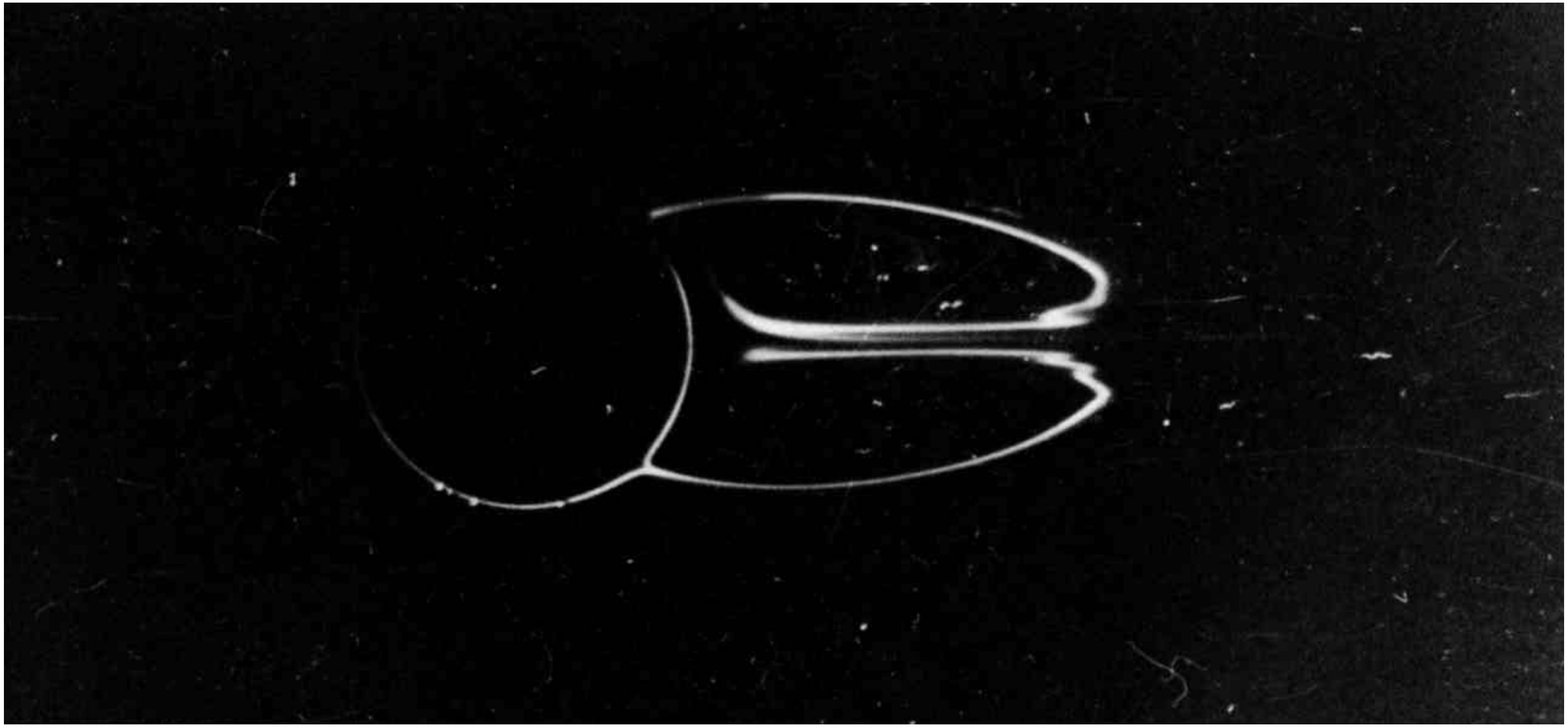


[Water and aluminum dust.]

Reynolds number Re = 26

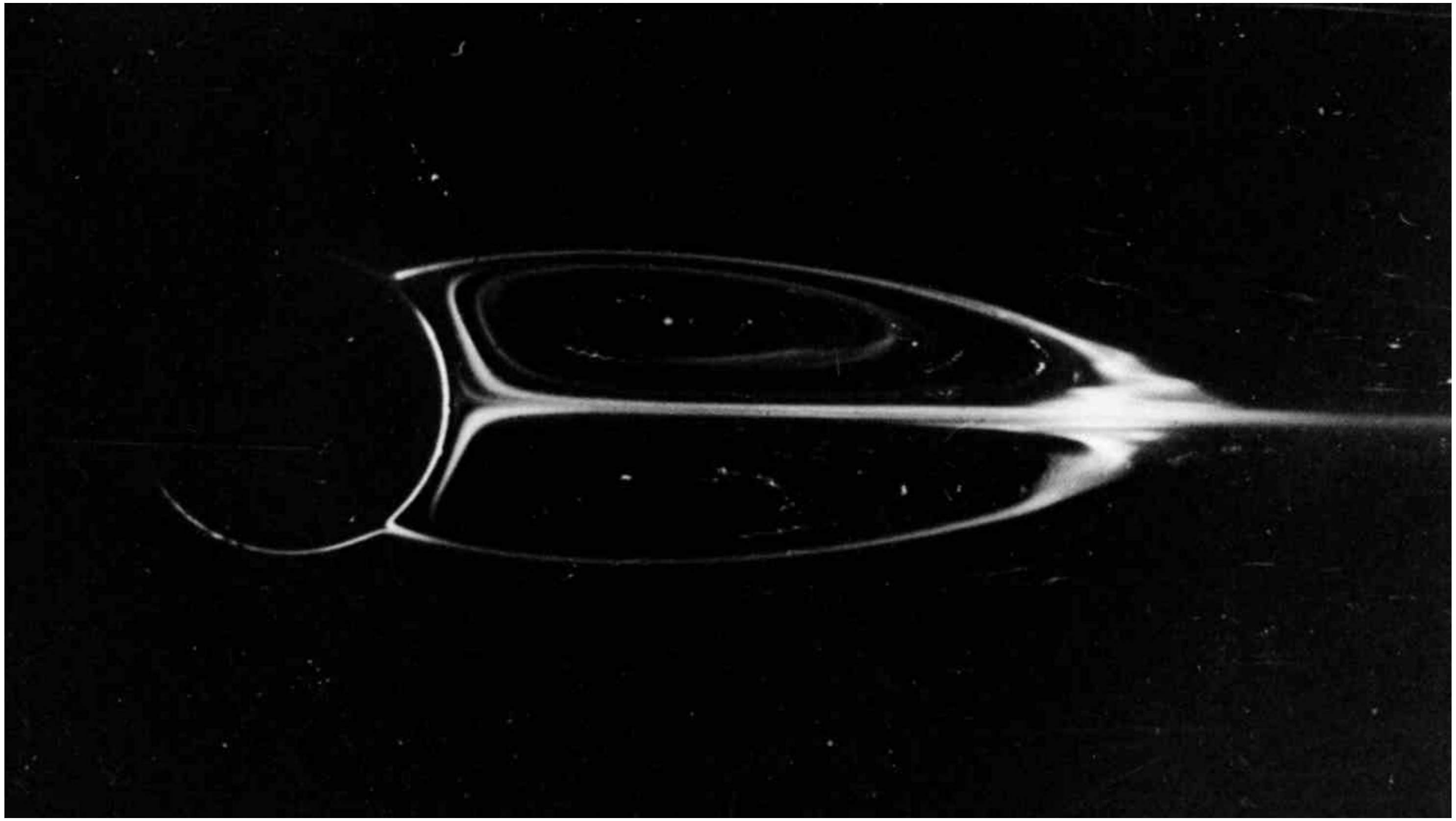
[Oil and magnesium.]

Reynolds number Re = 28.4



[Water and condensed milk.]

Reynolds number Re = 41



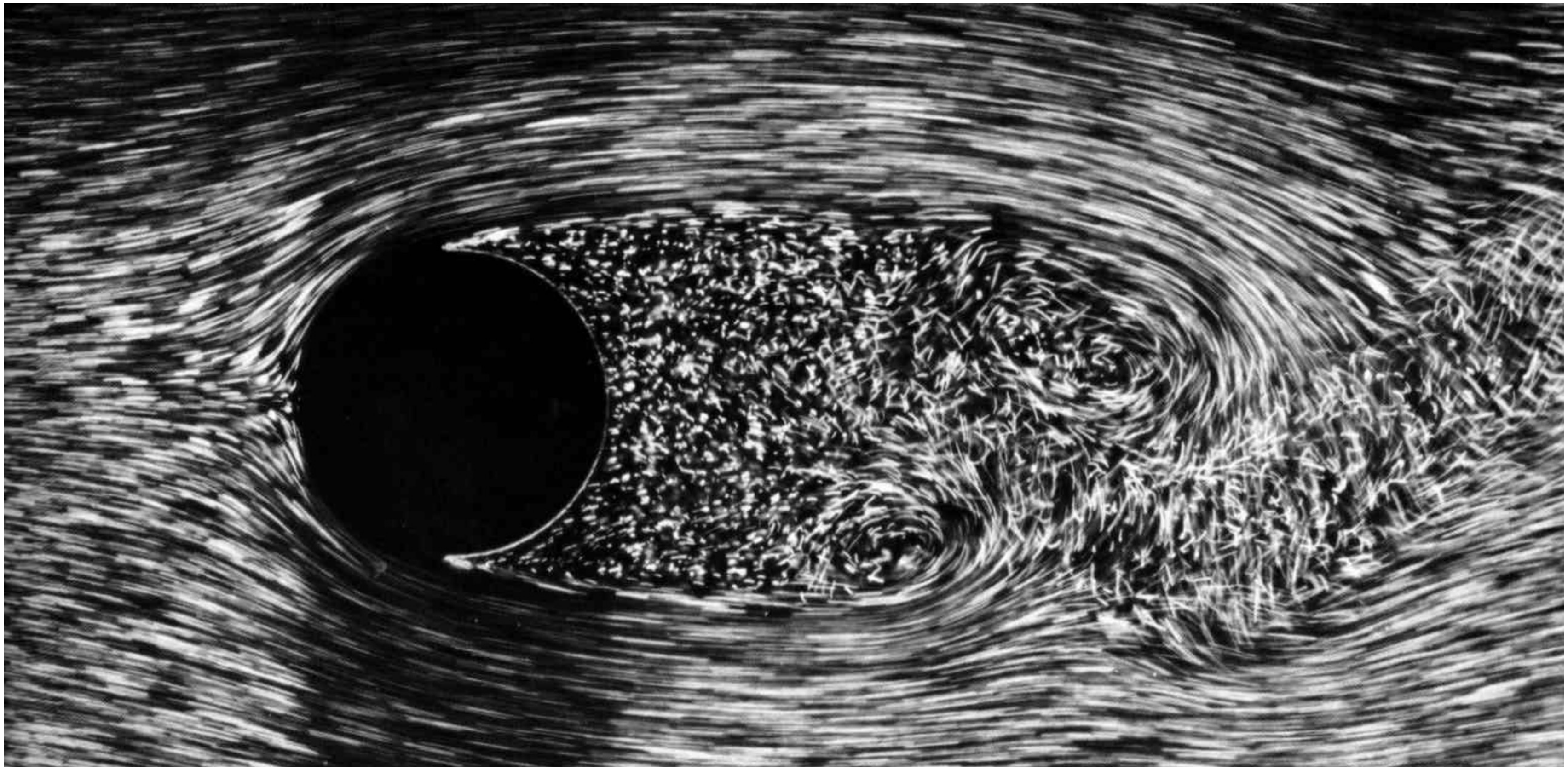
[Water and condensed milk.]

Reynolds number Re = 300

Karman vortex street

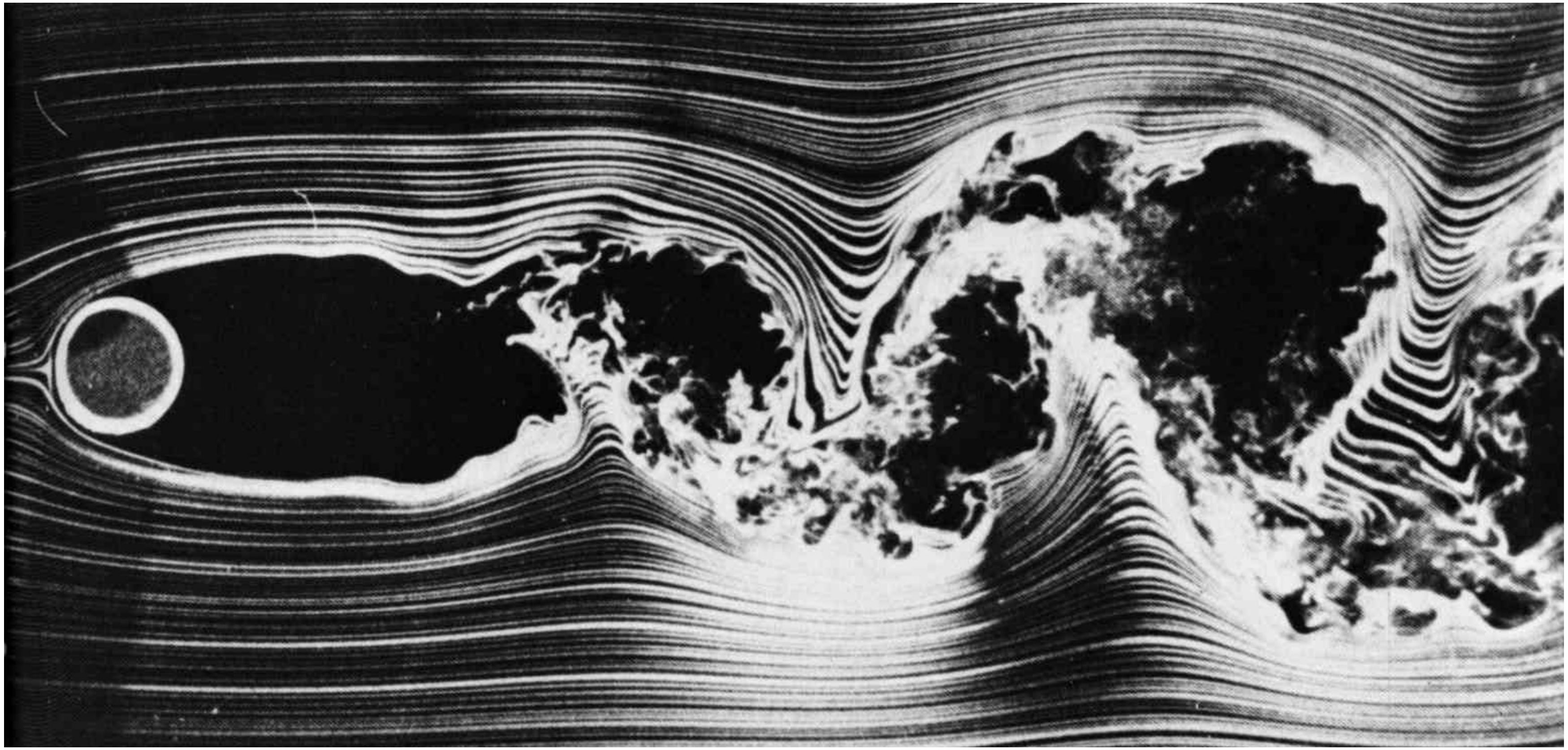
[Wind and smoke.]

Reynolds number Re = 2000



[Water and air bubbles.]

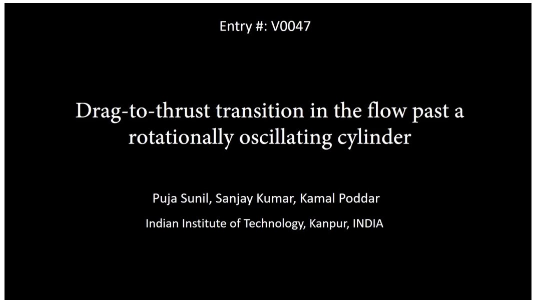
Reynolds number Re = 10 000



[Water and air bubbles.]

Vortex shedding

- https://www.youtube.com/watch?v=9FRTj6_1J2k



- https://gfm.aps.org/meetings/dfd-2020/5f5f0056199e4c091e67bd9e

[Water and air bubbles.]

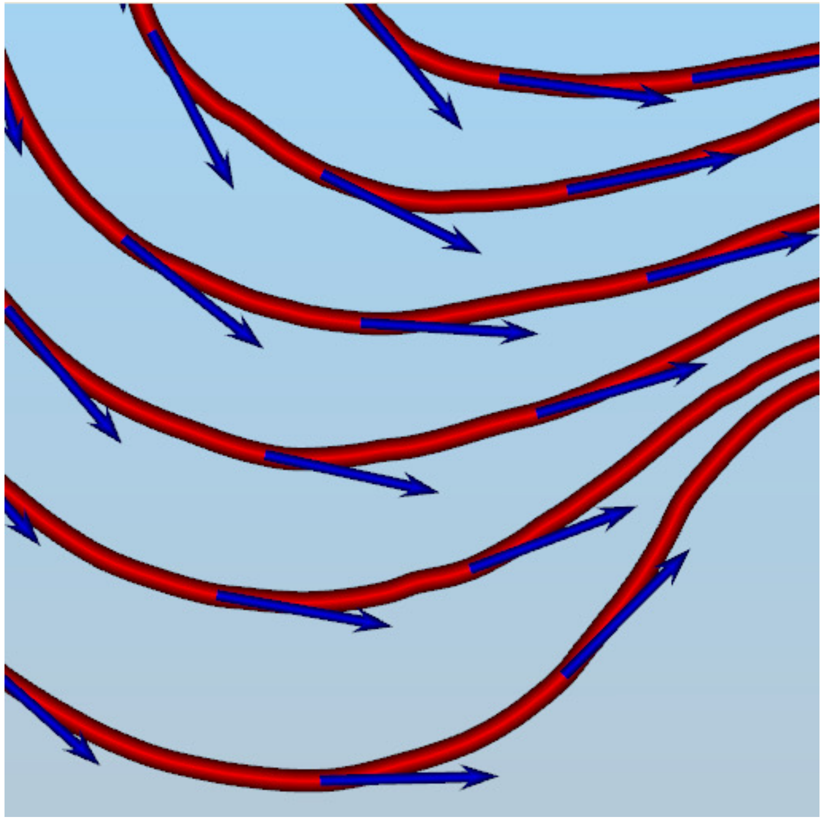
Flow visualization: pathlines

[https://en.wikipedia.org/wiki/Streamlines,_streaklines,_and_pathlines#/media/File:Kaberneeme_campfire_site.jpg]

Flow visualization: streaklines

[https://en.wikipedia.org/wiki/Streamlines,_streaklines,_and_pathlines#/media/File:Aeroakustik-Windkanal-Messhalle.JPG]

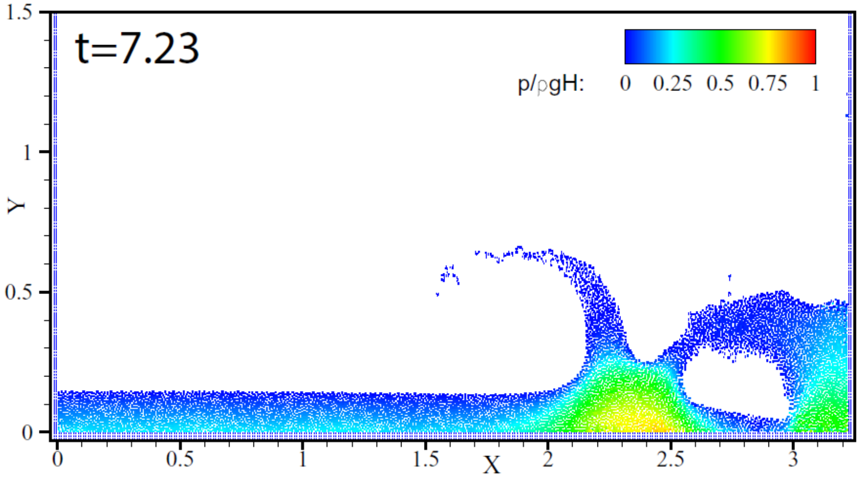
Flow visualization: streamlines



https://www3.nd.edu/~cwang11/2dflowvis.html

Representation of fluid flow

- Pathlines vs Streamlines
- Particles vs mesh/fixed coordinate system
- Lagrangian vs Eulerian representation
- Smooth particle hydrodynamics vs Finite element method



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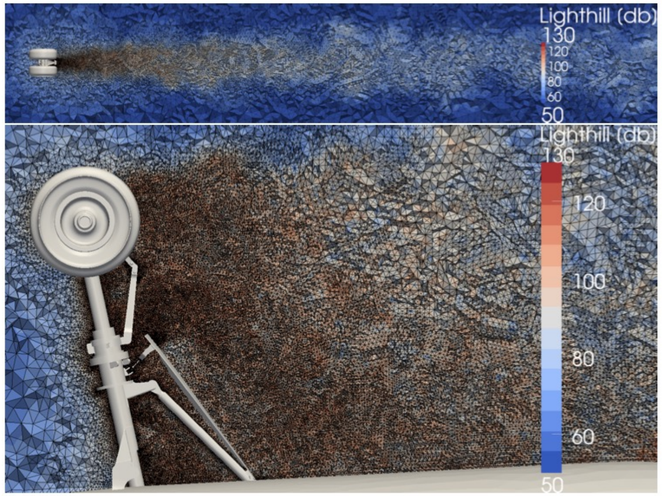

Representation of fluid flow

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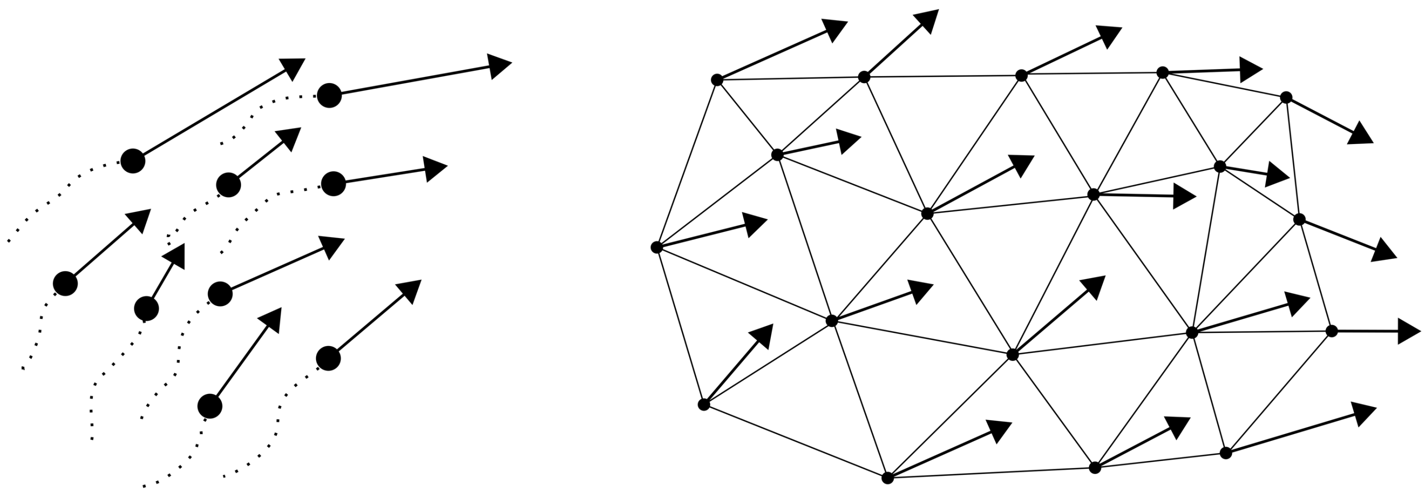
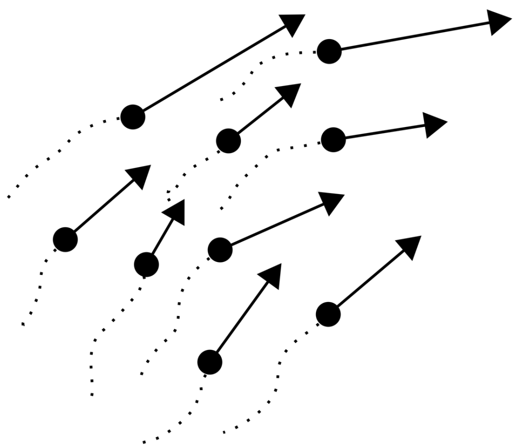
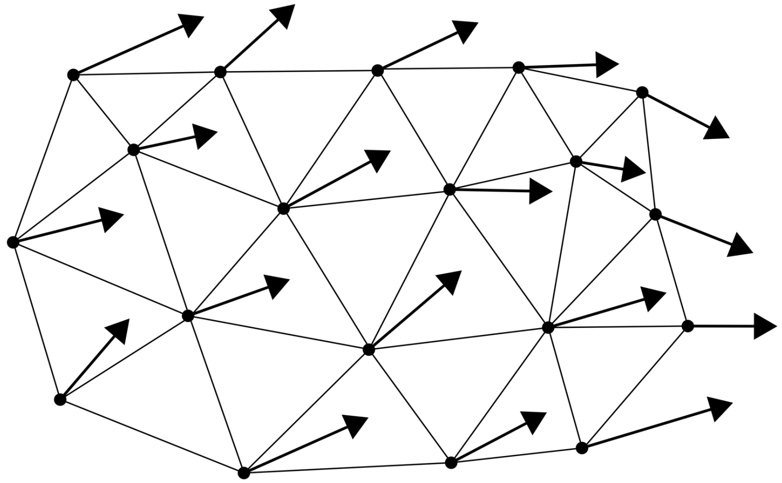


Representation of fluid flow

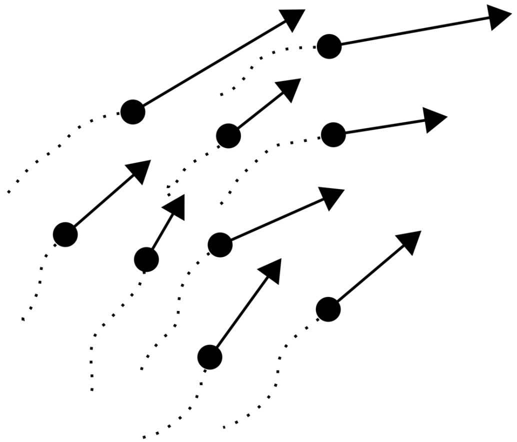
- Lagrangian representation: moving particle position $X(t)$, $X(0) = X_0$
- Eulerian representation: fixed position x, velocity $u(x,t)$

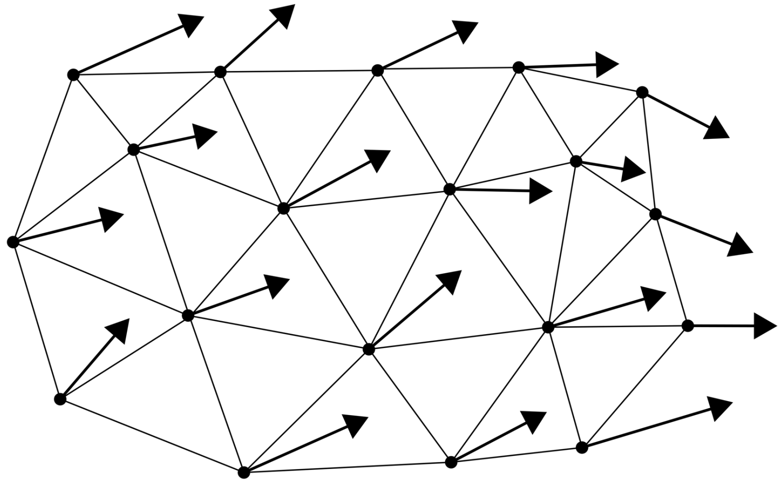




Representation of fluid flow

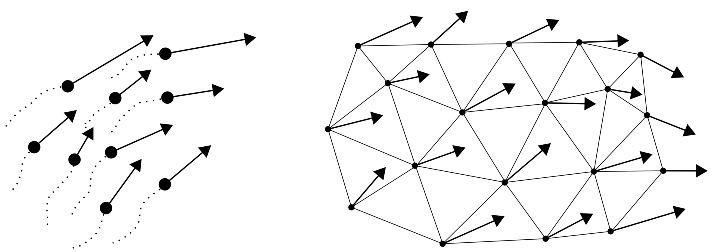
- Lagrangian representation: moving particle position $X(t), X(0) = X_0$
- Eulerian representation: fixed position x, velocity $u(x, t)$
- $u(X(t), t) = \frac{dX}{dt}$





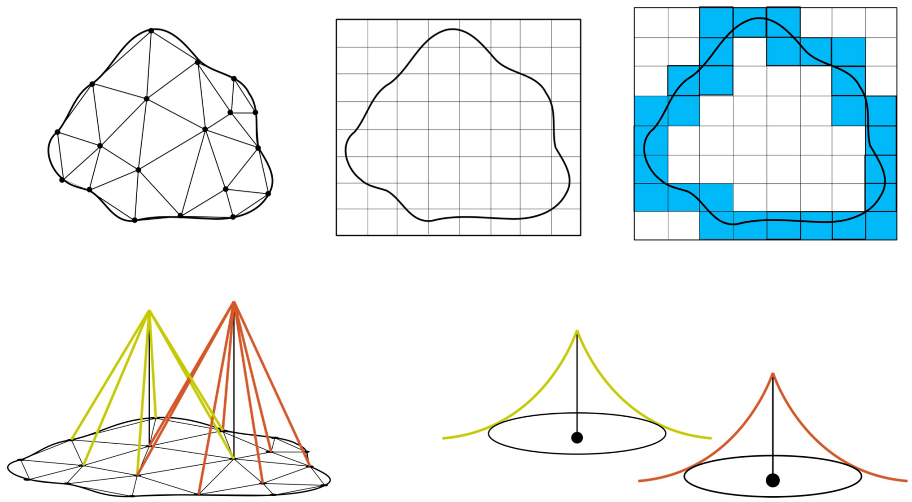
Representation of fluid flow

- Lagrangian representation: moving particle position $X(t), X(0) = X_0$
- Eulerian representation: fixed position x, velocity $u(x, t)$
- $u(X(t), t) = \frac{dX}{dt}$
- Material derivative: $\frac{Du}{Dt} = \left(\frac{dX}{dt} \cdot \nabla\right) u + \frac{\partial u}{\partial t} = \frac{\partial u}{\partial t} + (u \cdot \nabla) u$
- Acceleration along particle path



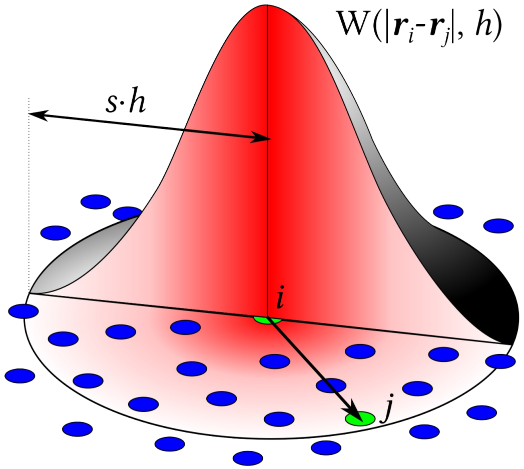
Discretization of DE: $R(u) = 0 \longrightarrow Ax = b$

- Particle system - mesh-free radial basis function
- Structured grid - stencil
- Unstructured mesh - basis function



- Minimization $min \parallel R(u) \parallel$
- Collocation $R\big(u(\mathrm{x_i})\big) = 0$, for all i
- Projection $(R(u), v) = 0$, for all v

Smooth particle hydrodynamics (SPH)



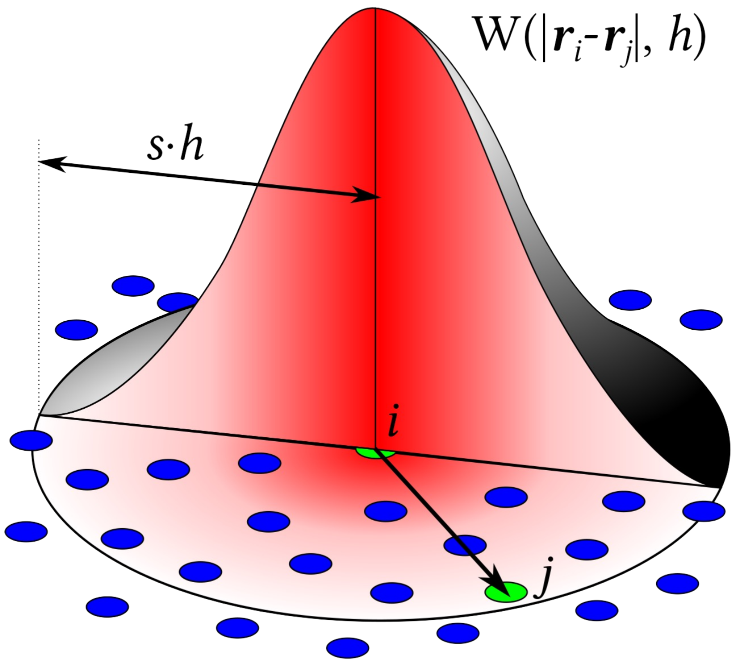
- Particle system $\{x_i\}$
- Kernel function W_h
- Smoothing length h
- Field representation $A(x) = \sum_i A_i W_h(\|x - x_i\|)$

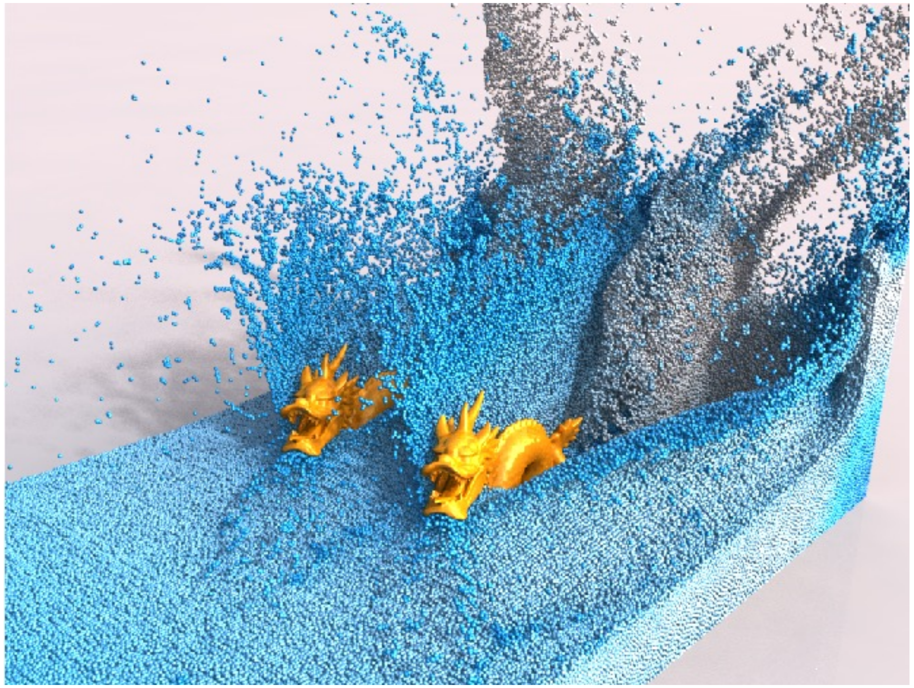
$$\frac{\partial}{\partial x_i} A(x) = \sum_i A_i \frac{\partial}{\partial x_i} W_h(\|x - x_i\|)$$

- Navier-Stokes equations (partial differential equation)

$$\rho(\dot{u} + (u \cdot \nabla)u) + \nabla p - \nu \Delta u = \rho f$$
$$\nabla \cdot u = 0$$

Smooth particle hydrodynamics (SPH)

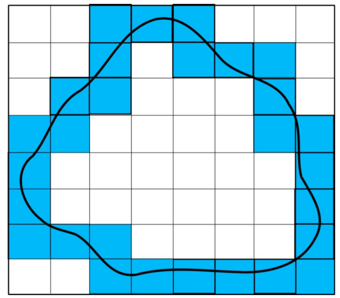


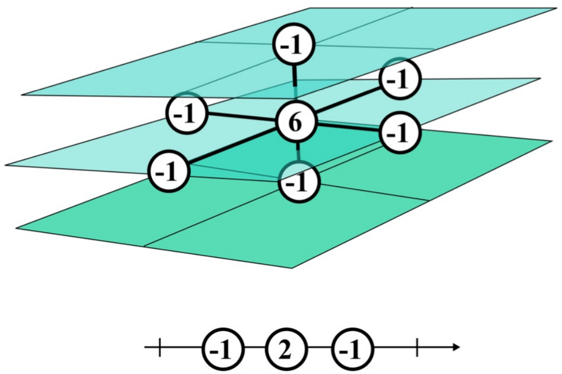


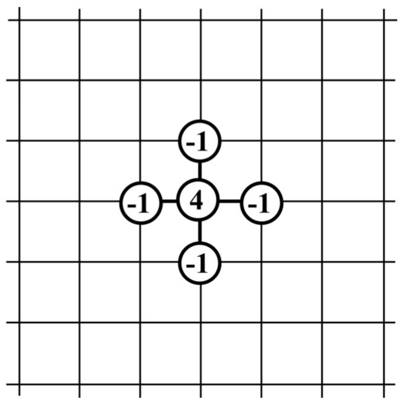
[Bender, Koschier, SIGGRAPH, 2015]

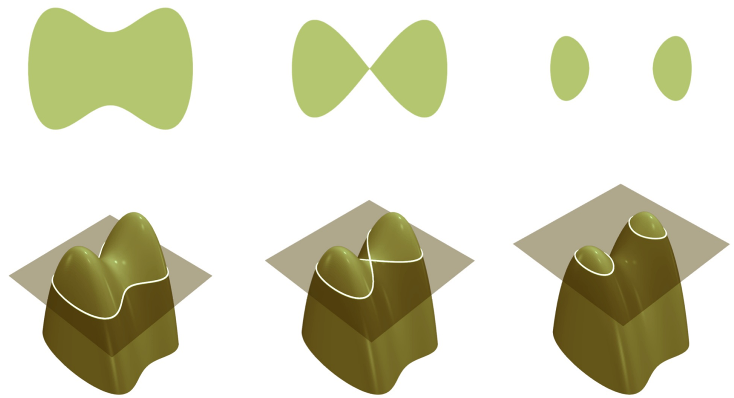
Finite difference method

- Structured grid - stencil (e.g. finite difference method)
- Collocation $R\left(u(\mathrm{x_i})\right) = 0$, for all i
- Level set function for complex geometry









[https://en.wikipedia.org/wiki/Level-set_method#/media/File:Level_set_method.png]

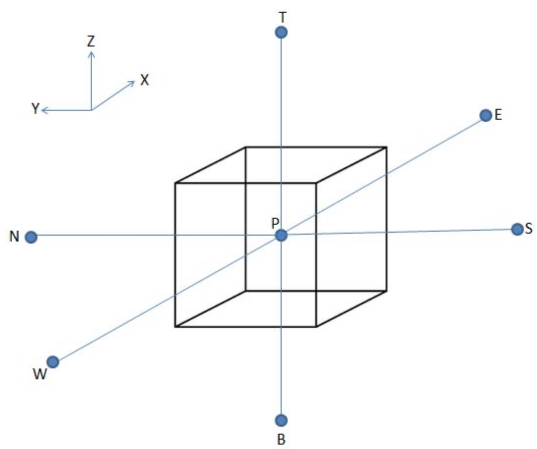
Finite volume method

- Based on local conservation laws over grid cells using Gauss theorem.

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{f}\left(\mathbf{u}\right) = \mathbf{0}.$$

$$\int_{v_i} \frac{\partial \mathbf{u}}{\partial t} \, dv + \int_{v_i} \nabla \cdot \mathbf{f}\left(\mathbf{u}\right) \, dv = \mathbf{0}.$$

$$\frac{d\bar{\mathbf{u}}_i}{dt} + \frac{1}{v_i} \oint_{S_i} \mathbf{f}\left(\mathbf{u}\right) \cdot \mathbf{n} \, dS = \mathbf{0}.$$

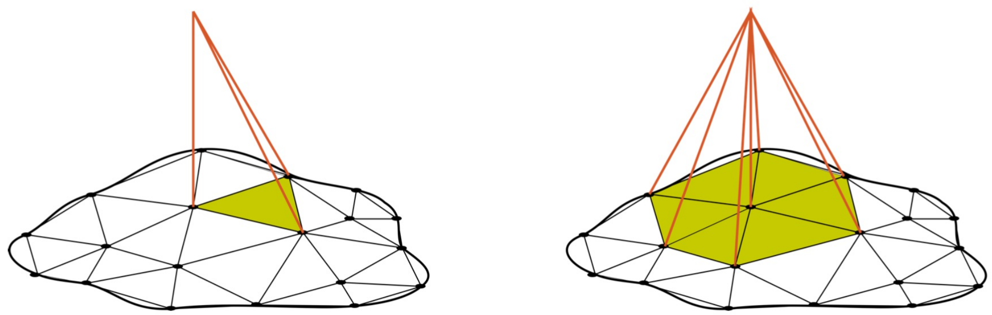


[https://en.wikipedia.org/wiki/Finite_volume_method]

Finite element method

- Unstructured mesh - basis function
- Fixed or deforming mesh
- Projection $(R(u), v) = 0$, for all v

$$u(x,t) \approx \sum_{i=1}^{N} U_i(t)\phi_i(x)$$



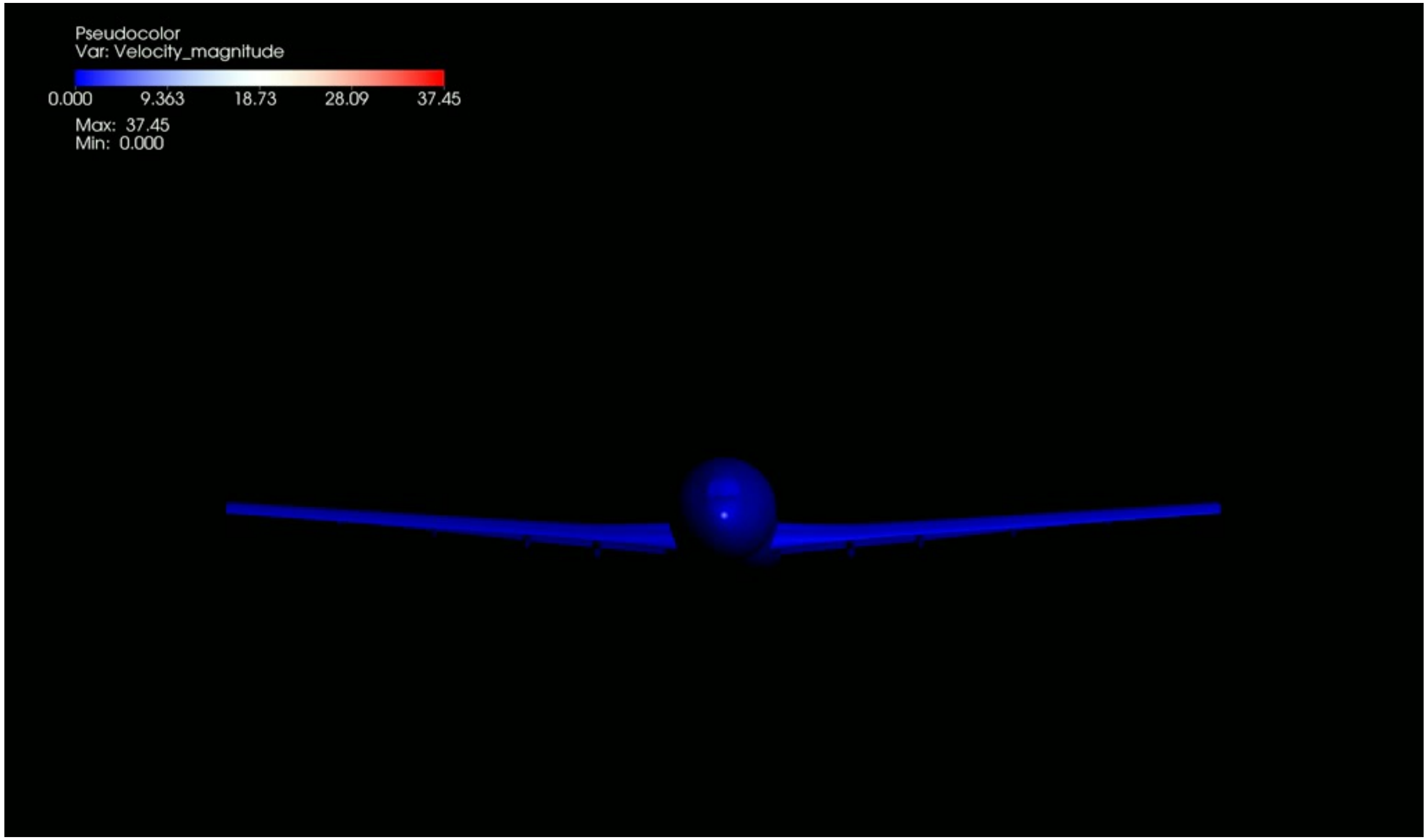
Algorithm 9.2. (A, b) = assemble_system(f).

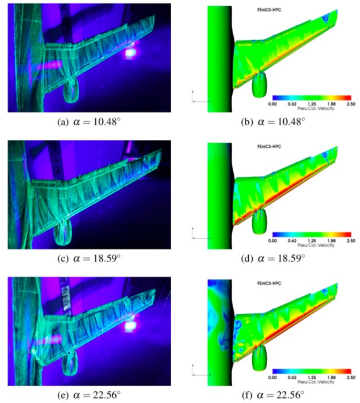
Input: function **f**

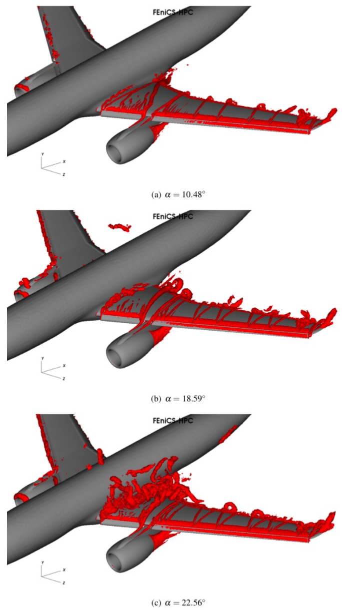
Output: assembled matrix **A** and vector **b**.

```
for k=0:no_elements-1 do
    q = get_no_local_shape_functions(k)
    loc2glob = get_local_to_global_map(k)
    for i=0:q do
        b[i] = integrate_vector(f, k, i)
        for j=0:q do
            a[i,j] = integrate_matrix(k, i, j)
        end for
    end for
    add_to_global_vector(b, loc2glob)
    add_to_global_matrix(a, loc2glob)
end for
return A, b
```

FEM simulation of air past airplane

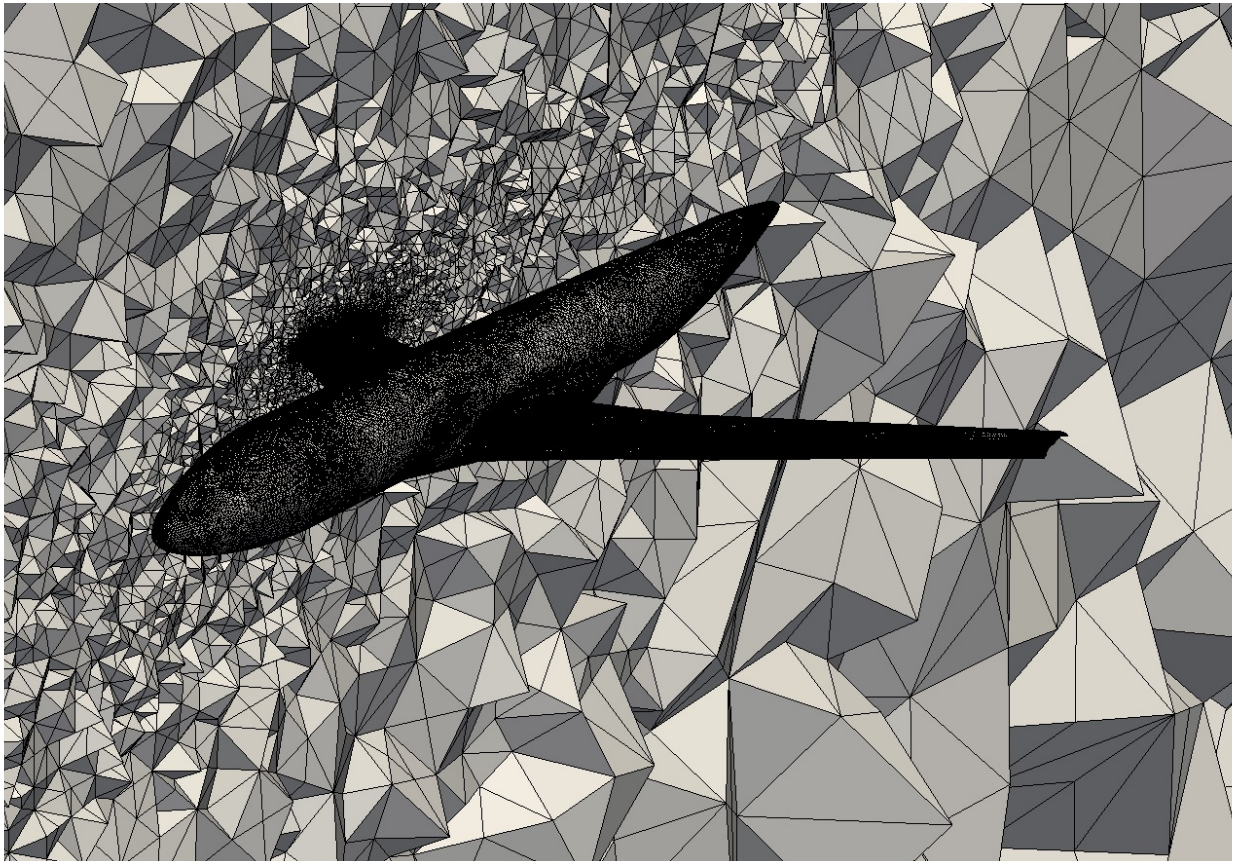


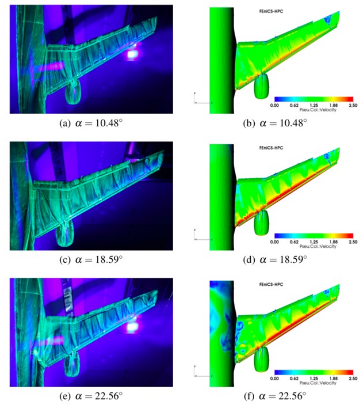


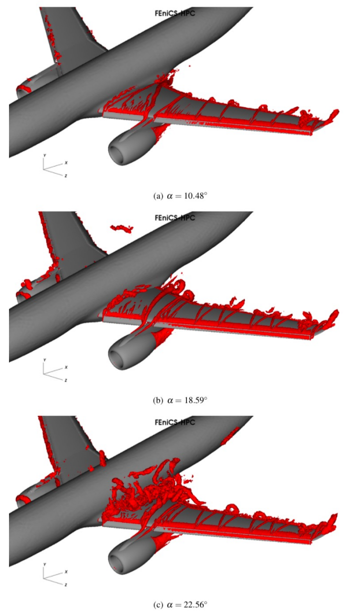


[Jansson et al., Springer, 2018]

Discretization by a mesh

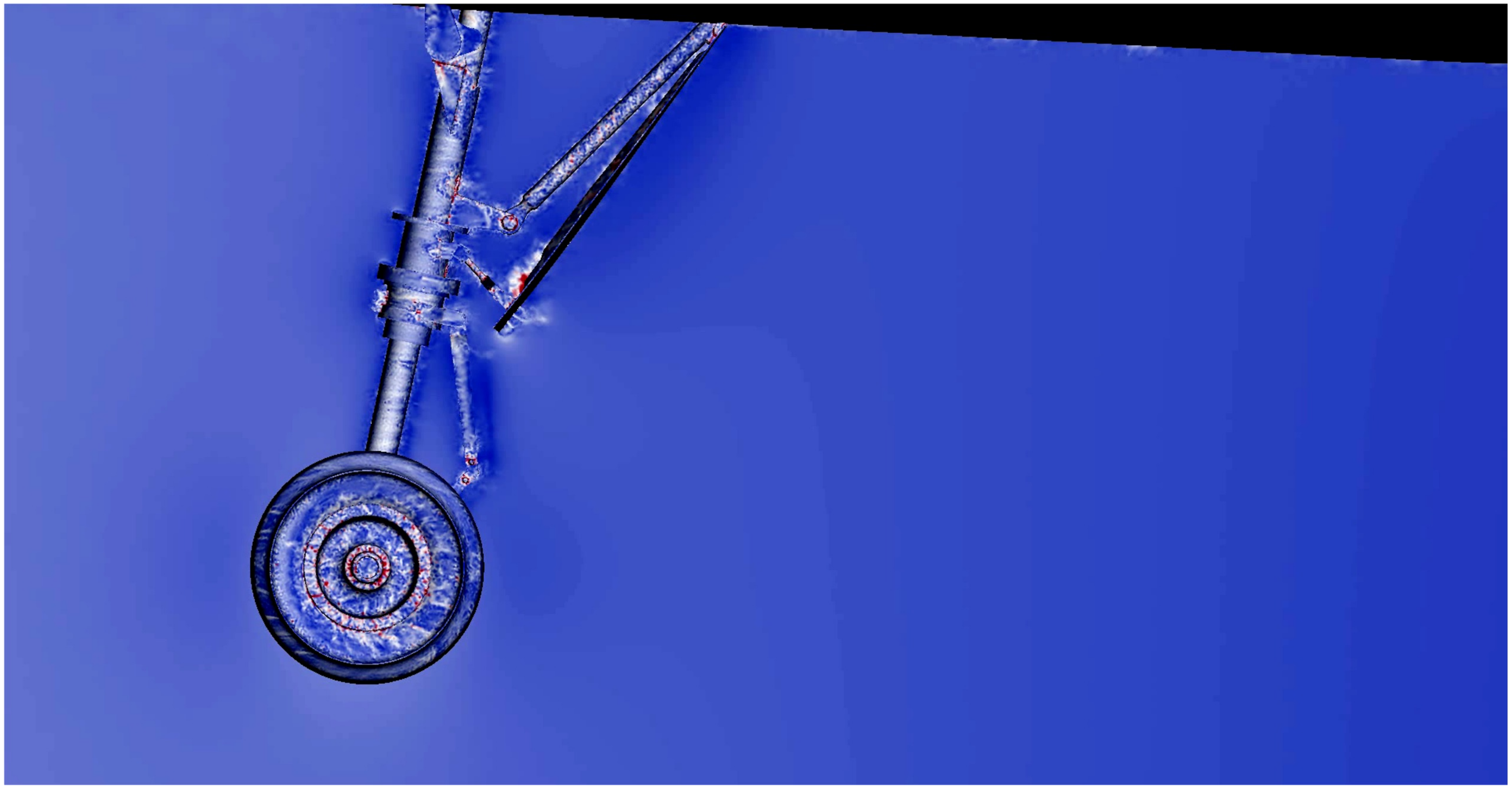






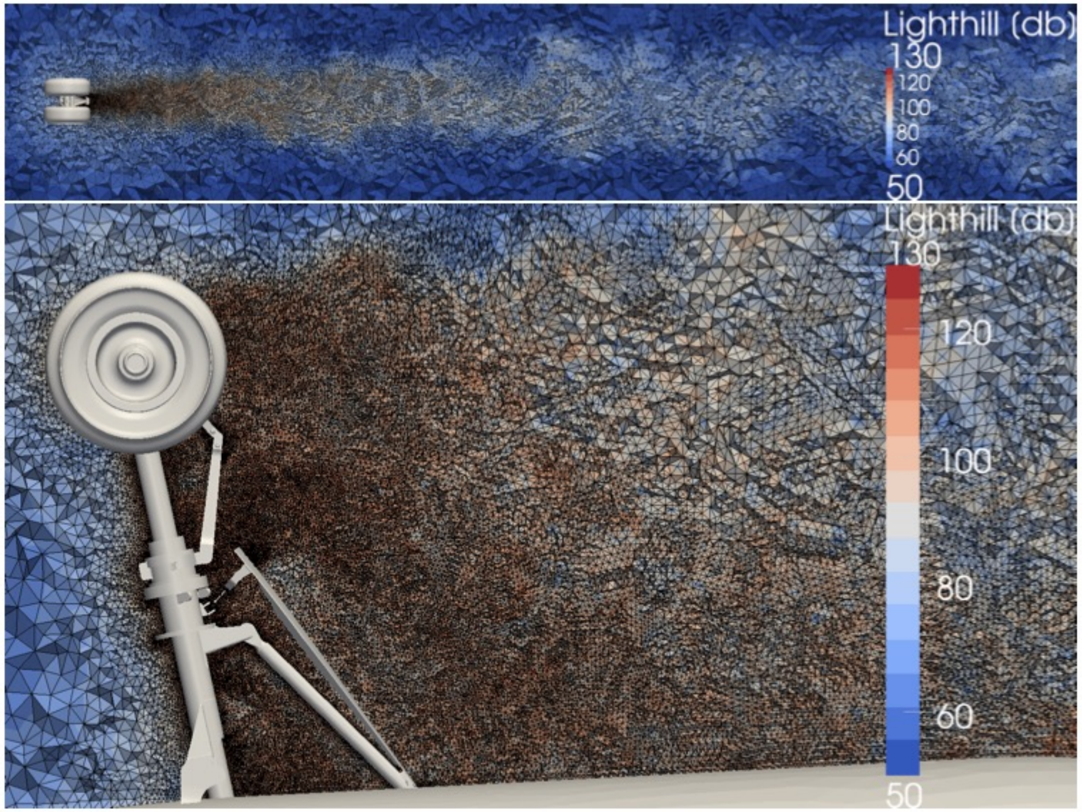
[Jansson et al., Springer, 2018]

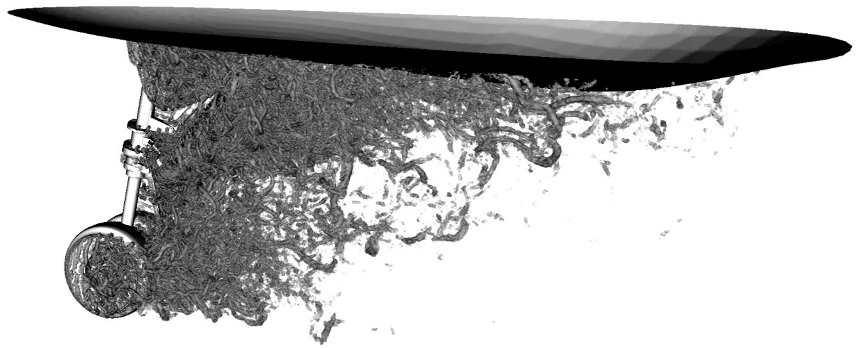
FEM simulation of airflow past landing gear



[De Abreu et al., Computers and Fluids, 2016]

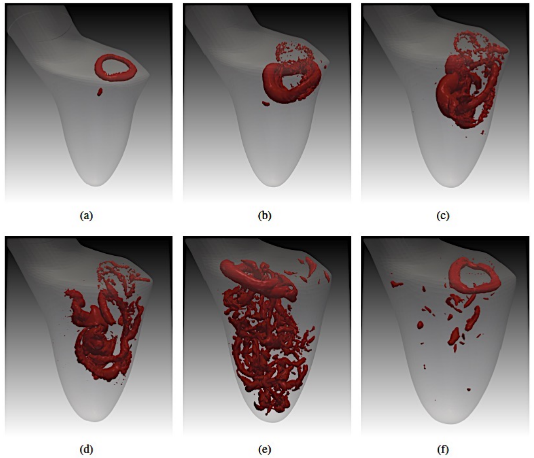
Acoustic sources and turbulent vortices

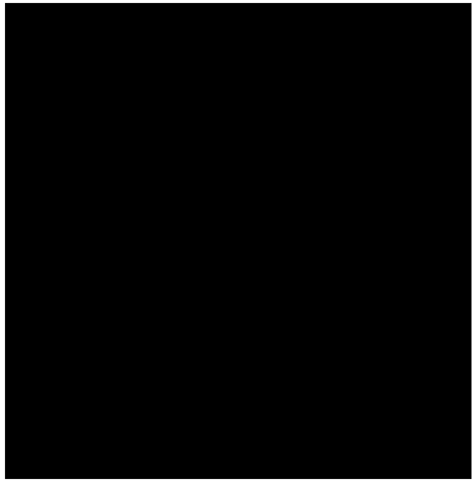


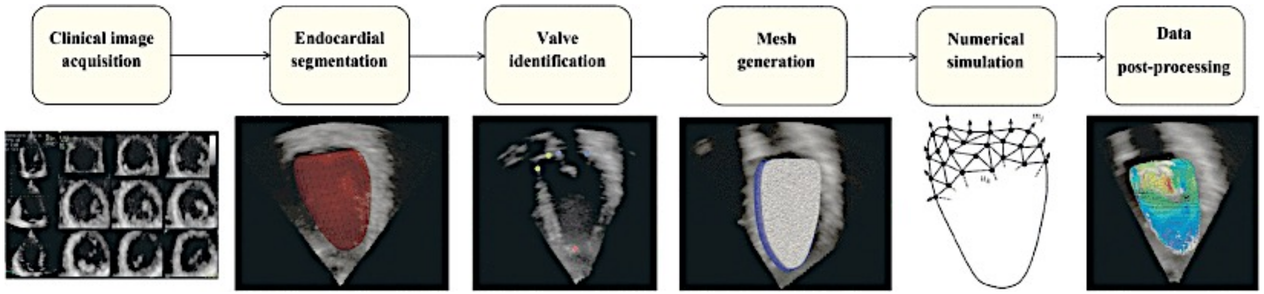


[De Abreu et al., Computers and Fluids, 2016]

Heart (deforming mesh) simulation







[Spühler et al., 2017, 2020]

Projection methods are optimal

Theorem 1.16 (Optimality of orthogonal projection). *The orthogonal projection $v_s \in S$, defined by*

$$(v - v_s, s) = 0, \quad \forall s \in S,$$

is the optimal approximation of $v \in V$ in $S \subset V$, in the sense that

$$\|v - v_s\| \leq \|v - s\|, \quad \forall s \in S,$$

for $\| \cdot \| = (\cdot, \cdot)^{1/2}$ the norm induced by the inner product in V.