DD2552 - Seminars on Theoretical Computer Science, Programming Languages and Formal Methods, Seminar 6

Karl Palmskog (palmskog@kth.se)

2021-09-16

Last Seminar and Today

Last seminar:

- Statistical verification basics for CSL

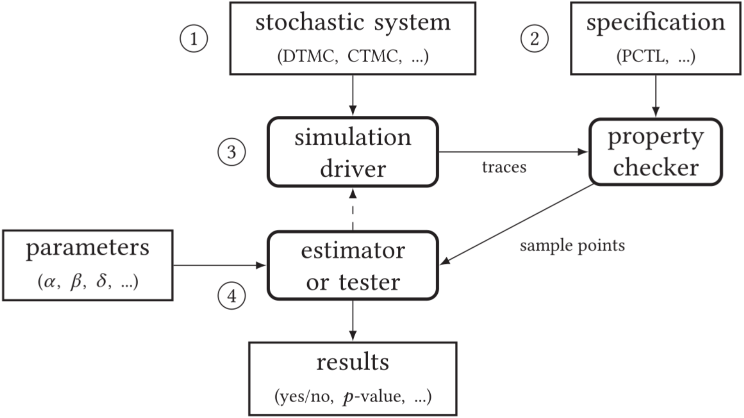
Today:

- More error control
- Estimation

Continuous Stochastic Logic (CSL) reminder

$$\phi ::= \top \mid a \mid \neg\phi \mid \phi \wedge \phi \mid P_{\geq\theta}(\psi)$$
$$\psi ::= \phi \, U^{\leq t} \, \phi$$
$$t \in R^{\geq 0}, \;\; \theta \in [0, 1]$$

Statistical checking reminder



Probability and hypotheses reminder

- consider formula $P_{\geq\theta}(\psi)$ for a model
- p is the (unknown) underlying probability (measure) for paths where ψ hold
- can either try to estimate p or pose *hypotheses*
  - $H_0 : p \geq \theta$ (null)
  - $H_1 : p < \theta$
- α: probability of Type I error
- β: probability of Type II error
- $\alpha + \beta \leq 1$

Sequential Probability Ratio Test reminder

- suppose we obtain m traces
- from traces we make observations $x_1, \ldots, x_m$
- define $d_m = \sum_{i=1}^{m} x_i$

$$f_m = \prod_{i=1}^{m} \frac{\Pr[X_i = x_i | p = p_1]}{\Pr[X_i = x_i | p = p_0]} = \frac{p_1^{d_m}(1-p_1)^{m-d_m}}{p_0^{d_m}(1-p_0)^{m-d_m}}$$

after computing f_m:

- accept H_0 if $f_m \leq \beta/(1-\alpha)$
- accept H_1 if $f_m \geq (1-\beta)/\alpha$
- otherwise, compute f_{m+1} and repeat

Indifference intervals

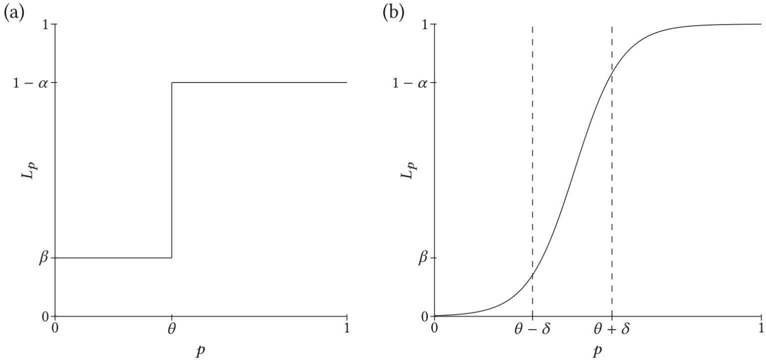


Fig. 4. Probability L_p of accepting the hypothesis $H_0 : p \geq \theta$, as function of p, with and without an indifference region.

Error control with indifference

Truth	**Decision** *accept* H_0, *reject* H_1	*reject* H_0, *accept* H_1
$p \geq \theta + \delta$: H_0 *true*, H_1 *false*	correct ($>1-\alpha$)	type I error ($\leq\alpha$)
$\theta - \delta < p < \theta + \delta$: H_0, H_1 *false*	indifferent	indifferent
$p \leq \theta - \delta$: H_0 *false*, H_1 *true*	type II error ($\leq\beta$)	correct ($>1-\beta$)

The conditions inside parentheses are on the probability for the given outcome.

Alternative: allow undecided results

- define two pairs of mutually exclusive hypotheses
- first pair:
  - $H_0^\perp : p \geq \theta$
  - $H_1^\perp : p \leq \theta - \delta$
- second pair:
  - $H_0^\top : p \geq \theta + \delta$
  - $H_1^\top : p \leq \theta$
- $P_{\geq\theta}(\psi)$ deemed true if both $H_0^\perp$ and $H_0^\top$ accepted
- $P_{\geq\theta}(\psi)$ deemed false if both $H_1^\perp$ and $H_1^\top$ accepted
- otherwise undecided

Error control for undecided results

Decision

Truth	*accept* $H_0^{\perp}, H_0^{\top}$	*accept* $H_0^{\perp\top}, H_1^{\perp\top}$	*accept* $H_1^{\perp}, H_1^{\top}$
$p \geq \theta$	correct ($>1 - \alpha - \gamma$)	undecided	type I error ($\leq\alpha + \gamma$)
$p < \theta$	type II error ($\leq\beta + \gamma$)	undecided	correct ($>1 - \beta - \gamma$)

The conditions inside parentheses are on the probability for the given outcome. "*accept* $H_i^{\perp\top}$" means accepting precisely one of $H_i^{\perp}$ and $H_i^{\top}$.

Best estimator of Bernoulli variable

$$\sum_{i=1}^{n} x_i / n \geq \theta$$

- problem: how to determine n to get error control

Estimation error control

- define a *precision* ϵ and a confidence α
- an estimate p' of p must have $|p' - p| \leq \epsilon$ with “confidence” α
- more formally, $Pr(|p' - p| \leq \epsilon) \geq 1 - \alpha$
- Chernoff bound: $Pr(|p' - p| \geq \epsilon) \geq 2e^{-2m\epsilon^2}$
- set $m = floor(ln(2\alpha)/(2\epsilon^2))$

Black Box System

- normally, we can ask for more samples/observations on demand
- in some cases