

Föreläsning 8 i ADK20

Algoritmkonstruktion: dekomposition

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Metod 3 - Dekomposition (Divide and conquer, Söndra och härska)

- Dela upp i mindre problem
- Lös delproblemen rekursivt
- Kombinera resultaten

Analys: Använd en rekursionsrelation

Exempel: Matrimultiplikation

Metod 3 - Dekomposition (Divide and conquer, Söndra och härska)

Matrismultiplikation

$$C = AB \Leftrightarrow \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

function MUL(A, B, n)

if $n = 1$ **then return** $A \cdot B$

$C_{11} = \text{ADD}(\text{MUL}(A_{11}, B_{11}, \frac{n}{2}), \text{MUL}(A_{12}, B_{21}, \frac{n}{2}))$

$C_{12} = \text{ADD}(\text{MUL}(A_{11}, B_{12}, \frac{n}{2}), \text{MUL}(A_{12}, B_{22}, \frac{n}{2}))$

$C_{21} = \text{ADD}(\text{MUL}(A_{21}, B_{11}, \frac{n}{2}), \text{MUL}(A_{22}, B_{21}, \frac{n}{2}))$

$C_{22} = \text{ADD}(\text{MUL}(A_{21}, B_{12}, \frac{n}{2}), \text{MUL}(A_{22}, B_{22}, \frac{n}{2}))$

return C

Fungerar om $n = 2^k$ för något k

$$\left. \begin{array}{l} \text{Analys: } T(1) = 1 \\ T(n) = 8T\left(\frac{n}{2}\right) + 4\left(\frac{n}{2}\right)^2 \end{array} \right\} \Rightarrow T(n) = \mathcal{O}(n^3)$$

Snabb multiplikation av 2×2 -matriser (strassen)

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

Kan beräknas genom:

$$m_1 = (a_{12} - a_{22}) \cdot (b_{21} + b_{22})$$

$$c_{11} = m_1 + m_2 - m_4 + m_6$$

$$m_2 = (a_{11} + a_{22}) \cdot (b_{11} + b_{22})$$

$$c_{12} = m_4 + m_5$$

$$m_3 = (a_{11} - a_{21}) \cdot (b_{11} + b_{12})$$

$$c_{21} = m_6 + m_7$$

$$m_4 = (a_{11} + a_{12}) \cdot b_{22}$$

$$c_{22} = m_2 - m_3 + m_8 - m_7$$

$$m_5 = a_{11} \cdot (b_{12} + b_{22})$$

$$m_6 = a_{22} \cdot (b_{21} + b_{11})$$

$$m_7 = (a_{21} + a_{22}) \cdot b_{11}$$

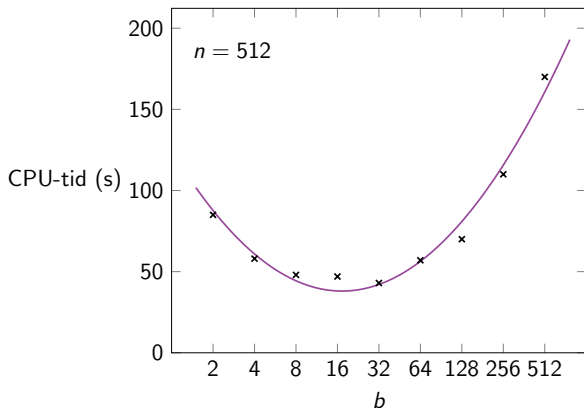
Sammanlagt 7 multiplikationer samt 18 additioner och subtraktioner

Multiplikation av två $n \times n$ -matriser tar tid:

$$\left. \begin{array}{l} T(1) = 1 \\ T(n) = 7T\left(\frac{n}{2}\right) + 18\left(\frac{n}{2}\right)^2 \end{array} \right\} \Rightarrow T(n) \sim n^{2.81}$$

Strassen i praktiken

- Använd Strassens multiplikationsalgoritm för stora matriser ($n > b$)
- Använd vanlig matrismultiplikation för mindre matriser ($n \leq b$)
- Hur ska brytpunkten b väljas?



Multiplikation av binära tal

- $x = \underbrace{x_{n-1}x_{n-2} \cdots x_{\frac{n}{2}}}_a \underbrace{x_{\frac{n}{2}-1} \cdots x_1x_0}_b = a \cdot 2^{\frac{n}{2}} + b$
- $y = \underbrace{y_{n-1}y_{n-2} \cdots y_{\frac{n}{2}}}_c \underbrace{y_{\frac{n}{2}-1} \cdots y_1y_0}_d = c \cdot 2^{\frac{n}{2}} + d$
- $xy = ac \cdot 2^n + (ad + bc)2^{\frac{n}{2}} + bd$
- Anta att $n = 2^k$

Multiplikation av binära tal

```
function MULT( $x, y, k$ )  
  if  $k = 0$  then return  $x \cdot y$   
  else  
     $[a, b] \leftarrow x$   
     $[c, d] \leftarrow y$   
     $p_1 \leftarrow \text{MULT}(a, c, k-1)$   
     $p_2 \leftarrow \text{MULT}(b, d, k-1)$   
     $p_3 \leftarrow \text{MULT}(a, d, k-1)$   
     $p_4 \leftarrow \text{MULT}(b, c, k-1)$   
    return  $p_1 \cdot 2^n + (p_3 + p_4)2^{\frac{n}{2}} + p_2$ 
```

$$\left. \begin{array}{l} T(1) = \Theta(1) \\ T(n) = 4T\left(\frac{n}{2}\right) + \Theta(n) \end{array} \right\} \Rightarrow T(n) = \mathcal{O}(n^{\log_2 4}) = \mathcal{O}(n^2)$$

Smartare multiplikation (Karatsuba)

- $A = a \cdot c$
- $B = b \cdot d$
- $C = (a + b) \cdot (c + d)$
- $D = A \cdot 2^n + (C - A - B) \cdot 2^{\frac{n}{2}} + B$
- $D = ac \cdot 2^n + (ad + bc)2^{\frac{n}{2}} + bd = x \cdot y$

Smartare multiplikation (Karatsuba)

```
function SMARTMULT( $x, y, k$ )  
  if  $k \leq 4$  then return  $x \cdot y$   
  else  
     $[a, b] \leftarrow x$   
     $[c, d] \leftarrow y$   
     $A \leftarrow \text{SMARTMULT}(a, c, k-1)$   
     $B \leftarrow \text{SMARTMULT}(b, d, k-1)$   
     $C \leftarrow \text{SMARTMULT}(a+b, c+d, k-1)$   
    return  $A \cdot 2^n + (C - A - B)2^{\frac{n}{2}} + B$ 
```

$$\left. \begin{array}{l} T(1) = \Theta(1) \\ T(n) = 3T\left(\frac{n}{2}\right) + \Theta(n) \end{array} \right\} \Rightarrow T(n) = \mathcal{O}(n^{\log_2 3}) = \mathcal{O}(n^{1.58})$$

(Bästa kända algoritmen: $\mathcal{O}(n \log n \log \log n)$)