

SF1661 HT20

Föreläsning 10

29/9

Komplexa tal

$$\bullet \mathbb{C} = \{z = x + iy ; x, y \in \mathbb{R}, i^2 = -1\}$$

$$\bullet z + w, z - w, z \cdot w, \frac{z}{w}$$

$$\bullet |z|, \operatorname{Re}(z), \operatorname{Im}(z), \overline{z}$$

$$|z + w| \leq |z| + |w|$$

$$\bullet z = |z| (\cos \theta + i \sin \theta) = |z| e^{i\theta}$$

• Ekvationslösning och faktorisering
av polynom

Varför komplexa tal?

• Historia

• Matematiska skäl

• Tillämpningar

Hösten Lösning av 3:e grads ekv.

Cardanos formel (1500-tal):

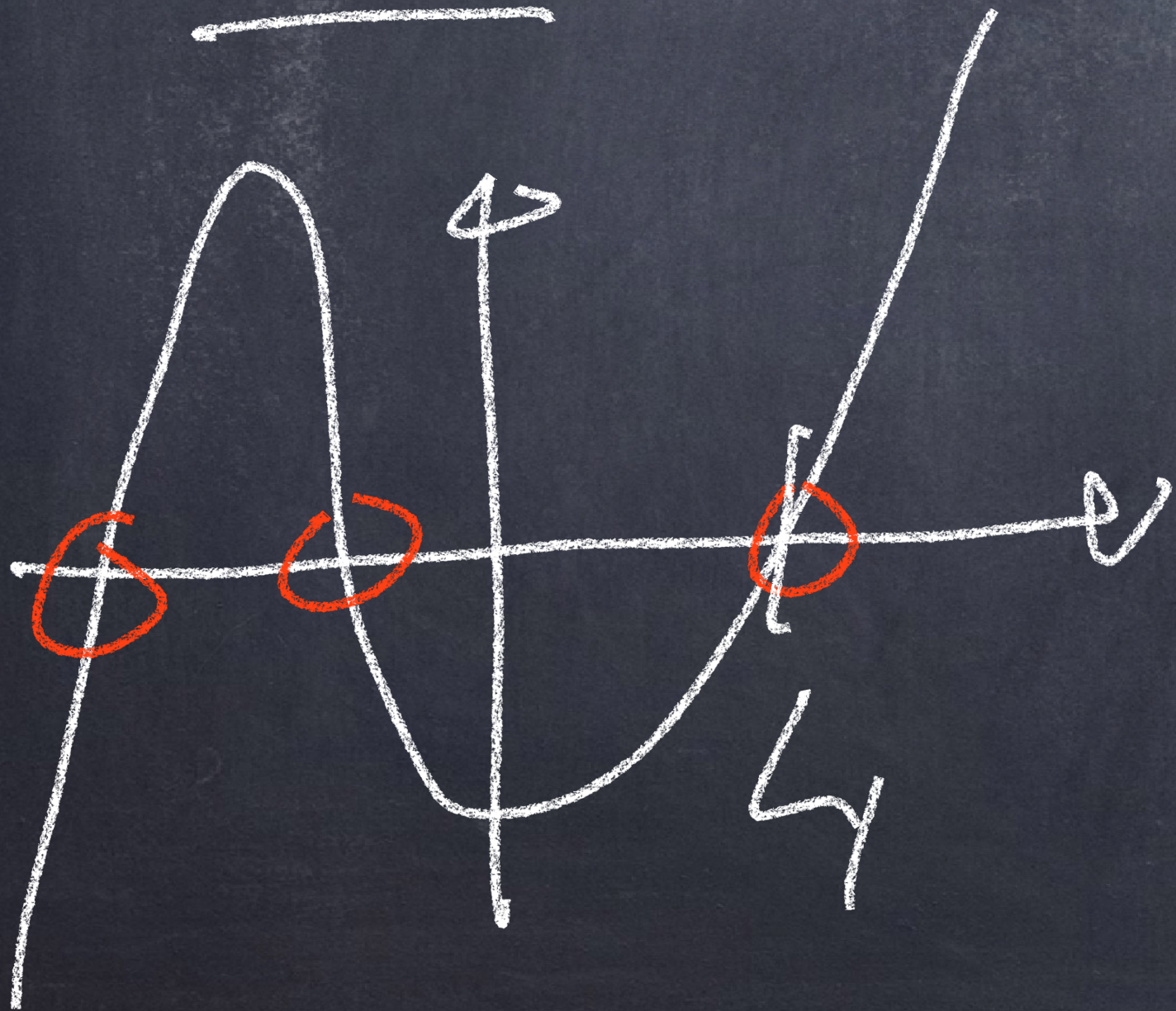
$$X^3 + Px + Q = 0 \quad \text{har lösningar}$$

$$X = \sqrt[3]{-\frac{Q}{2} + \sqrt{\left(\frac{Q}{2}\right)^2 + \left(\frac{P}{3}\right)^3}} + \sqrt[3]{-\frac{Q}{2} - \sqrt{\left(\frac{Q}{2}\right)^2 + \left(\frac{P}{3}\right)^3}}$$

behöver ni
inte kunna :-)

Även vid ekvationen med reella koefficienter
och reella lösningar ger Cardanos formel
räkning med "imaginära tal" på vägen."

Ex: $x^3 - 15x - 4 = 0$ | $p = -15$, $q = -4$



$$\left(\frac{q}{2}\right)^2 = 4 \quad \left(\frac{p}{3}\right)^3 = -125$$

$$x = \sqrt[3]{2 + \sqrt{4 - 125}} + \sqrt[3]{2 - \sqrt{4 - 125}}$$
$$= \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i} = 4$$

Matematiska skäl

$$\bullet P(z) = a_n z^n + \dots + a_1 z + a_0 = 0, \quad a_i \in \mathbb{C}$$
$$\iff$$

$$P(z) = (z - \alpha_1)(z - \alpha_2) \dots (z - \alpha_n), \quad \alpha_i \in \mathbb{C}$$

$$\bullet e^{i\pi} = -1$$

"Kortaste vägen mellan två utsagor"

- om reella tal går ofta genom komplexa talplanet

Tillämpningar

• "jw-metoden" växelströmskretsar

• Kvantmekanikens grundekvation
("Schrödinger equation")


$$i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x) \psi$$

KOMPLEXA TAL

$$\mathbb{C} = \{a + ib; a, b \in \mathbb{R}\}$$

$+$, $-$, $\cdot$, $/$

med tillägg

"som vanligt"

$$i^2 = -1$$

$$(a + ib) + (c + id) = (a + c) + (b + d)i$$

$$(a + ib) + (c + id) = (a + c) + (b + d)i$$

$$(a + ib)(c + id) = ac + iad + ibc + i^2 bd$$

$$= (ac - bd) + i(ad + bc)$$

$$\frac{a + ib}{c + id}$$

$$\stackrel{||}{=} \frac{(a + ib)(c - id)}{(c + id)(c - id)} =$$

$$\frac{a+ib}{c+id} \stackrel{||}{=} \frac{(a+ib)(c-id)}{(c+id)(c-id)} \stackrel{||}{=} \frac{(ac+bd) + i(bc-ad)}{c^2 - (id)^2} \stackrel{||}{=}$$

$$\stackrel{||}{=} \frac{ac+bd}{c^2+d^2} + i \frac{bc-ad}{c^2+d^2}$$

$(c+id \neq 0)$

$$\text{L\ddot{a}t } Z = \left(\frac{7}{2} + i \right), \quad w = 1 - 2i$$

Berechnen

$$Z + w = \left(\frac{7}{2} + i \right) + (1 - 2i) = \left(\frac{7}{2} + 1 \right) + i(1 - 2) = \frac{9}{2} - i$$

$$2Z - w = 2\left(\frac{7}{2} + i \right) - (1 - 2i) = 7 + 2i - 1 + 2i = 6 + 4i$$

$$Z \cdot w = \left(\frac{7}{2} + i \right)(1 - 2i) = \left(\frac{7}{2} \cdot 1 + i(-2i) \right) + \left(\frac{7}{2}(-2i) + (1 \cdot i) \right) = \dots$$

$$\frac{Z}{w} = \frac{\left(\frac{7}{2} + i \right)(1 + 2i)}{(1 - 2i)(1 + 2i)} = \frac{\left(\frac{7}{2} - 2 \right) + i(7 + 1)}{1^2 + 2^2} = \frac{3}{10} + i \frac{8}{5}$$

$$\text{Låt } Z = \left(\frac{7}{2} + i \right), \quad W = 1 - 2i$$

Beräkna

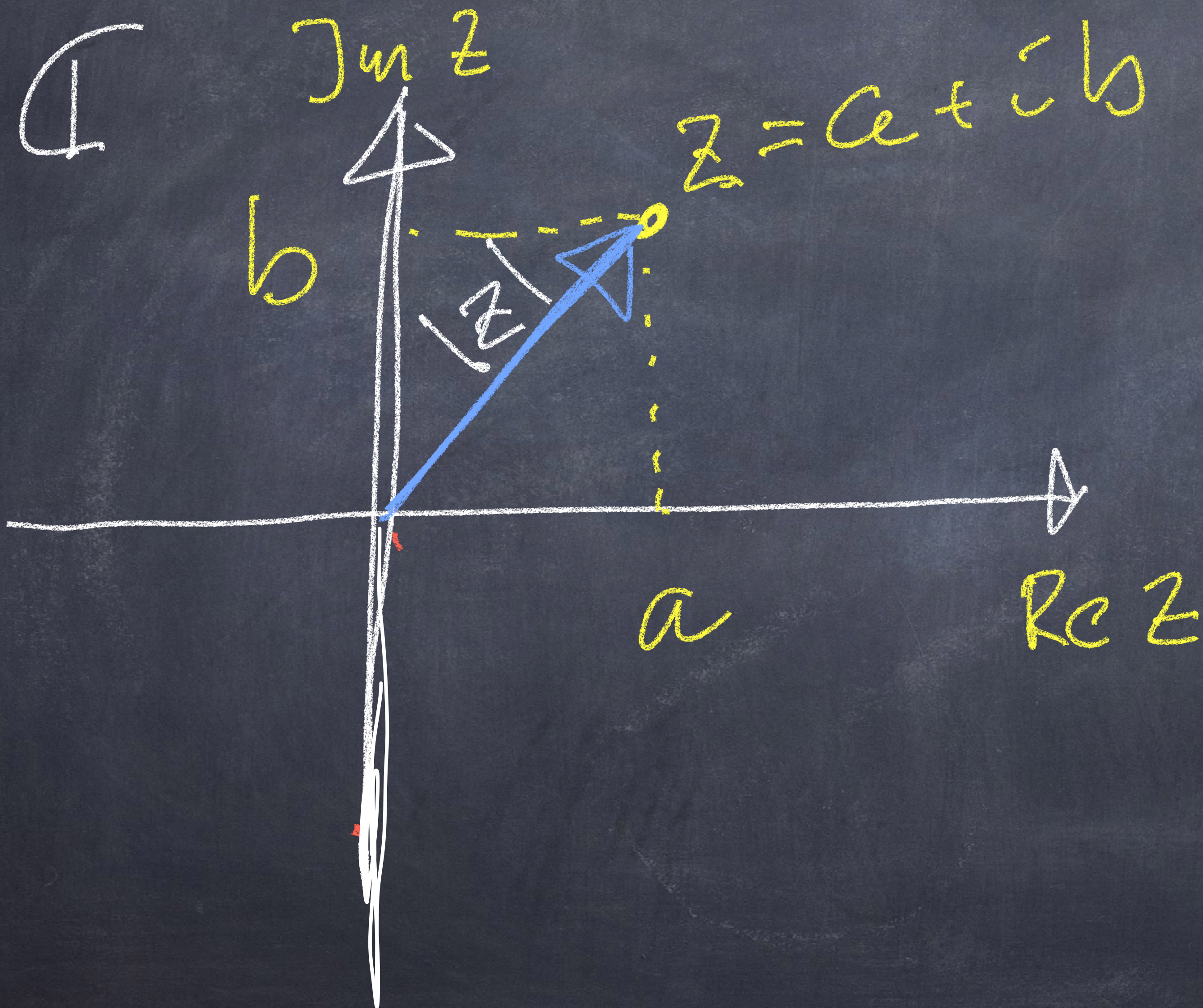
$$Z + W = \frac{9}{2} - i$$

$$2Z - W = 6 + 4i$$

$$Z \cdot W = \frac{11}{2} - 6i$$

$$\frac{Z}{W} = \frac{3}{10} + \frac{8}{5}i$$

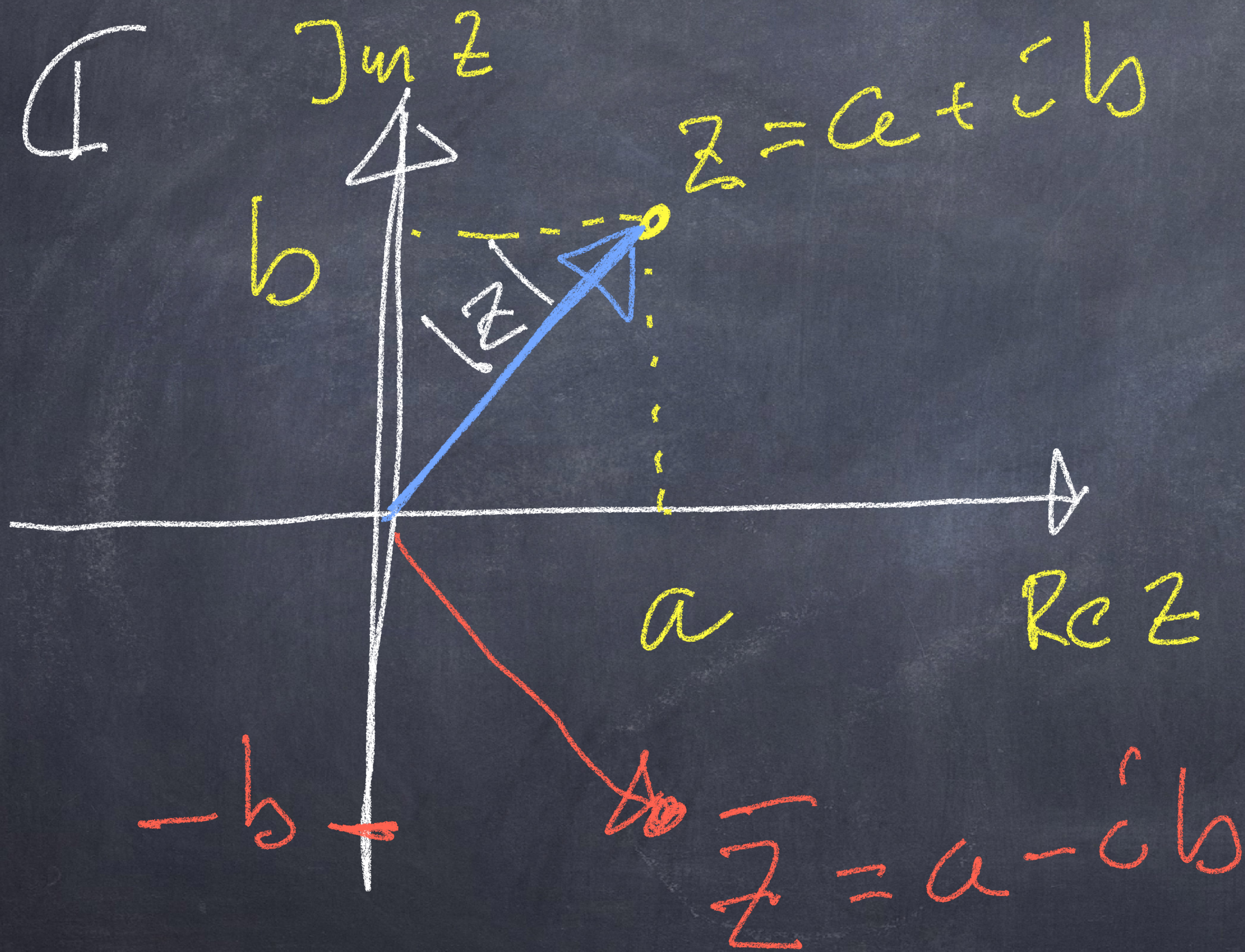
Geometrisch LÖSUNG



$$\text{Re}(z) = a$$

$$\text{Im}(z) = b$$

$\operatorname{Re}(z)$, $\operatorname{Im}(z)$, $\overline{z}$ och $|z|$.



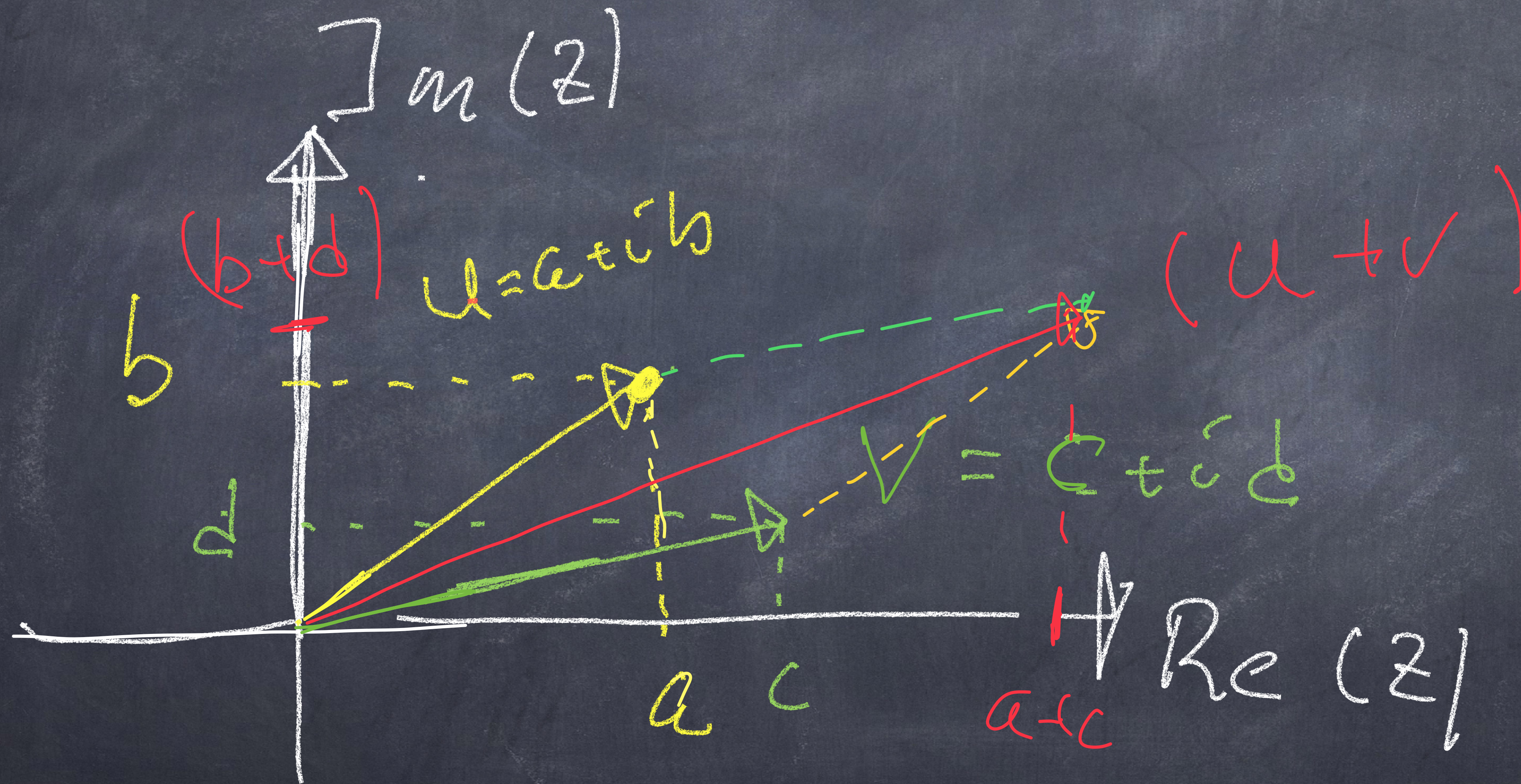
$$\operatorname{Re}(z) = a$$

$$\operatorname{Im}(z) = b$$

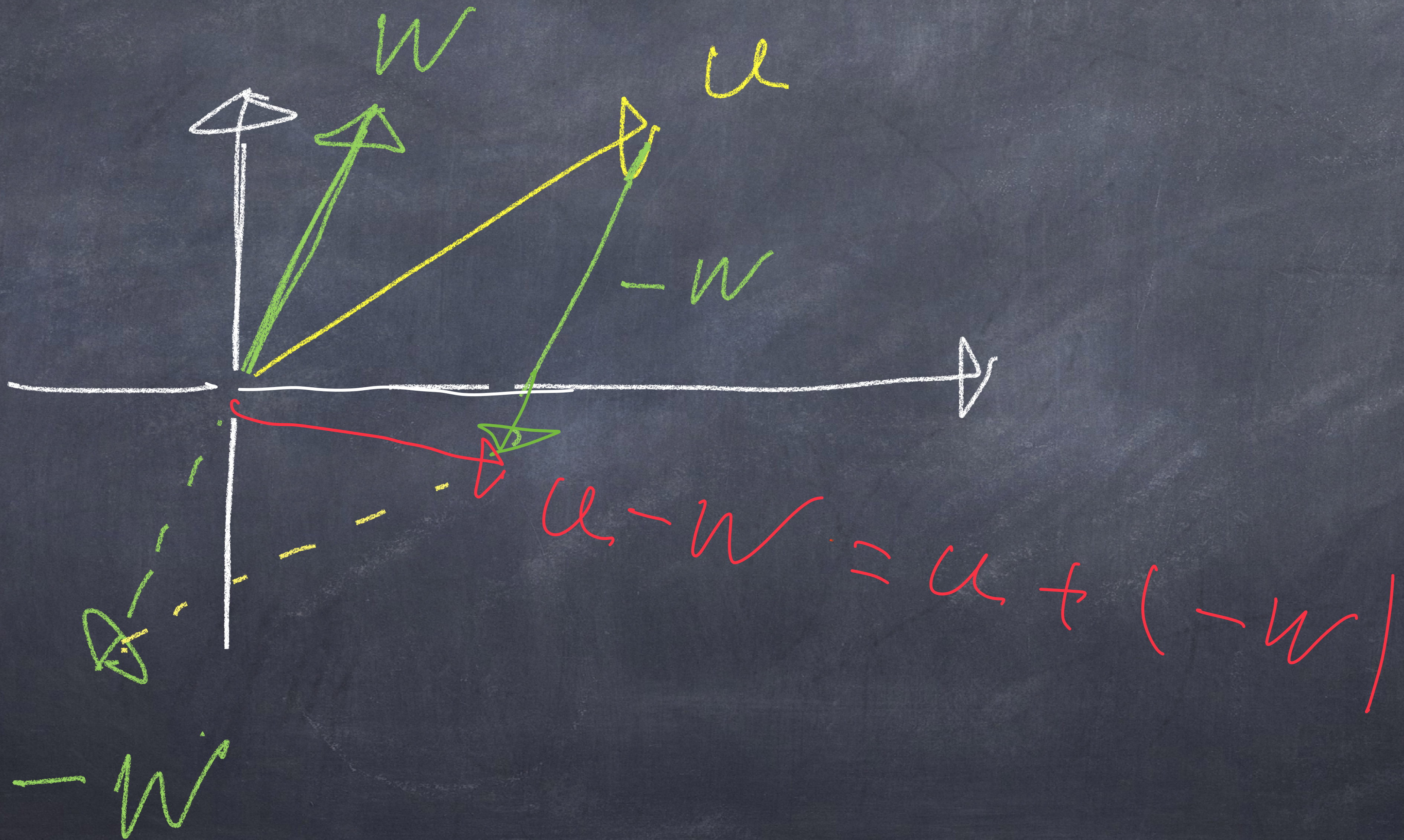
$$|z| = \sqrt{a^2 + b^2}$$

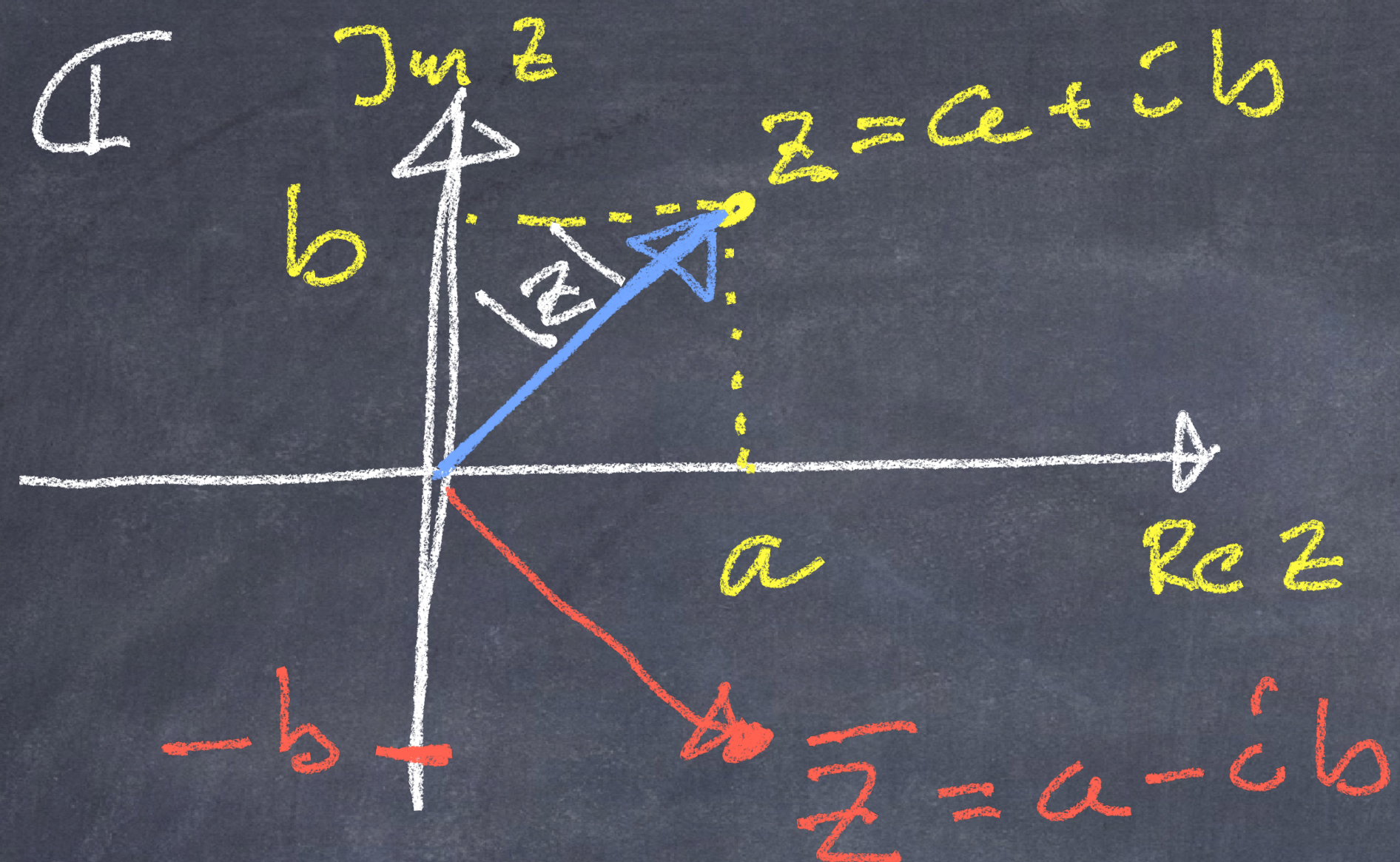
$$\neq 0$$

Geometrisch bild an addition

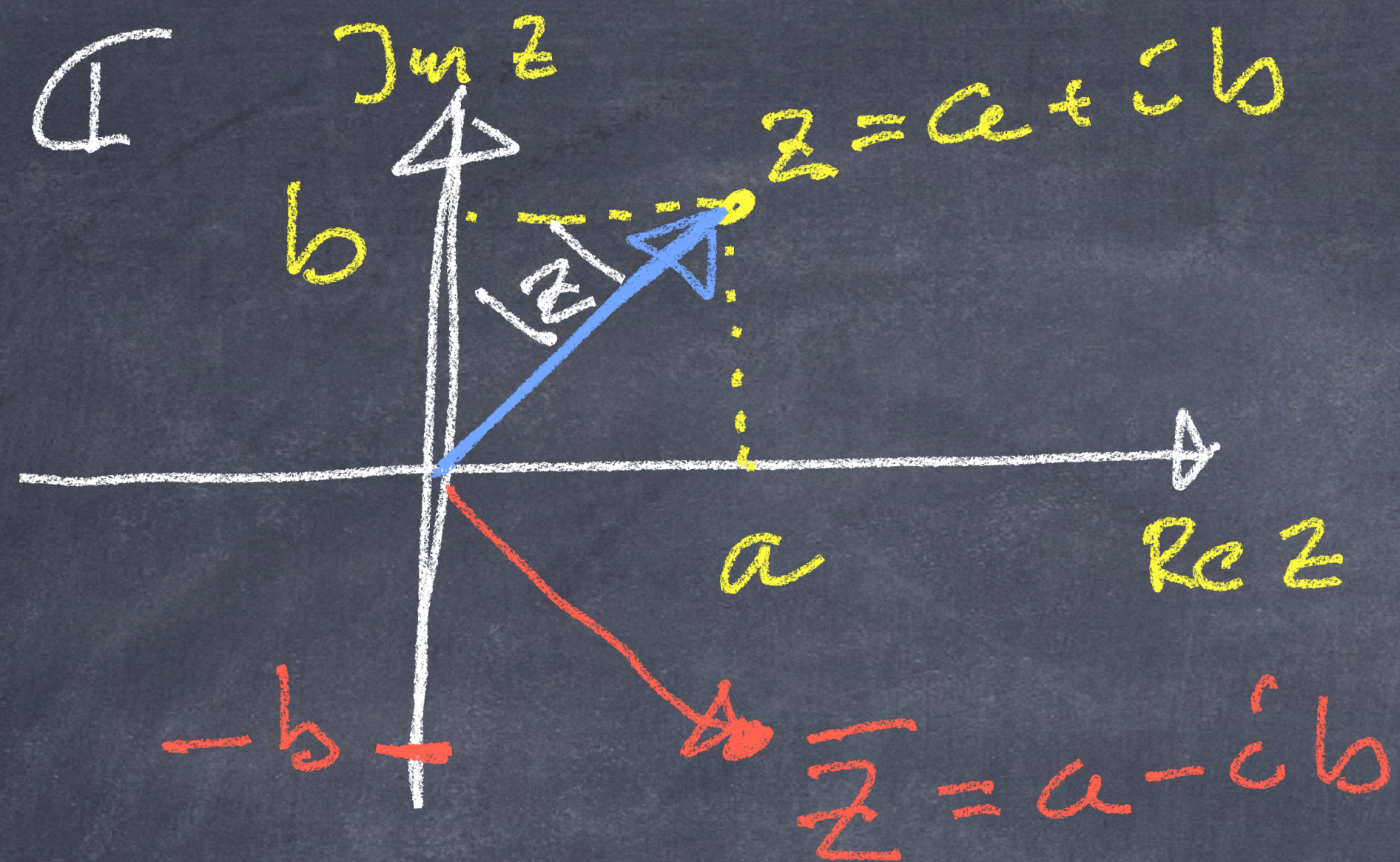


Geometrisch bild an subtraktion





$$|z| = \sqrt{a^2 + b^2}$$



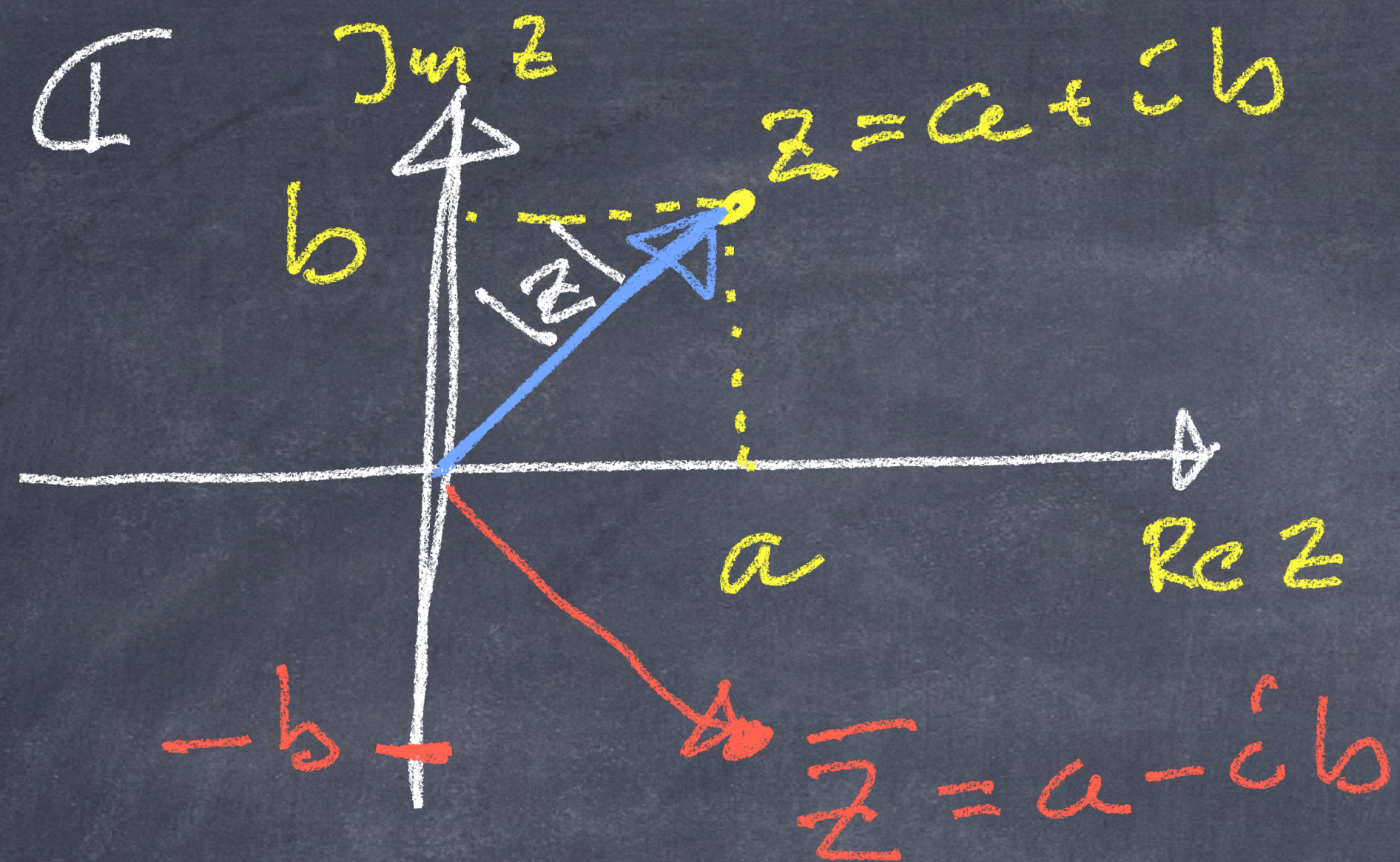
$$|z| = \sqrt{a^2 + b^2}$$

Sats:

$$\overline{z \pm w} = \bar{z} \pm \bar{w}$$

$$\overline{z \cdot w} = \bar{z} \cdot \bar{w}$$

$$\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}, \quad w \neq 0$$



$$\operatorname{Re}(z) = a$$

$$\operatorname{Im}(z) = b$$

$$|z| = \sqrt{a^2 + b^2}$$

Satz:

Satz: $\overline{z \pm w} = \overline{z} \pm \overline{w}$

$$\overline{z \cdot w} = \overline{z} \cdot \overline{w}$$

$$\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}, \quad w \neq 0$$

$$|z| |w| = |zw|$$

$$\frac{|z|}{|w|} = \left| \frac{z}{w} \right|, \quad w \neq 0$$

$$|z + w| \leq |z| + |w|$$

(TRIANGELÖSLICHKEITEN)

Beweis $\left(\overline{zw} = \bar{z} \cdot \bar{w} \right)$

Lass $z = a + ib$, $w = c + id$

V.L. = $\overline{zw} = \overline{(a + ib)(c + id)} =$

$= \overline{(ac - bd) + i(ad + bc)} = \underline{\underline{(ac - bd) - i(ad + bc)}}$

Beweis $\left(\overline{zw} = \bar{z} \cdot \bar{w} \right)$

Lass $z = a + ib$, $w = c + id$

V.L. $= \overline{zw} = \overline{(a + ib)(c + id)} =$

$= \overline{(ac - bd) + i(ad + bc)} = \underline{(ac - bd) - i(ad + bc)}$

H.L. $= (a - ib)(c - id) =$

$= (ac - bd) + i(-ad - bc) = \underline{(ac - bd) - i(ad + bc)}$

$= \underline{\underline{V.L.}}$

Beris $(|zw| = |z||w|)$

V.L. & H.L. $\in \mathbb{R}$, V.L., H.L. ≥ 0 ,

Så ekvivalent att visa $|zw|^2 = |z|^2 |w|^2$.

Sätt $z = a + ib$, $w = c + id$.

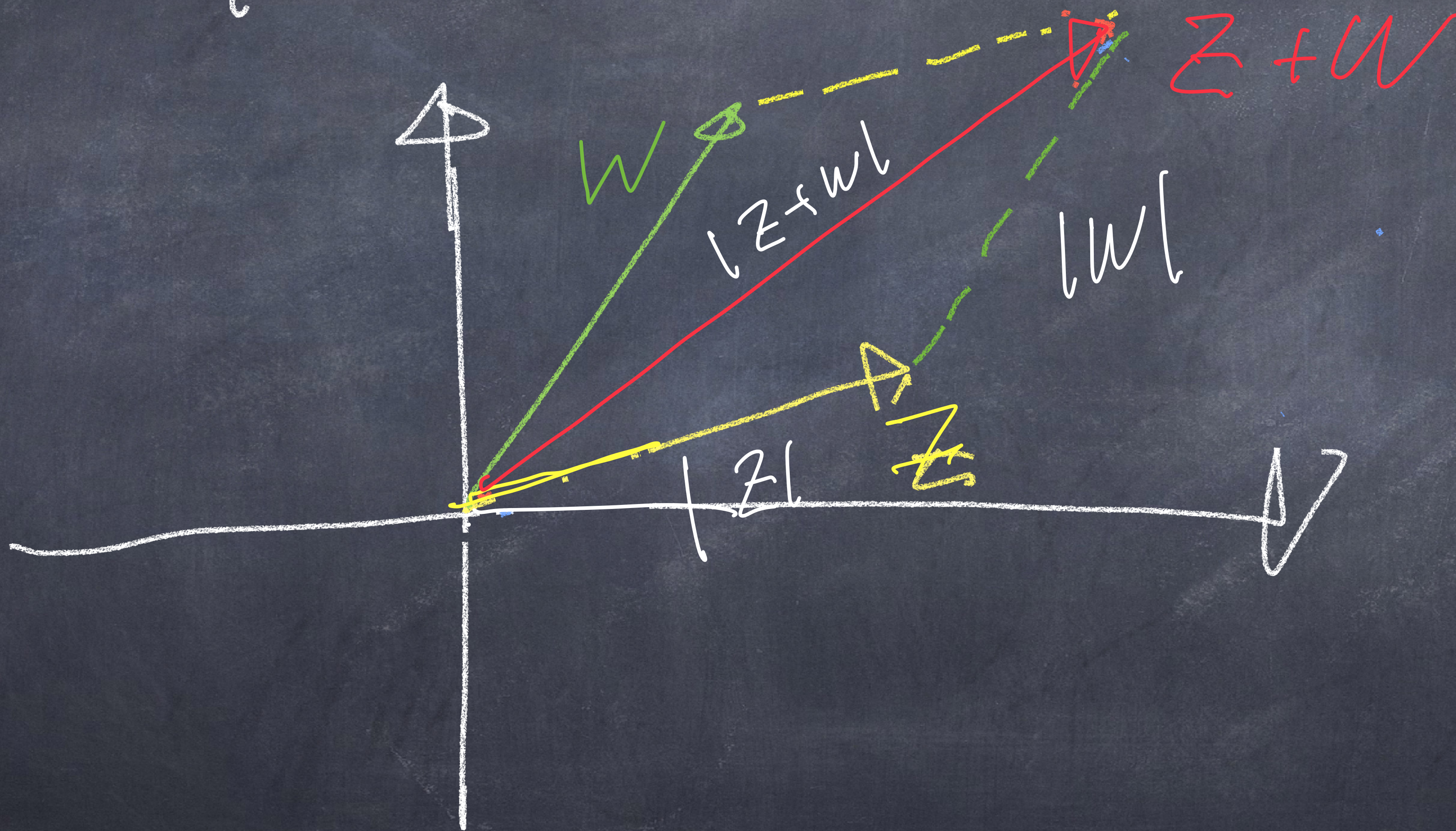
$$(V.L.)^2 = |z \cdot w|^2 = |(ac - bd) + i(ad + bc)|^2$$

$$= (ac - bd)^2 + (ad + bc)^2 = \dots$$

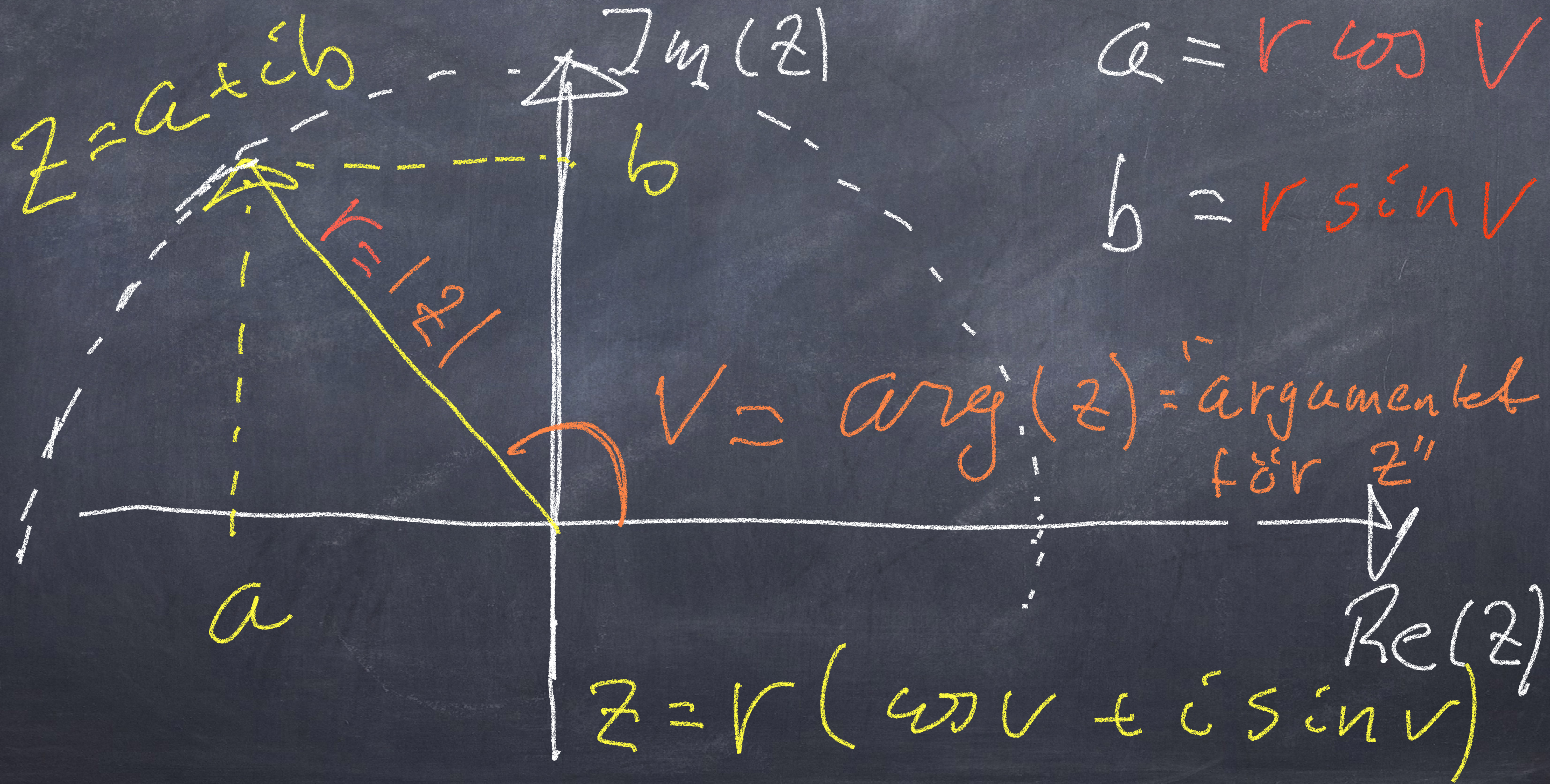
$$(H.L.)^2 = |z|^2 |w|^2 = (a^2 + b^2)(c^2 + d^2) = \dots = (V.L.)^2$$

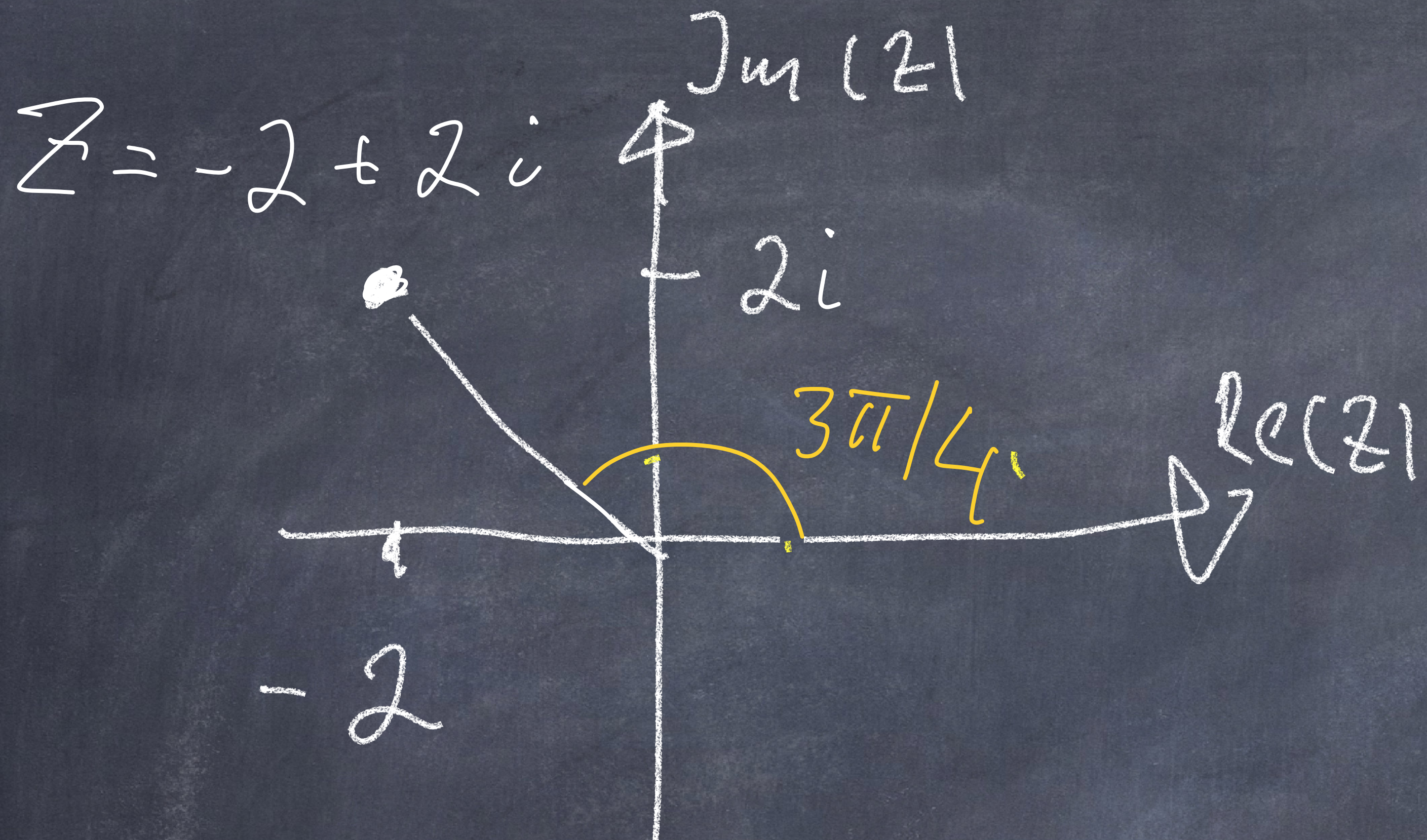
Trüngerelicheten

$$|Z+U| \leq |Z| + |U|$$



Polär form





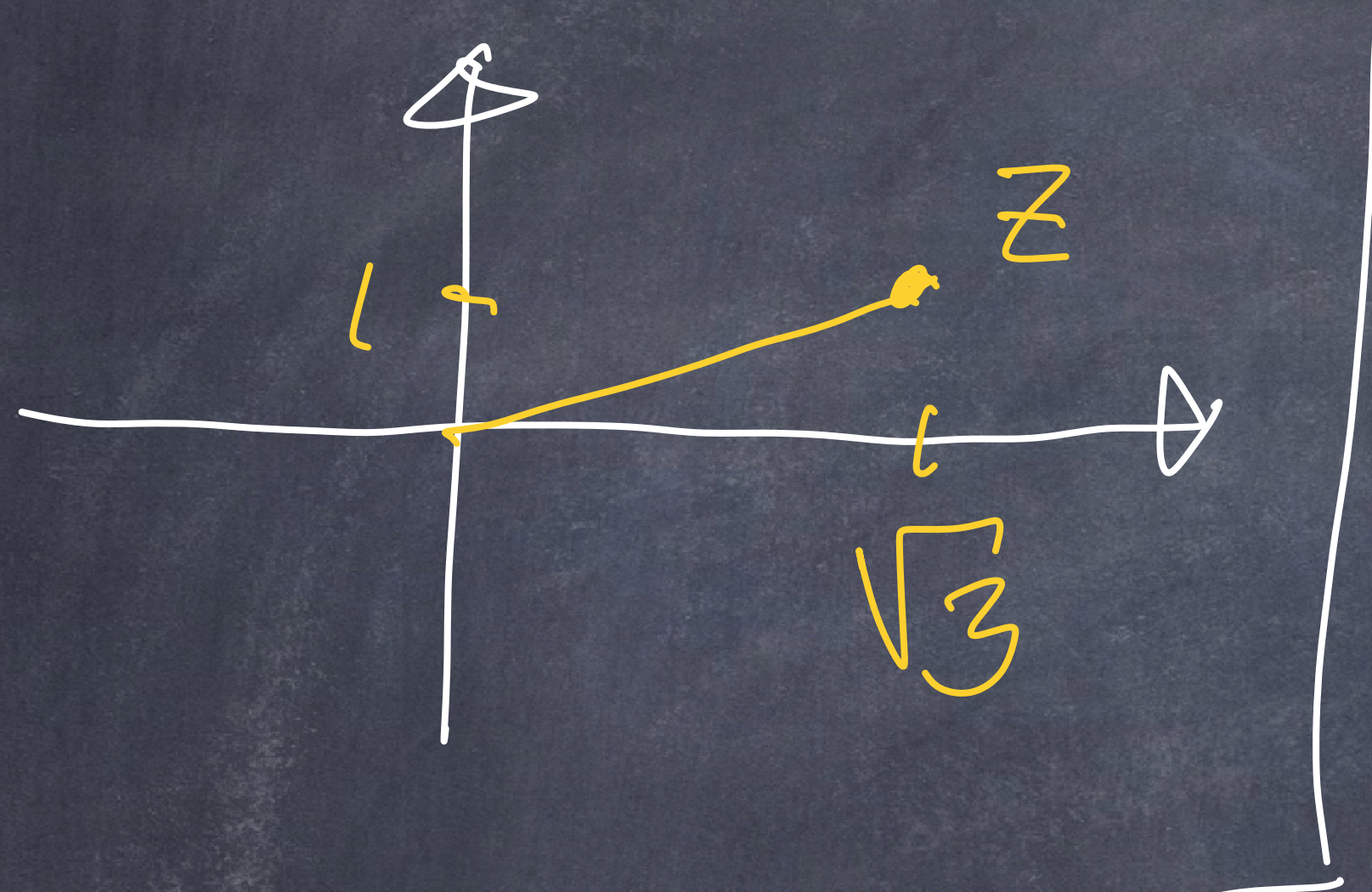
$$r = |Z| = \sqrt{(-2)^2 + 2^2}$$
$$= \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}$$

$$\varphi = \arg(Z) = \frac{3\pi}{4}$$

$$Z = 2\sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$$

$$\text{Koll.: } = 2\sqrt{2} \left(-\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) = -2 + 2i$$

Skriv $z = \sqrt{3} + i$ på polar form.



$$|z| = \sqrt{\sqrt{3}^2 + 1^2} = \sqrt{3+1} = 2$$

$$z = \sqrt{3} + i = 2 \left(\frac{\sqrt{3}}{2} + \frac{i}{2} \right)$$

Sök nu φ s.a.
$$\begin{cases} \cos \varphi = \sqrt{3}/2 \\ \sin \varphi = 1/2 \end{cases} \quad \text{I.ex } \varphi = \frac{\pi}{6}$$

Alltså är
$$z = 2 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$$

Skriv $z = -\sqrt{3} - i$ på polär form



$$|z| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = 2$$

$$z = -\sqrt{3} - i = 2 \left(-\frac{\sqrt{3}}{2} - \frac{i}{2} \right)$$

$$|z| \left(\cos \varphi + i \sin \varphi \right)$$

$$z = 2 \left(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6} \right)$$

Sök φ s.a.

$$\cos \varphi = -\sqrt{3}/2$$

$$\sin \varphi = -1/2$$

Ta t.ex $\varphi = 7\pi/6$, så

Polär form & multiplikation & division

$$\text{Om } z = |z| (\cos u + i \sin u)$$

$$w = |w| (\cos v + i \sin v) \quad \text{Så}$$

$$z \cdot w =$$

Polär form & multiplikation & division

$$\text{Om } z = |z| (\cos u + i \sin u)$$

$$w = |w| (\cos v + i \sin v) \quad \text{Så}$$

$$z \cdot w = |z| |w| (\cos u + i \sin u) \cdot (\cos v + i \sin v)$$

Polär form & multiplikation & division

$$\text{Om } z = |z| (\cos u + i \sin u)$$

$$w = |w| (\cos v + i \sin v) \quad \text{Så}$$

$$\begin{aligned} z \cdot w &= |z| |w| (\cos u + i \sin u) \cdot (\cos v + i \sin v) \\ &= |z| |w| \left(\begin{aligned} &(\cos u \cos v - \sin u \sin v) \\ &+ i (\sin v \cos u + \cos v \sin u) \end{aligned} \right) \end{aligned}$$

Polär form & multiplikation & division

$$\text{Om } z = |z| (\cos u + i \sin u)$$

$$w = |w| (\cos v + i \sin v) \quad \text{Så}$$

$$z \cdot w = |z| |w| (\cos u + i \sin u) \cdot (\cos v + i \sin v)$$

$$= |z| |w| \left(\begin{aligned} & \cos u \cos v - \sin u \sin v \\ & + i (\sin v \cos u + \cos v \sin u) \end{aligned} \right)$$

Add.

$$= |z| |w| (\cos(u+v) + i \sin(u+v))$$

formelerna

$\underline{\text{Sats:}}$ $z = |z| (\cos u + i \sin u)$ och $w = |w| (\cos v + i \sin v)$

$z \cdot w = |z| \cdot |w| = (\cos(u+v) + i \sin(u+v))$

$\frac{z}{w} = \frac{|z|}{|w|} (\cos(u-v) + i \sin(u-v))$

Exempel: $Z = (1 + i) = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$

$$W = -2 + 2i = 2\sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$$

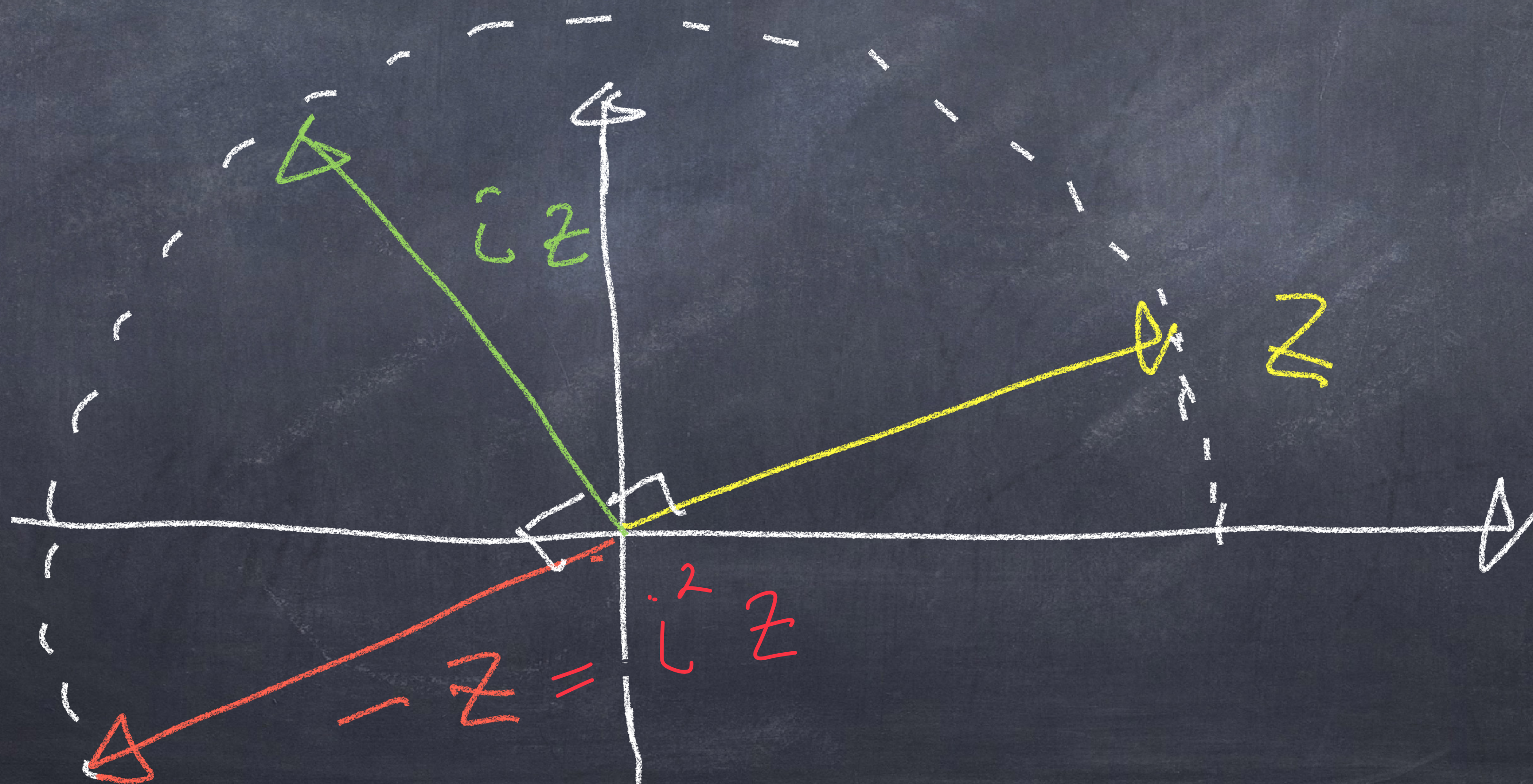
$$Z \cdot W = \sqrt{2} \cdot 2\sqrt{2} \left(\cos \left(\frac{\pi}{4} + \frac{3\pi}{4} \right) + i \sin \left(\frac{\pi}{4} + \frac{3\pi}{4} \right) \right)$$

$$= 4 \left(\underbrace{\cos \pi}_{-1} + i \underbrace{\sin \pi}_{=0} \right) = -4$$

Äfr: $(1 + i)(-2 + 2i) = (-2 + \underbrace{2i^2}_{-2}) + (2i - 2i) = \underline{\underline{-4}}$

Ex: $\dot{L} = \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)$

$\dot{L}^2 = -1 = \cos \pi + i \sin \pi$



de Moïvres formel :

$$\text{Om } z = a + ib = |z|(\cos v + i \sin v)$$

$$\text{Så } z^n = (a + ib)^n =$$

$$= |z|^n (\cos nv + i \sin nv)$$

för alla $n \in \mathbb{Z}$.

Berús se su. Wikipedía
sōdan "de Moivres formel".

Example:
$$\frac{(1+i)^{10}}{(1-i)^8} = (1+i)^{10} \cdot (1-i)^{-8}$$

$$\left\{ \begin{aligned} 1+i &= \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \\ 1-i &= \sqrt{2} \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right) \end{aligned} \right\} =$$

$$= \left(\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right)^{10} \cdot \left(\sqrt{2} \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right) \right)^{-8}$$

$$= (\sqrt{2})^{10} \cdot (\sqrt{2})^{-8} \left(\cos \frac{10\pi}{4} + i \sin \frac{10\pi}{4} \right) \left(\cos \left(-\frac{8\pi}{4} \right) + i \sin \left(-\frac{8\pi}{4} \right) \right)$$

$$= \sqrt{2}^2 \left(\cos \left(\frac{10\pi}{4} - \frac{8\pi}{4} \right) + i \sin \left(\frac{10\pi}{4} - \frac{8\pi}{4} \right) \right) = 2 \left(\underbrace{\cos \frac{2\pi}{4}}_0 + i \underbrace{\sin \frac{2\pi}{4}}_1 \right) = \underline{\underline{2i}}$$

$$L_a \in \quad E(v) = \omega v + i s i n v$$

Enlight over

$$E(u) \cdot E(v) = E(u+v)$$

$$\frac{E(u)}{E(v)} = E(u-v)$$

$$[E(u)]^n = E(n \cdot u)$$

3 fr:

$$e^u \cdot e^v = e^{u+v}$$

$$\frac{e^u}{e^v} = e^{u-v}$$

$$(e^u)^n = e^{n \cdot u}$$

Vi definierar (Gl.a.) därför

$$e^{iy} = \cos y + i \sin y, \quad y \in \mathbb{R}$$

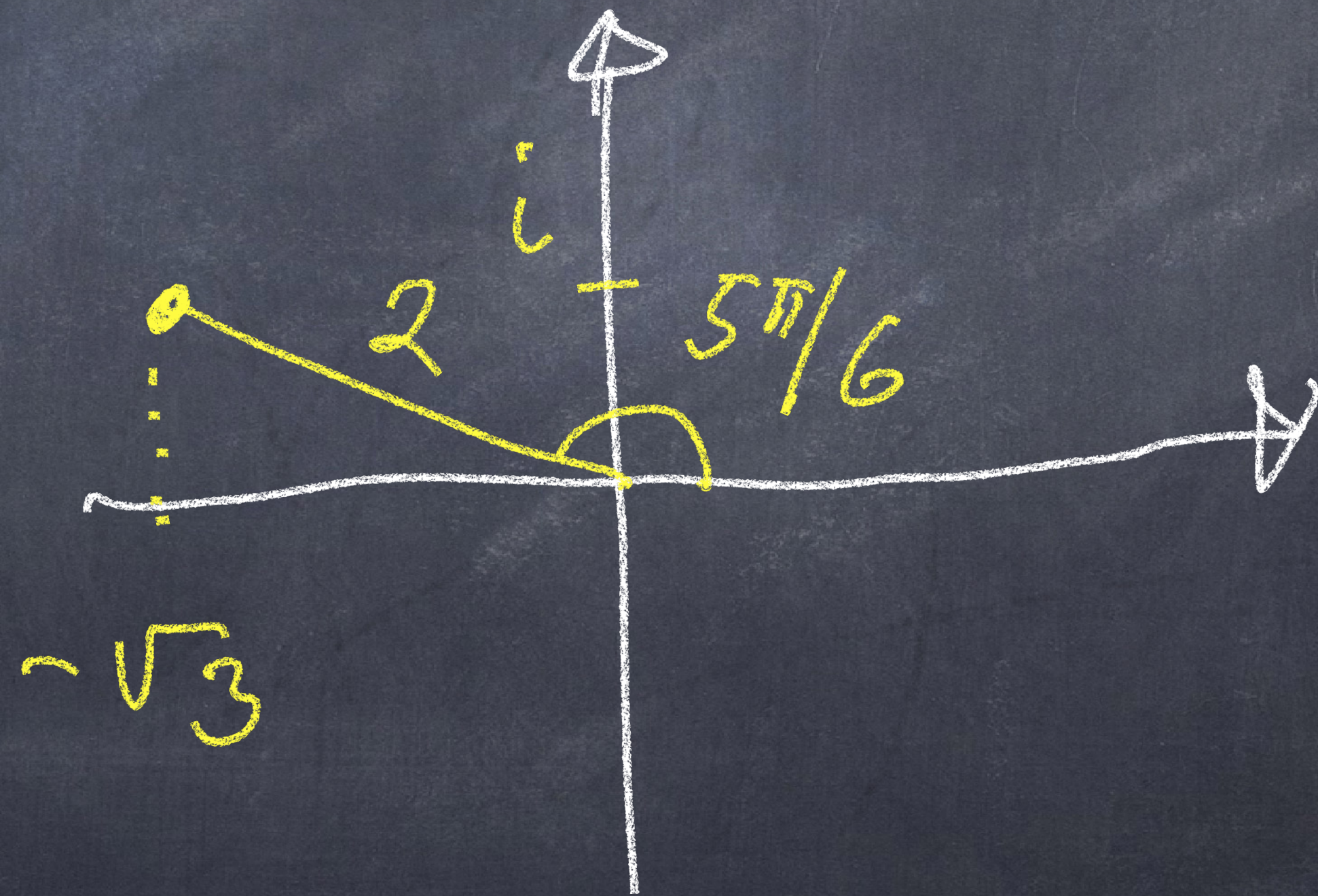
Ex.: $e^{i\pi} = \cos \pi + i \sin \pi = -1$

De Moivre's: $(e^{iv})^n = e^{in v}$

Ex: $-\sqrt{3} + i = 2 \left(-\frac{\sqrt{3}}{2} + \frac{1}{2} i \right) = 2 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)$

$= 2 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)$

$\Rightarrow 2 e^{i 5\pi/6}$



0 m $z = x + iy$ definieren wir

$$\underline{e^z = e^{x+iy} = e^x \cdot e^{iy}}$$

$$\underline{= e^x (\cos y + i \sin y)}$$

$$e^{x+iy} = e^x (\cos y + i \sin y)$$

Följer att

$$|e^{x+iy}| = e^x, \quad \arg(e^{x+iy}) = y$$

Exempel Om $z = \ln 2 + i \frac{5\pi}{6}$ så är

$$\begin{aligned} e^z &= e^{\ln 2 + i \frac{5\pi}{6}} \\ &= e^{\ln 2} \cdot e^{i \frac{5\pi}{6}} = 2 e^{i \frac{5\pi}{6}} \end{aligned} \quad |e^z| = e^{\ln 2} = 2, \quad \arg(e^z) = \frac{5\pi}{6}$$

Komplexa trigonometrisk

$$\begin{cases} e^{iy} = \cos y + i \sin y \\ e^{-iy} = \cos y - i \sin y \end{cases} \Rightarrow \begin{cases} \cos y = \frac{e^{iy} + e^{-iy}}{2} \\ \sin y = \frac{e^{iy} - e^{-iy}}{2i} \end{cases}$$

för $y \in \mathbb{R}$

$$\begin{cases} \cos z \stackrel{\text{def}}{=} \frac{e^{iz} + e^{-iz}}{2} \\ \sinh z \stackrel{\text{def}}{=} \frac{e^{iz} - e^{-iz}}{2i} \end{cases}$$

Fördjupning / ^u Överkurs



$$\cos(x+iy) =$$

$$\frac{1}{2} \left(e^{i(x+iy)} + e^{-i(x+iy)} \right)$$

$$= \frac{1}{2} \left(e^{-y+ix} + e^{y-ix} \right)$$

$$= \frac{1}{2} e^{-y} (\cos x + i \sin x) + \frac{1}{2} e^y (\cos x - i \sin x)$$

$$\cosh z = \cosh(\bar{z})$$

$$i \sinh z = \sinh(\bar{z})$$

$$\cosh iz = \sinh z$$

$$i \sinh iz = -\sinh z$$

$$\cosh t = \frac{e^t + e^{-t}}{2}$$

läßt $z \in \mathbb{C}$,

$$\sinh t = \frac{e^t - e^{-t}}{2}$$

Kann man zeigen $(x \in \mathbb{R})$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$x = iy$ setzen $e^{iy} = \cos y + i \sin y$