SF1661 HT21

Föreläsning 9

23/9 - 21

TRIGONOMETRIC IDENTITIES

$\sin^2 x + \cos^2 x = 1$

$\sec^2 x = 1 + \tan^2 x$

$\csc^2 x = 1 + \cot^2 x$

$\sin(-x) = -\sin x$

$\sin(\pi - x) = \sin x$

$\sin\left(\frac{\pi}{2} - x\right) = \cos x$

$\cos(-x) = \cos x$

$\cos(\pi - x) = -\cos x$

$\cos\left(\frac{\pi}{2} - x\right) = \sin x$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

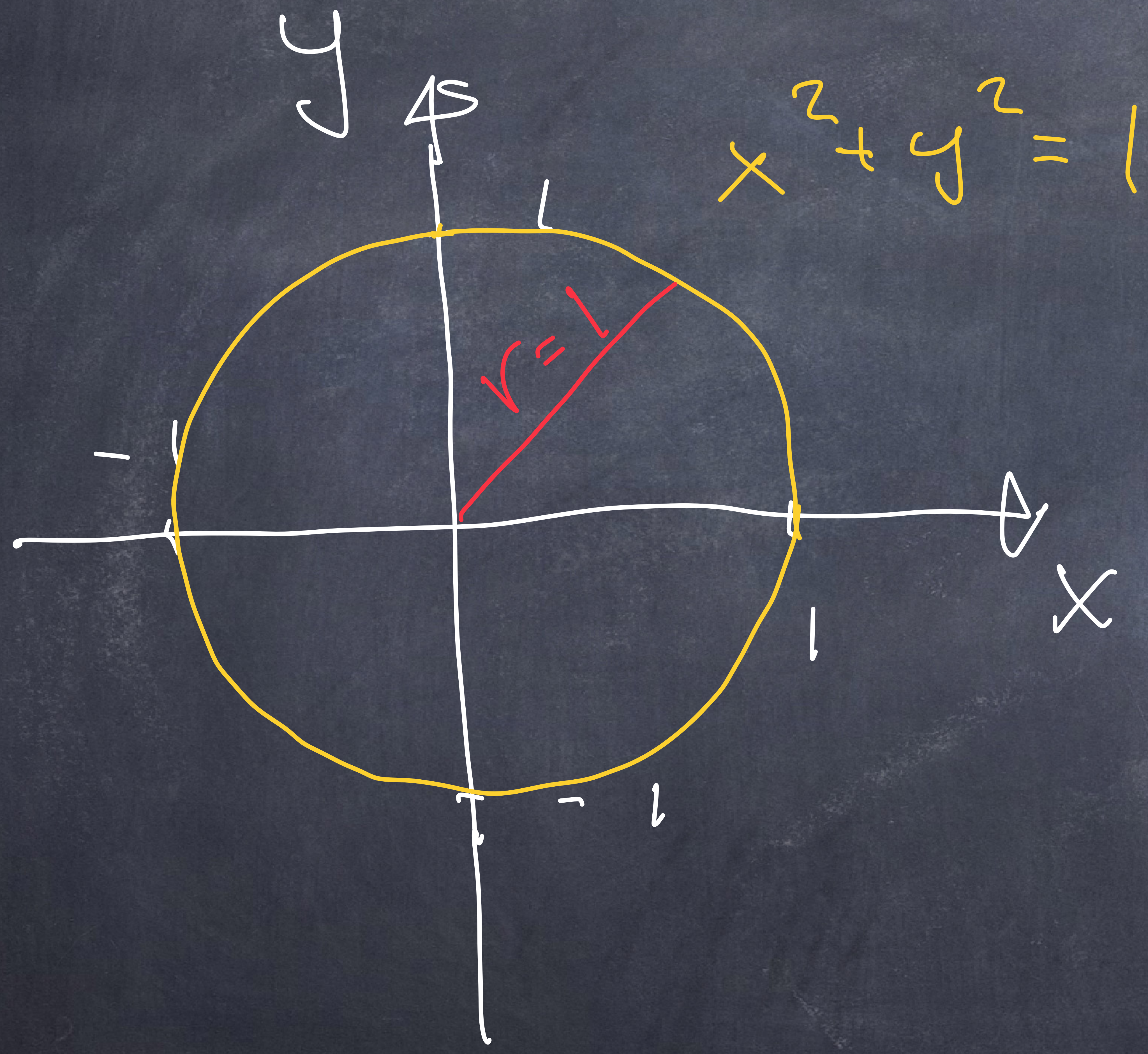
$\sin 2x = 2 \sin x \cos x$

$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

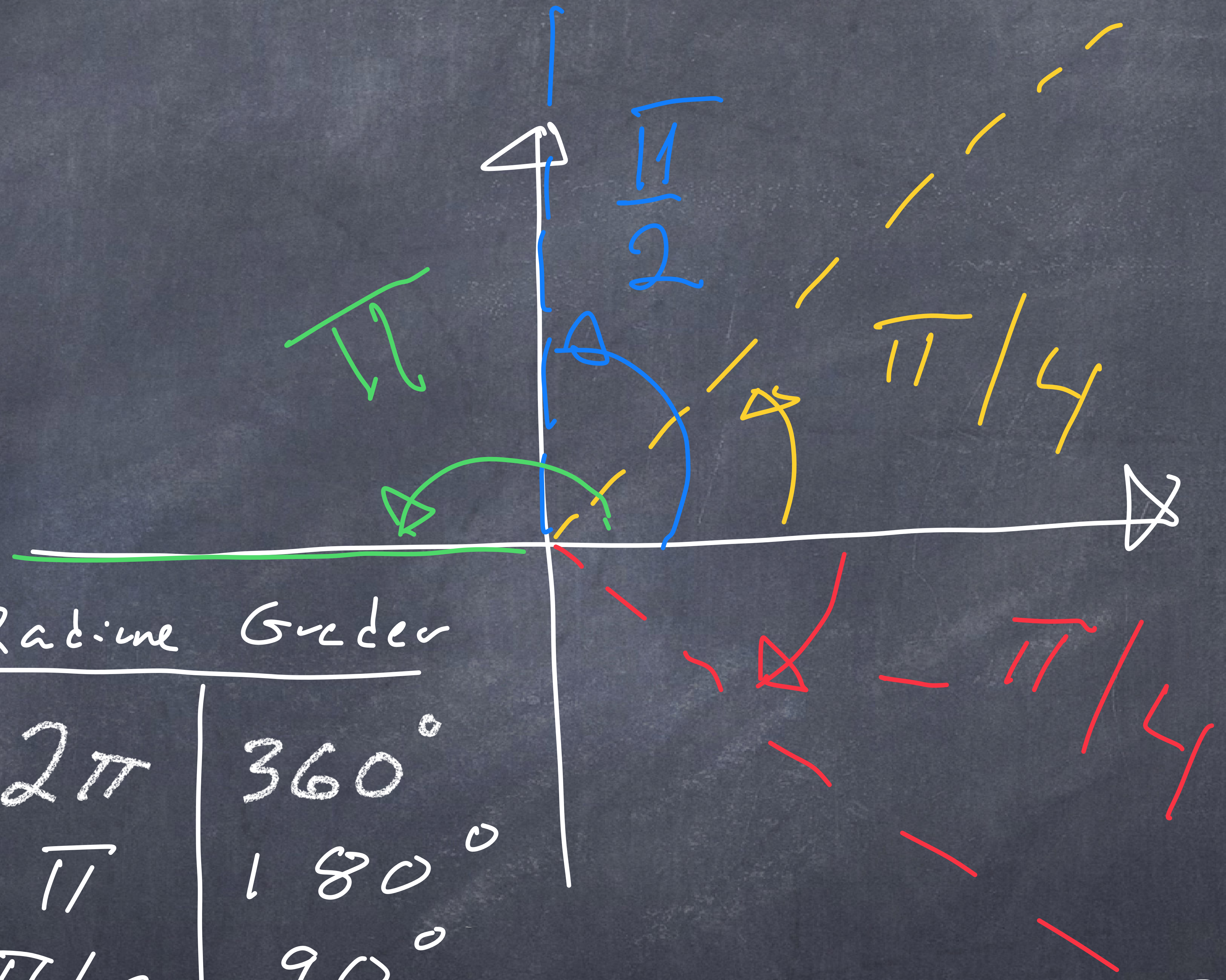
$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

Enhetscirkeln



Radianer



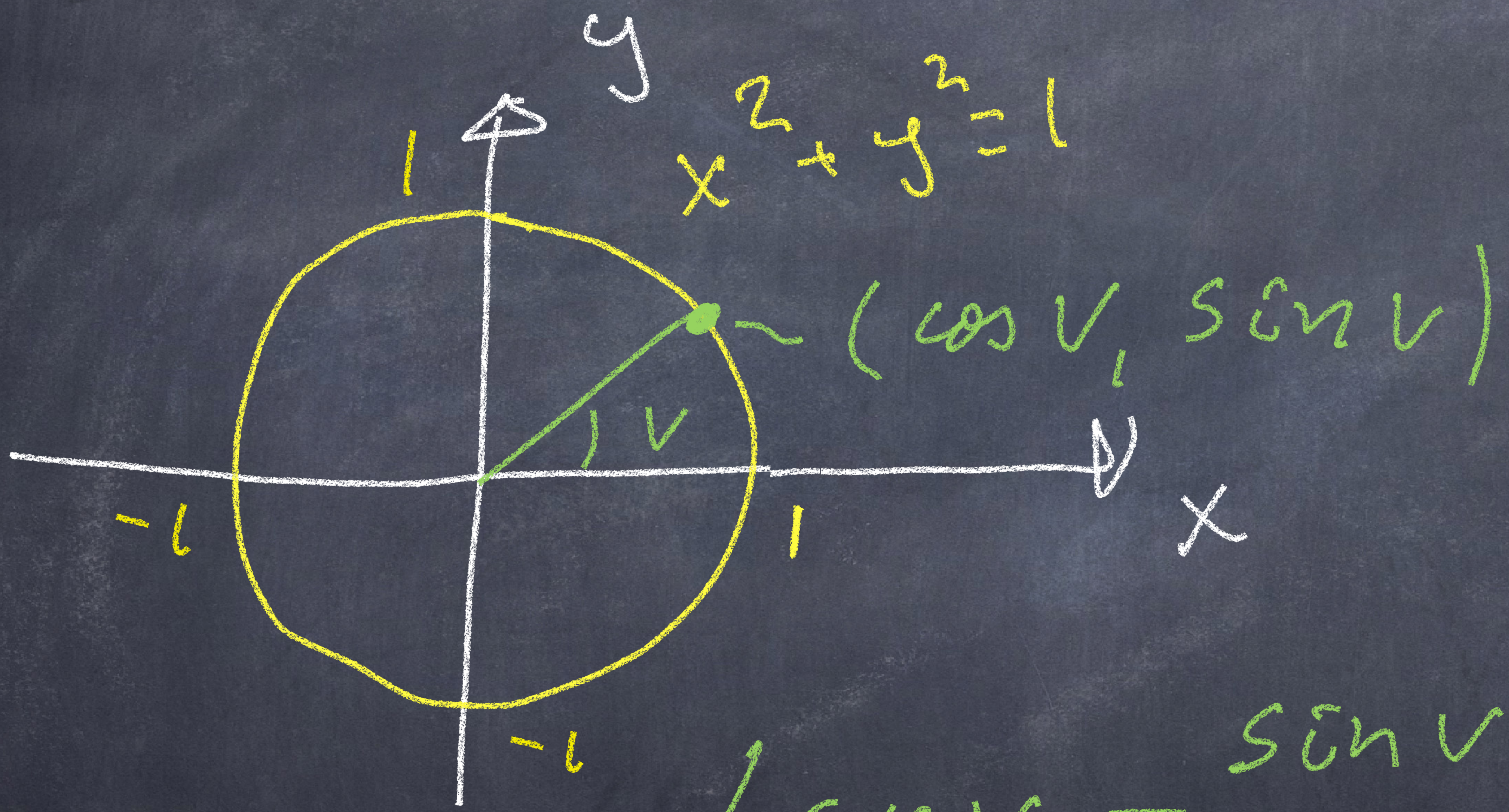
Radiane	Grader
2π	360°
π	180°
$\pi/2$	90°
$\pi/3$	60°
$\pi/4$	45°
$\pi/6$	30°

$$2\pi\ (\text{rad}) = 360^\circ$$

$$1\ (\text{rad}) = \frac{180^\circ}{\pi}$$

$$\frac{\pi}{180}\ (\text{rad}) = 1^\circ$$

Definitioner cos, sin & tan

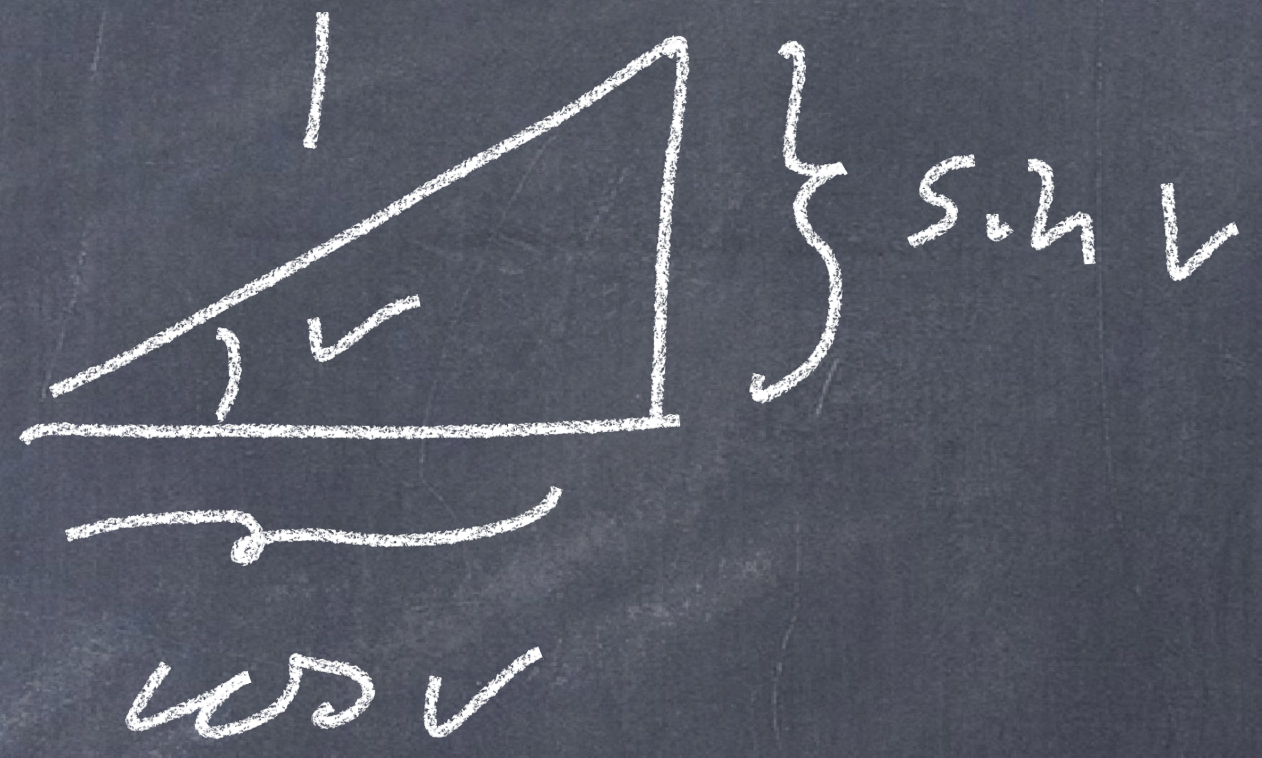


$$\tan v = \frac{\sin v}{\cos v}$$

Konsekvenser

$-1 \leq \cos v \leq 1$

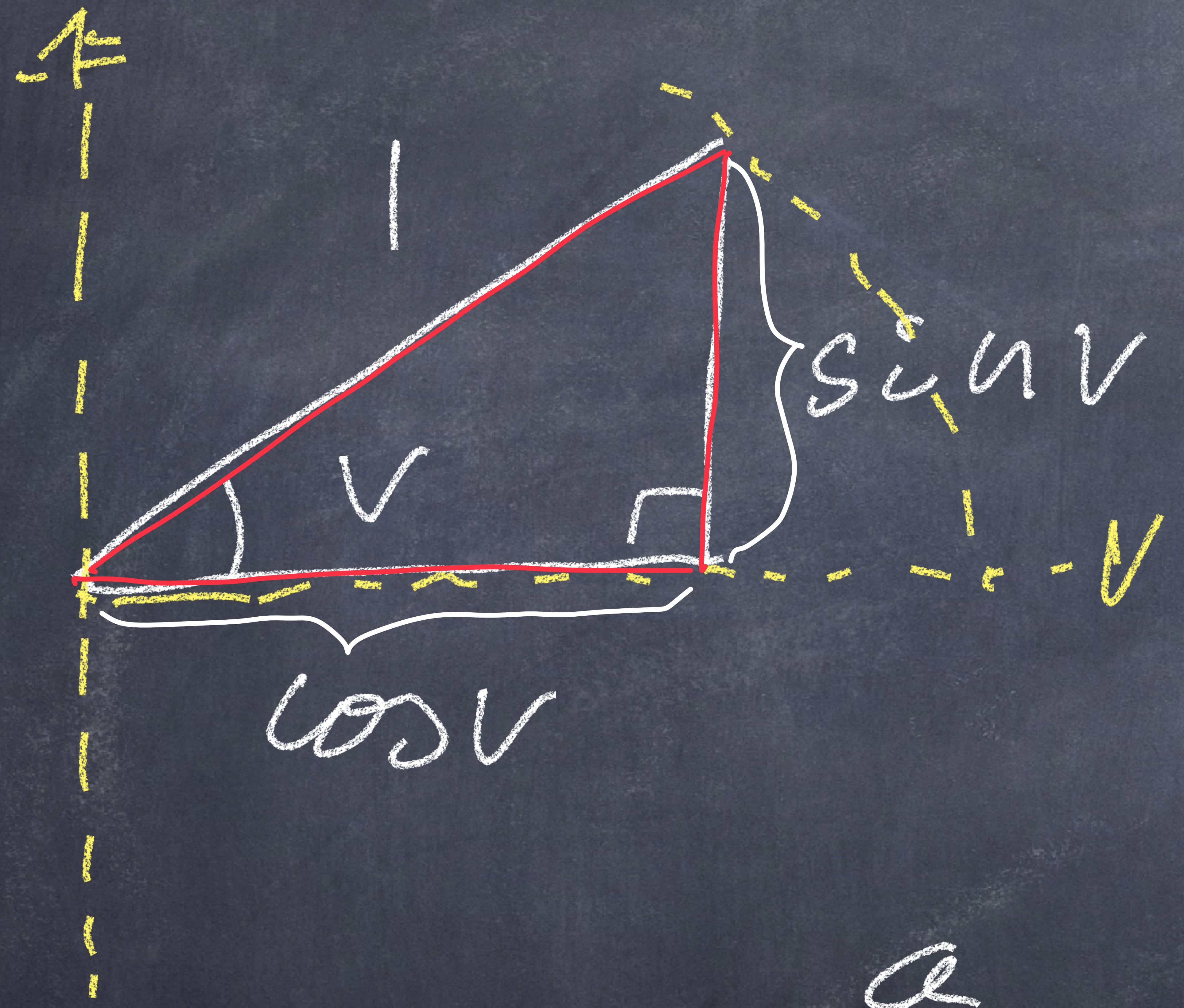
$-1 \leq \sin v \leq 1$



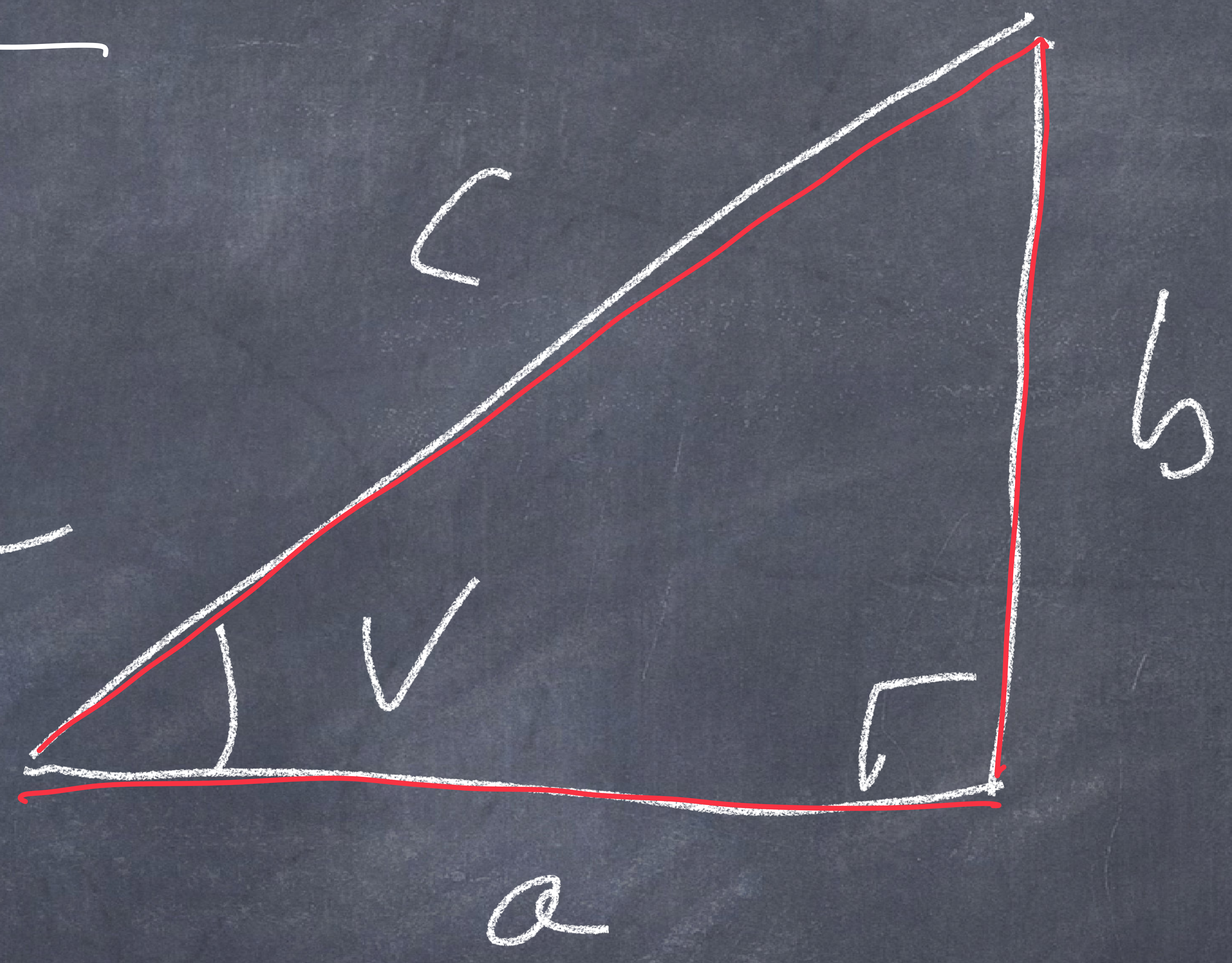
$\cos^2 v + \sin^2 v = 1$

OBS: $\cos^2 v$ betyder $(\cos v)^2$

Om $\underline{0 < v < \frac{\pi}{2}}$

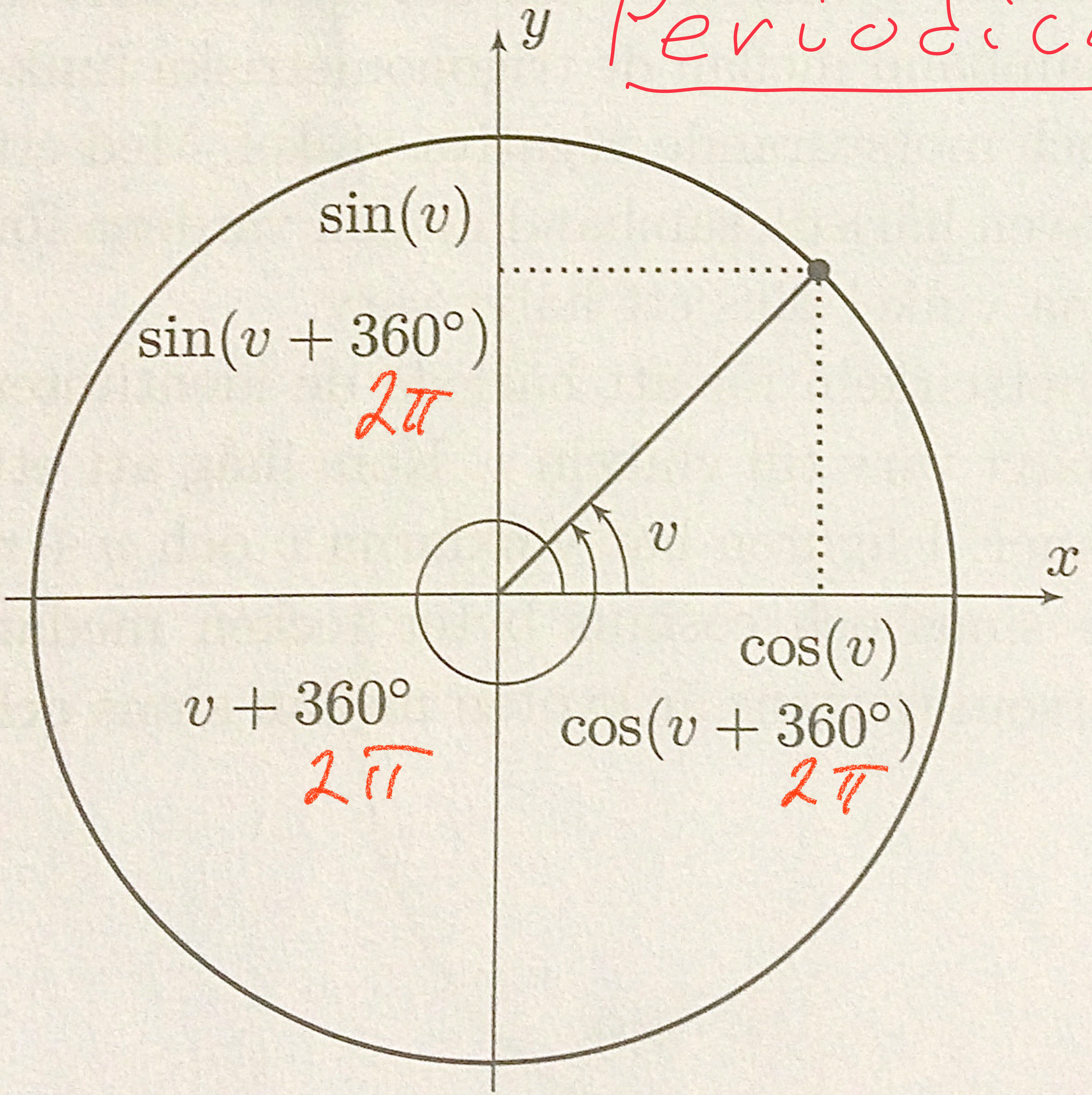


$\sim$



$$\cos v = \frac{a}{c}, \quad \sin v = \frac{b}{c}, \quad \tan v = \frac{b}{a}$$

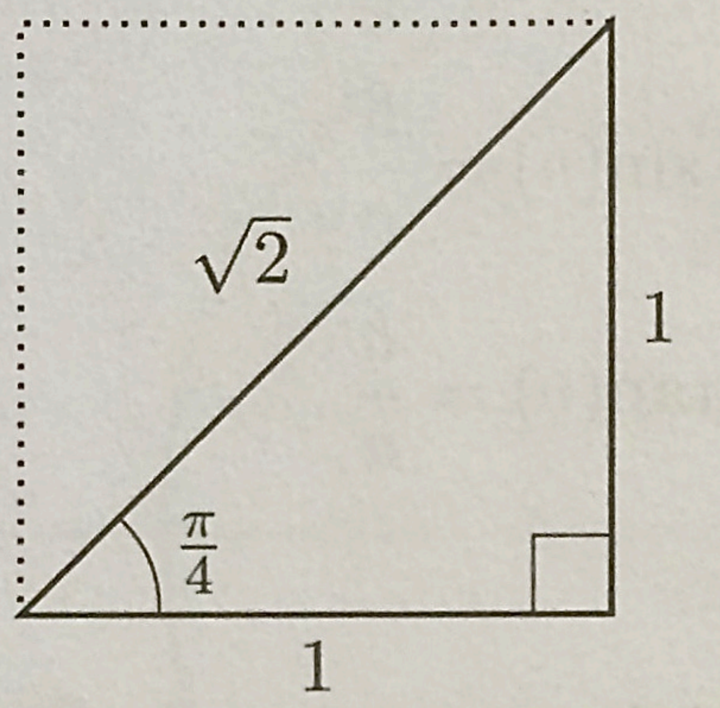
Periodicitet



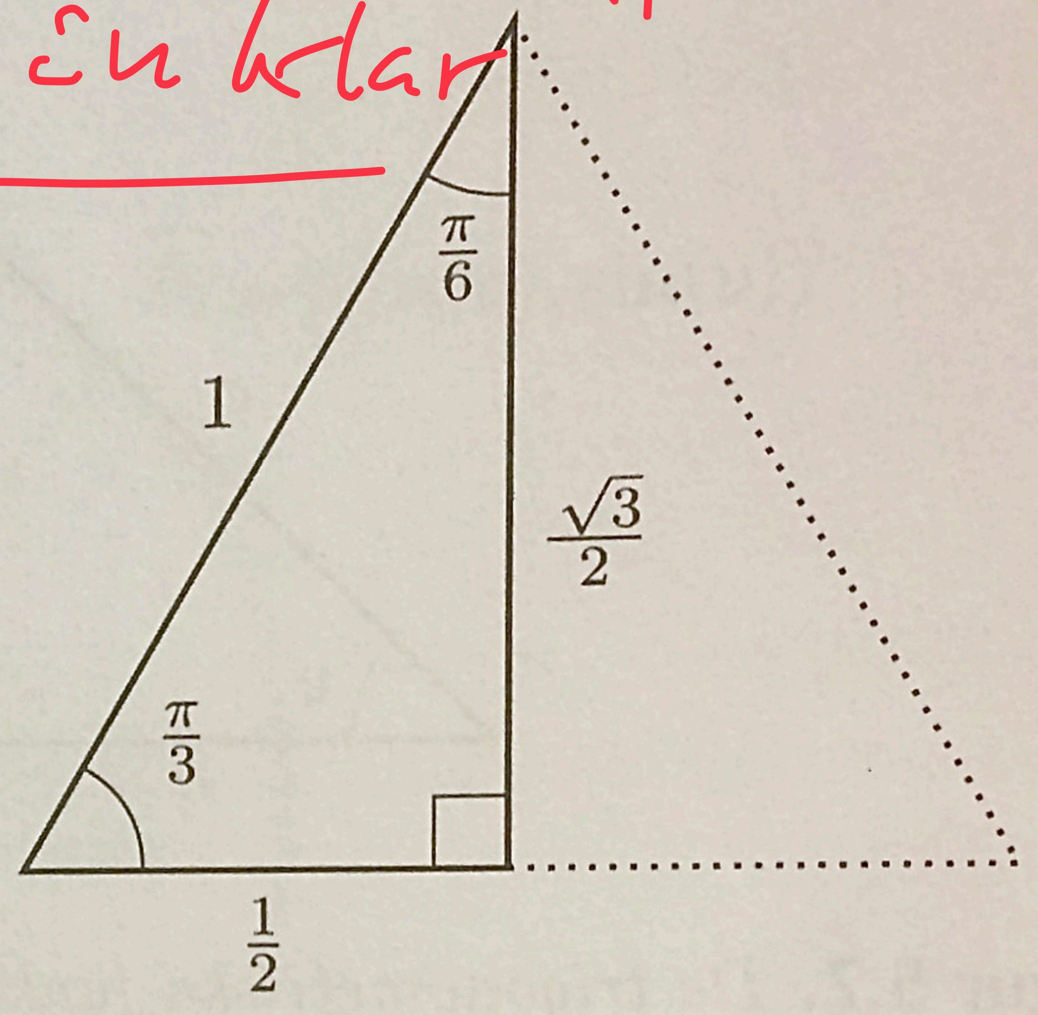
2π

$$\cos(v + 360^\circ) = \cos(v)$$
$$\sin(v + 360^\circ) = \sin(v)$$
$$\tan(v + 360^\circ) = \tan(v)$$
$$= \tan(v + \underbrace{180^\circ}_{\pi})$$

"Standardvinklar"



Figur 5.8. *En halv kvadrat.*

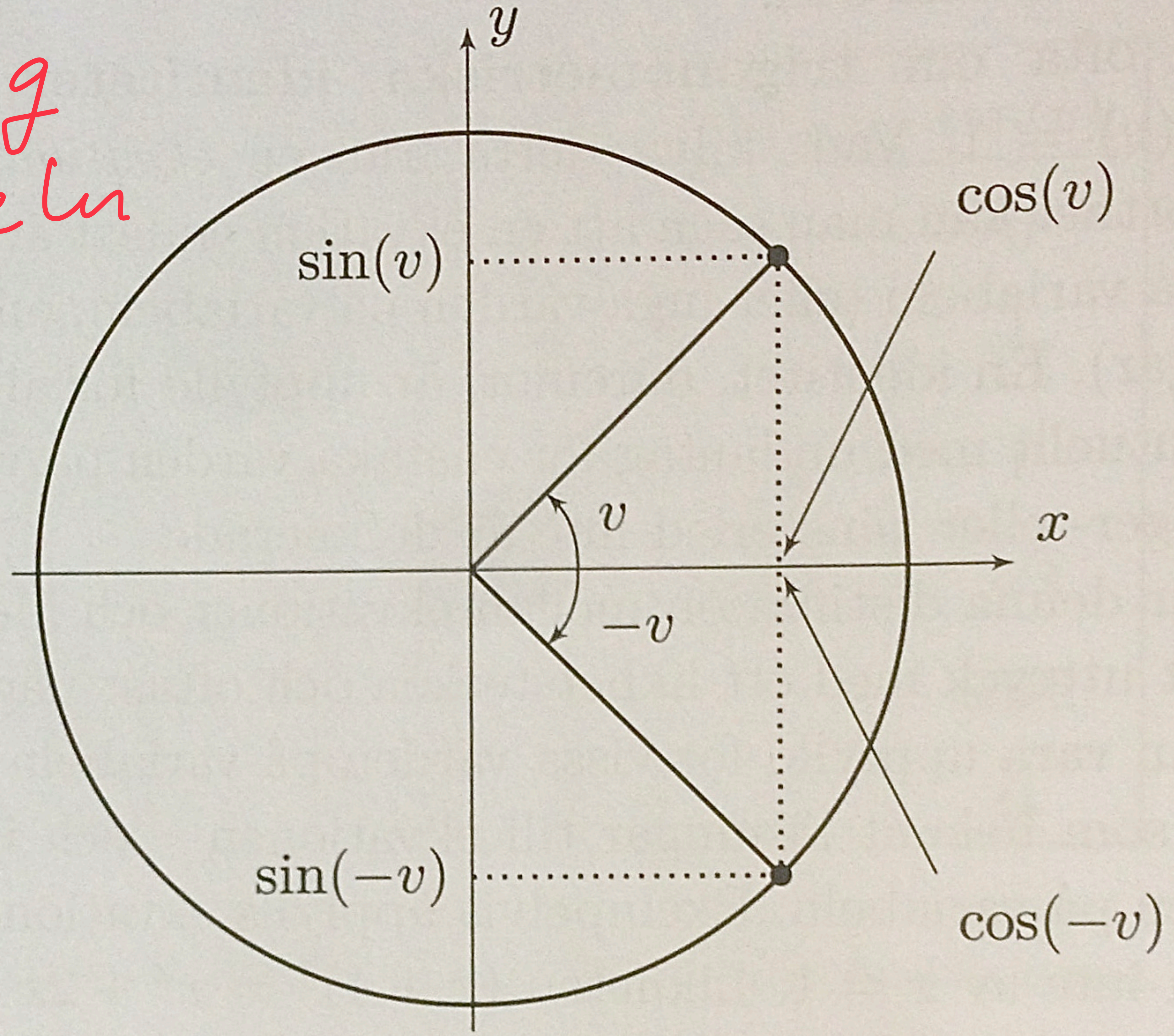


Figur 5.9. *En halv liksidig triangel.*

v rad	v°	$\cos(v)$	$\sin(v)$	$\tan(v)$
0	0	1	0	0
$\frac{\pi}{6}$	30	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	45	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	60	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	90	0	1	odef.
π	180	-1	0	0

Figur 5.10. *Några värden för de trigonometriska funktionerna.*

Spegling i x-axeln

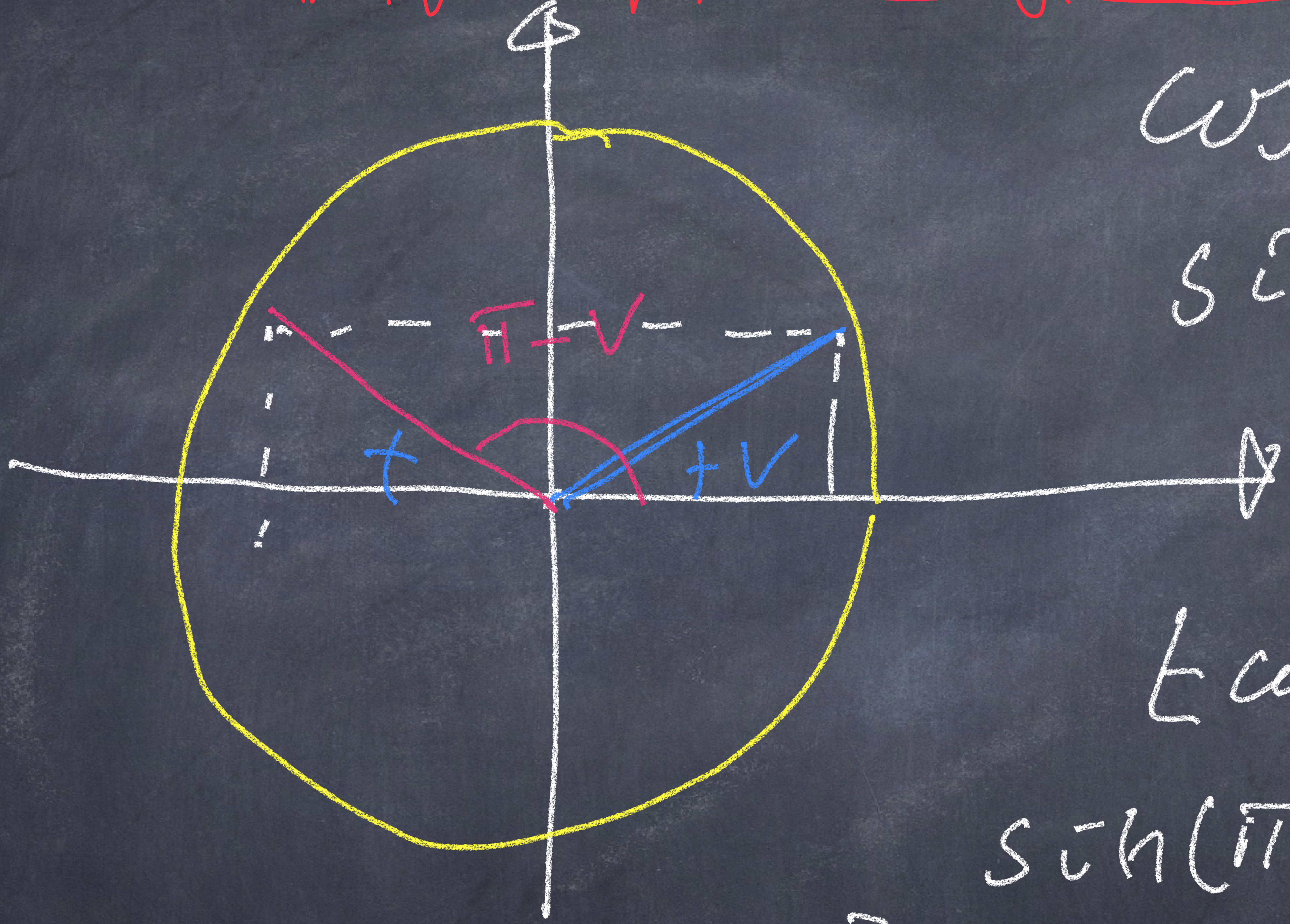


Figur 5.11. *De trigonometriska funktionernas värden för negativa vinklar.*

Vi sammanfattar detta med tre identiteter.

$$\begin{aligned}\cos(-v) &= \cos(v)\\ \sin(-v) &= -\sin(v)\\ \tan(-v) &= -\tan(v)\end{aligned}$$

Spegling i y-axeln



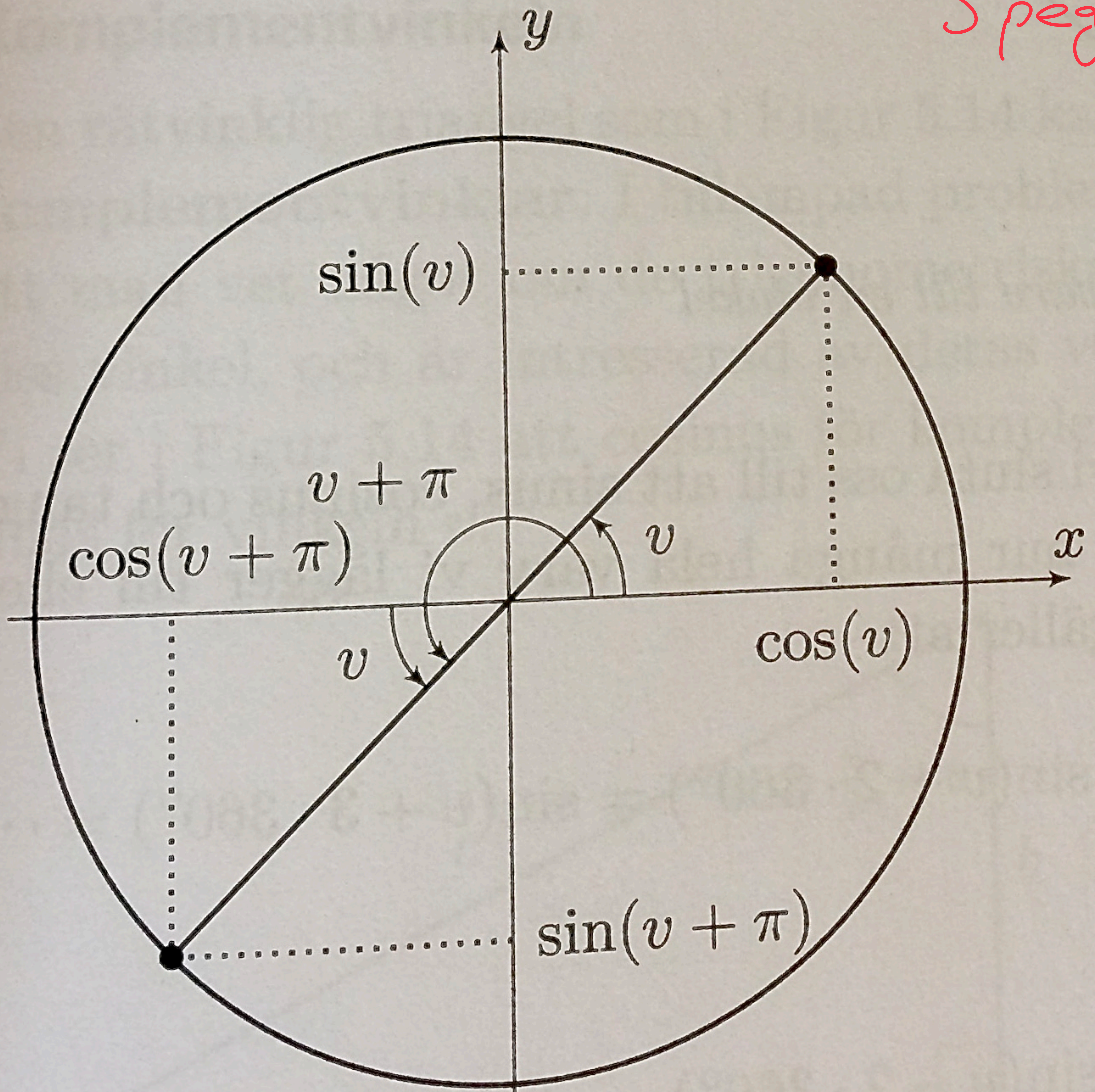
$$\cos(\pi - v) = -\cos v$$

$$\sin(\pi - v) = \sin v$$

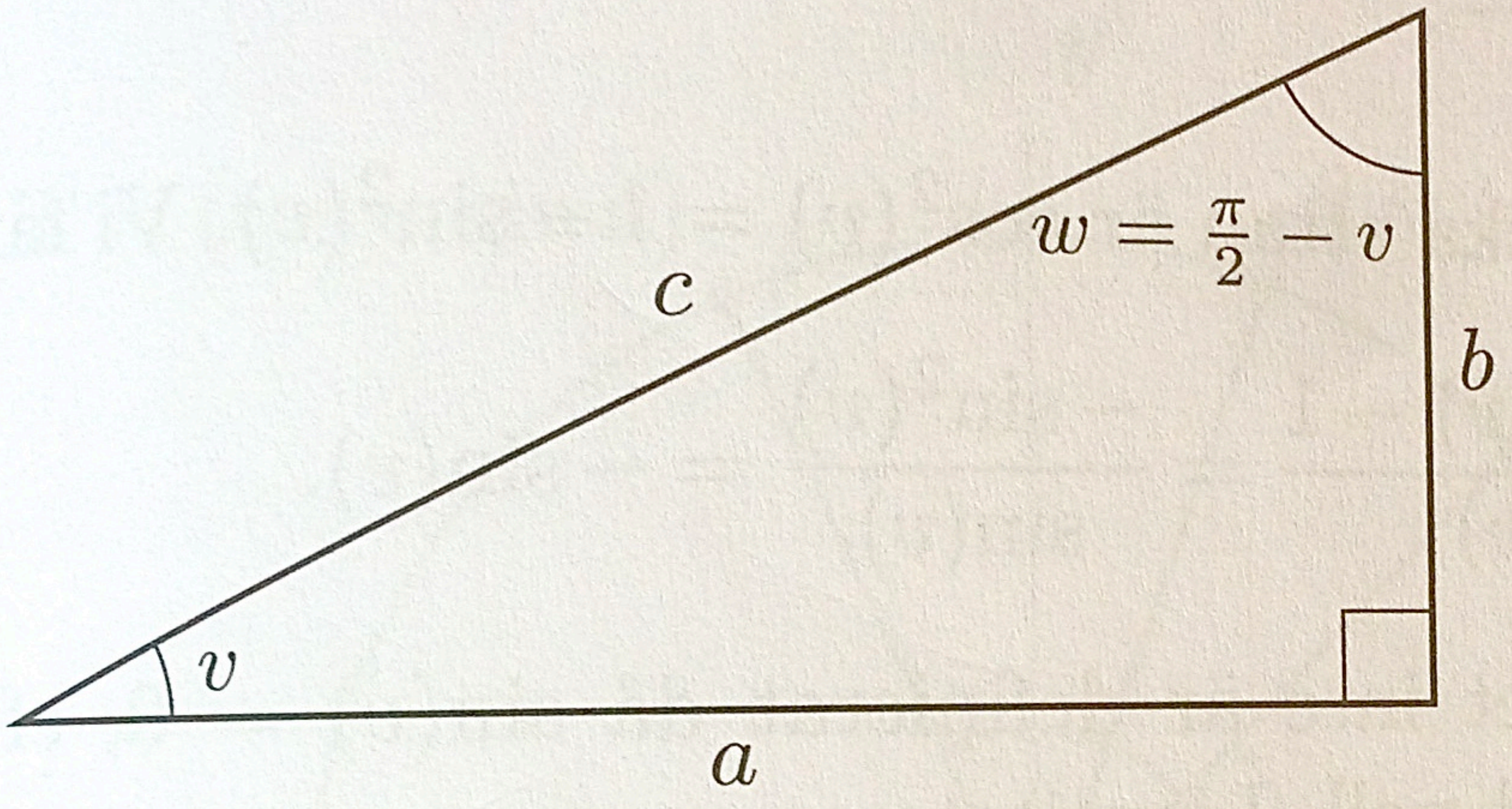
$$\tan(\pi - v) =$$

$$= \frac{\sin(\pi - v)}{\cos(\pi - v)} = \frac{\sin v}{-\cos v} = -\tan v$$

Spegling i Origo



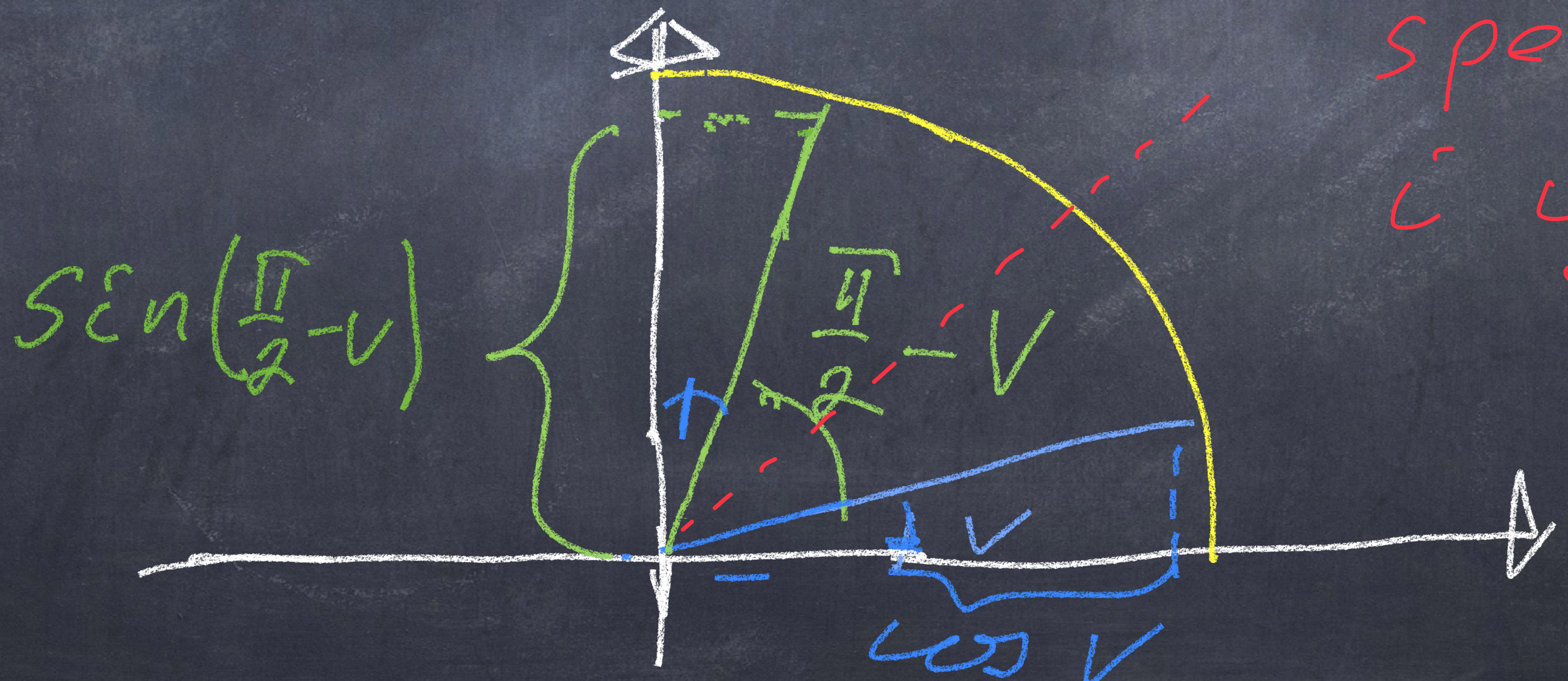
$$\cos(v+\pi) = -\cos(v)$$
$$\sin(v+\pi) = -\sin(v)$$
$$\tan(v+\pi) = \tan(v)$$



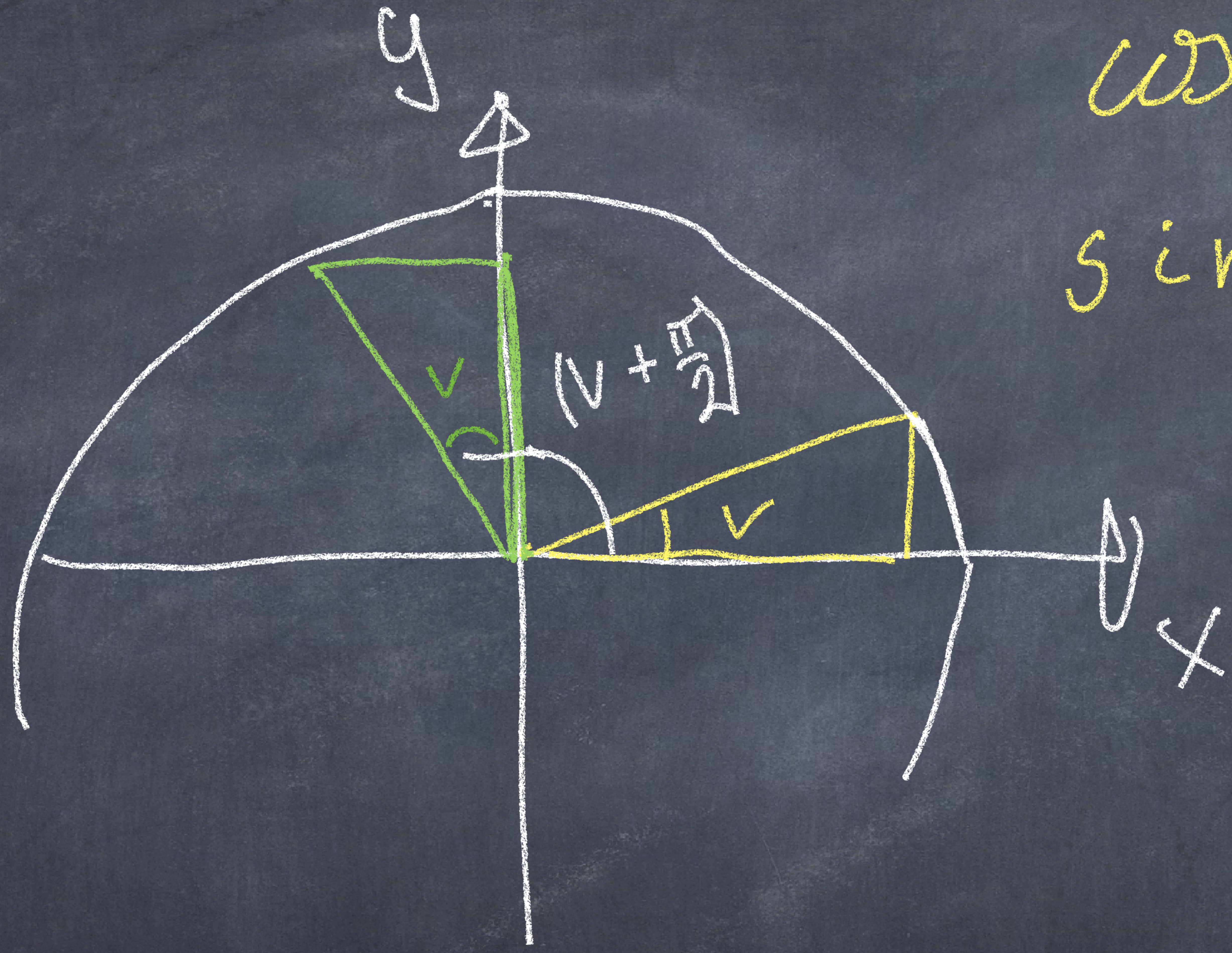
$$\cos\left(\tfrac{\pi}{2} - v\right) = \frac{b}{c} = \sin(v)$$

$$\sin\left(\tfrac{\pi}{2} - v\right) = \frac{a}{c} = \cos(v)$$

$$\tan\left(\tfrac{\pi}{2} - v\right) = \frac{a}{b} = \frac{1}{\tan(v)}$$



spegling
i y=x



$$\cos v = \sin\left(v + \frac{\pi}{2}\right)$$

$$\sin v = -\cos\left(v + \frac{\pi}{2}\right)$$

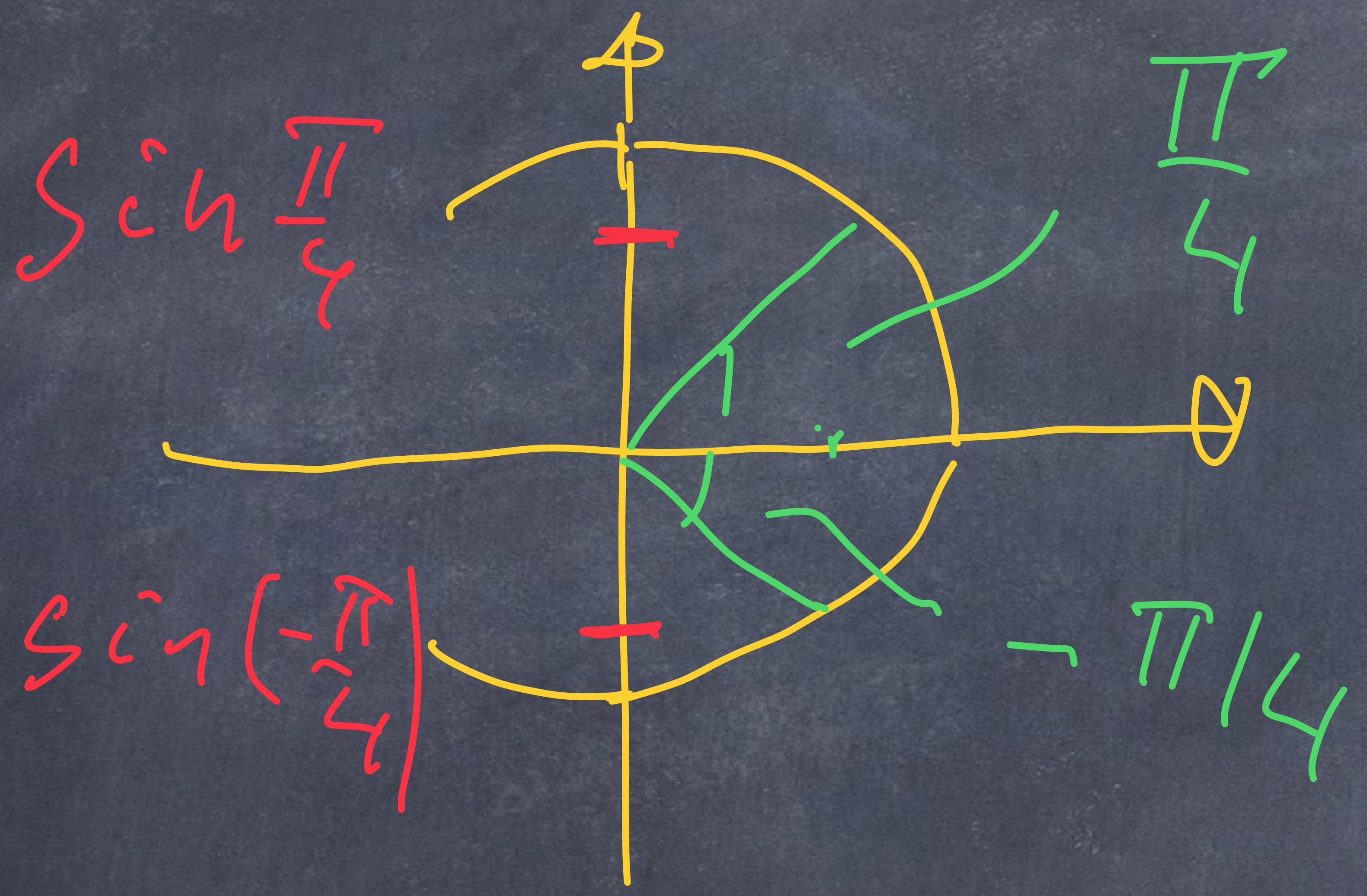
Uppgift: Bestäm

$\sin\left(-\frac{\pi}{4}\right)$

$\cos\left(\frac{7\pi}{6}\right)$

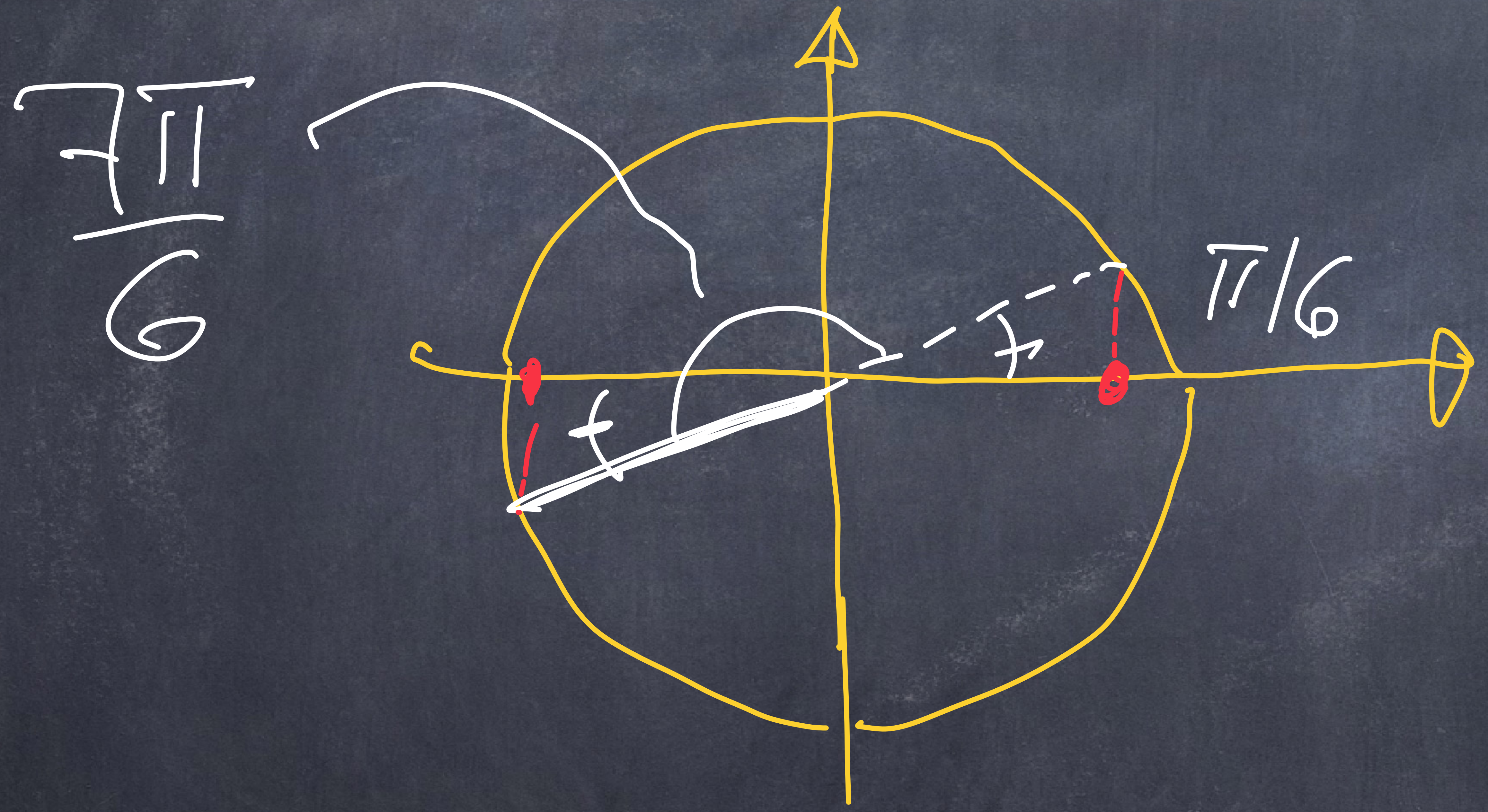
$\tan\left(\frac{11\pi}{3}\right)$

$$\sin\left(-\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$



$$\cos\left(\frac{7\pi}{6}\right) = \cos\left(\pi + \frac{\pi}{6}\right) = -\cos\frac{\pi}{6}$$

$$= -\frac{\sqrt{3}}{2}$$

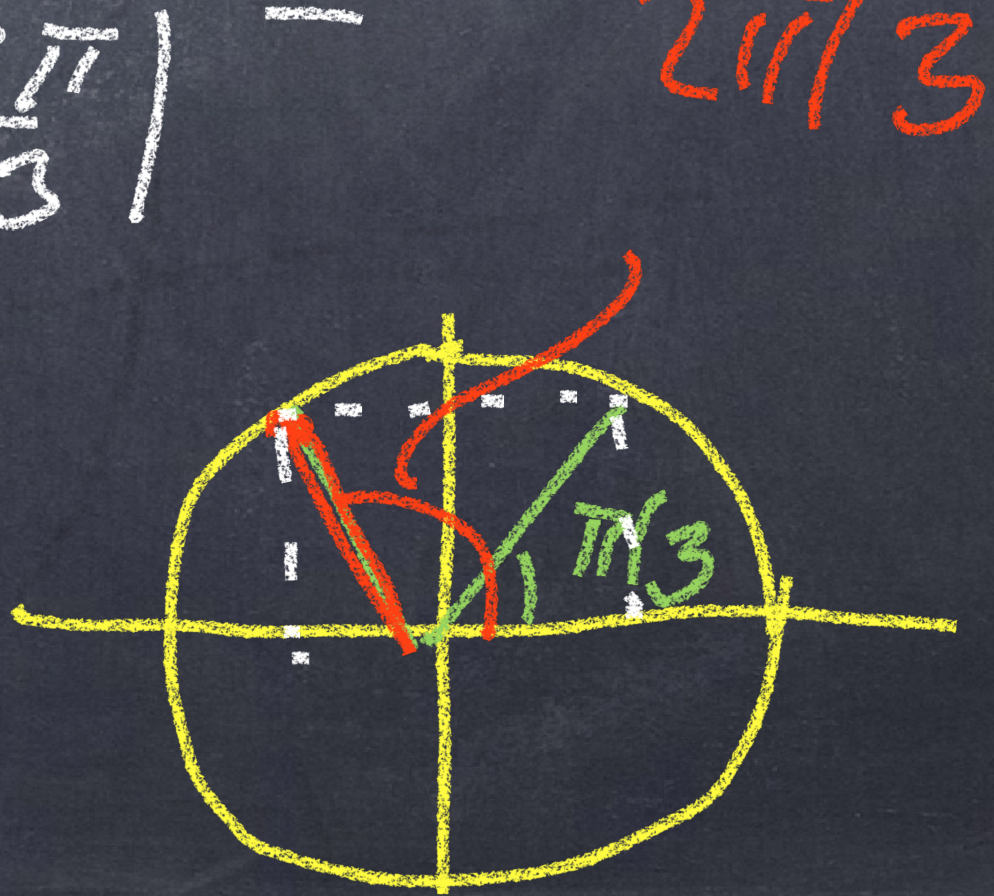


$$\tan\left(\frac{11\pi}{3}\right) = \tan\left(\frac{9\pi}{3} + \frac{2\pi}{3}\right)$$

$$= \tan\left(3\pi + \frac{2\pi}{3}\right) = \left\{ \begin{array}{l} \tan(\pi + v) = \tan v \\ \Rightarrow \tan(n \cdot \pi + v) = \tan v \\ \quad n \in \mathbb{Z} \end{array} \right\}$$

$$= \tan\frac{2\pi}{3} = \frac{\sin\frac{2\pi}{3}}{\cos\frac{2\pi}{3}} = \frac{\sin\left(\pi - \frac{2\pi}{3}\right)}{-\cos\left(\pi - \frac{2\pi}{3}\right)} =$$

$$= \frac{\sin \pi/3}{-\cos \pi/3} = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}$$



Additionsformlerna

$$\cos(u+v) = \cos u \cos v - \sin u \sin v \quad (1)$$

$$\cos(u-v) = \cos u \cos v + \sin u \sin v \quad (2)$$

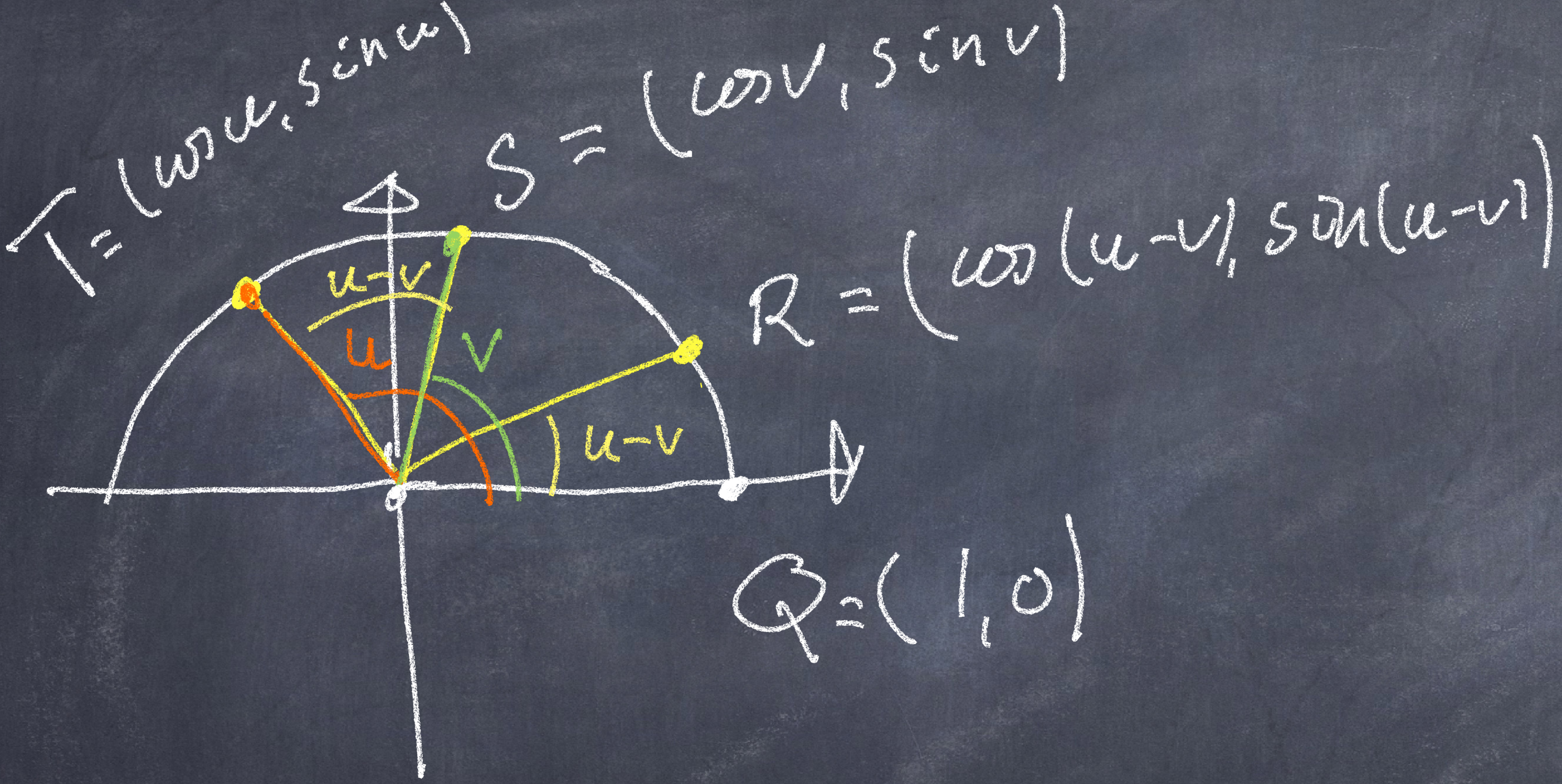
$$\sin(u+v) = \sin u \cos v + \cos u \sin v \quad (3)$$

$$\sin(u-v) = \sin u \cos v - \cos u \sin v \quad (4)$$

Först bevisas (2). Sedan $(2) \Rightarrow (1)$

$\Rightarrow (3) \Rightarrow 4$

Bevis av $\cos(u-v) = \cos u \cos v - \sin u \sin v$.



$$|TS| = |QR|$$

$$|TS|^2 = |QR|^2 \iff (\cos u - \cos v)^2 + (\sin u - \sin v)^2$$

$$= (1 - \cos(u-v))^2 + (\sin(u-v))^2$$

$$(\cos u - \cos v)^2 + (\sin u - \sin v)^2 = (1 - \cos(u-v))^2 + (\sin(u-v))^2$$

$$\Longleftrightarrow$$

$$\cos^2 u + \cos^2 v - 2\cos u \cos v + \sin^2 u + \sin^2 v - 2\sin u \sin v$$

$$= 1 + \cos^2(u-v) - 2\cos(u-v) + \sin^2(u-v)$$

$$\Longleftrightarrow$$

$$2 - 2\cos u \cos v - 2\sin u \sin v = 2 - 2\cos(u-v)$$

$$\Longleftrightarrow$$

$$\cos(u-v) = \cos u \cos v + \sin u \sin v$$

Bevis av $\cos(u+v) = \cos u \cos v - \sin u \sin v$ (1)

Sätt $v = -t$

$$\cos(u+v) = \cos(u-t) \overset{\text{enl (2)}}{=}$$

$$= \cos u \cos t + \sin u \sin t =$$

$$= \cos u \cos(-v) + \sin u \sin(-v)$$

$$= \{ \cos(-v) = \cos v, \quad \sin(-v) = -\sin v \}$$

$$= \cos u \cos v - \sin u \sin v$$

$$\boxed{\cos(u-v) = \cos u \cos v + \sin u \sin v \quad (2)}$$

Bevis av $\sin(u+v) = \sin u \cos v + \sin v \cos u$ (3)

$$\sin(u+v) = \cos\left(\frac{\pi}{2} - (u+v)\right) = \cos\left(\left(\frac{\pi}{2} - u\right) - v\right)$$

$$\overset{\text{enl}}{\underset{(2)}{=}} \cos\left(\frac{\pi}{2} - u\right)\cos v + \sin\left(\frac{\pi}{2} - u\right)\sin v$$

$$= \sin u \cos v + \cos u \sin v$$

Bevis av $\sin(u+v) = \sin u \cos v + \sin v \cos u$ (3)

$$\sin(u+v) = \cos\left(\frac{\pi}{2} - (u+v)\right) = \cos\left(\left(\frac{\pi}{2} - u\right) - v\right)$$

$$\overset{\text{enl}}{\underset{(2)}{=}} \cos\left(\frac{\pi}{2} - u\right)\cos v + \sin\left(\frac{\pi}{2} - u\right)\sin v$$

$$= \sin u \cos v + \cos u \sin v$$

Uppgift: Bevisa att

$$\sin(u-v) = \sin u \cos v - \sin v \cos u$$

(4)

$$V.L. = \sin(u-v) = \sin(u+(-v))$$

Enl. (3)

$$= \sin u \cos(-v) + \sin(-v) \cos u$$

$$= \left\{ \begin{array}{l} \cos(-v) = \cos v \\ \sin(-v) = -\sin v \end{array} \right\} =$$

$$= \sin u \cos v - \sin v \cos u = H.L.$$

Från additionsformlerna fås

$$\cos 2x = \cos^2 x - \sin^2 x \quad (5)$$

$$\sin 2x = 2 \sin x \cos x \quad (6)$$

Visa att

$$\cos^2 x = \frac{1 + \cos 2x}{2} \quad (7)$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad (8)$$

Beräkna

- $\cos \frac{\pi}{8}$
- $\cos \frac{5\pi}{8}$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$ skall visas

$$H.L. = \frac{1 + \cos^2 x - \sin^2 x}{2}$$

$\pi/8$

0

$$\cos^2 x = \frac{1+\cos 2x}{2}$$ skall visas

$$H.L. = \frac{1+\cos^2 x - \sin^2 x}{2}$$

$$= \frac{\cos^2 x + \cancel{\sin^2 x} + \cos^2 x - \cancel{\sin^2 x}}{2} = \cos^2 x$$

$$= V.L.$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$ skall visas

$$\text{H.L.} = \frac{1+\cos^2 x - \sin^2 x}{2}$$

$$= \frac{\cos^2 x + \cancel{\sin^2 x} + \cos^2 x - \cancel{\sin^2 x}}{2} = \cos^2 x$$

$$= \text{V.L.}$$

$$\cos^2\frac{\pi}{8} = \frac{1+\cos 2\cdot\frac{\pi}{8}}{2} = \frac{1+\cos\frac{\pi}{4}}{2} = \frac{1+1/\sqrt{2}}{2}$$

$$= \frac{\sqrt{2}+1}{2\sqrt{2}} \Rightarrow \cos\frac{\pi}{8} = +\cancel{-}\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

$$= \sqrt{2+\sqrt{2}}\,/\,2$$

$\cos \pi/8 > 0$

$$\cos\frac{5\pi}{8} = \cos\left(\frac{\pi}{2}+\frac{\pi}{8}\right) = -\sin\frac{\pi}{8}.$$

$$\sin^2\frac{\pi}{8} = \frac{1-\cos 2\cdot\frac{\pi}{8}}{2} = \frac{1-\cos\frac{\pi}{4}}{2}$$

$$= \frac{1-\frac{1}{\sqrt{2}}}{2} = \frac{\sqrt{2}-1}{2\sqrt{2}} = \frac{2-\sqrt{2}}{4}$$

Alltså är $\sin\frac{\pi}{8} = (+)\sqrt{\frac{2-\sqrt{2}}{4}}$ $\left(\sin\frac{\pi}{8}>0\right)$

och $\cos\frac{5\pi}{8} = -\sin\frac{\pi}{8} = -\sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{2}$

Trigonometriska ekvationer

$\sin x = \sin v$

$\Updownarrow$

$$x = \begin{cases} v + n \cdot 2\pi \\ (\pi - v) + n \cdot 2\pi \end{cases}$$

$\cos x = \cos v$

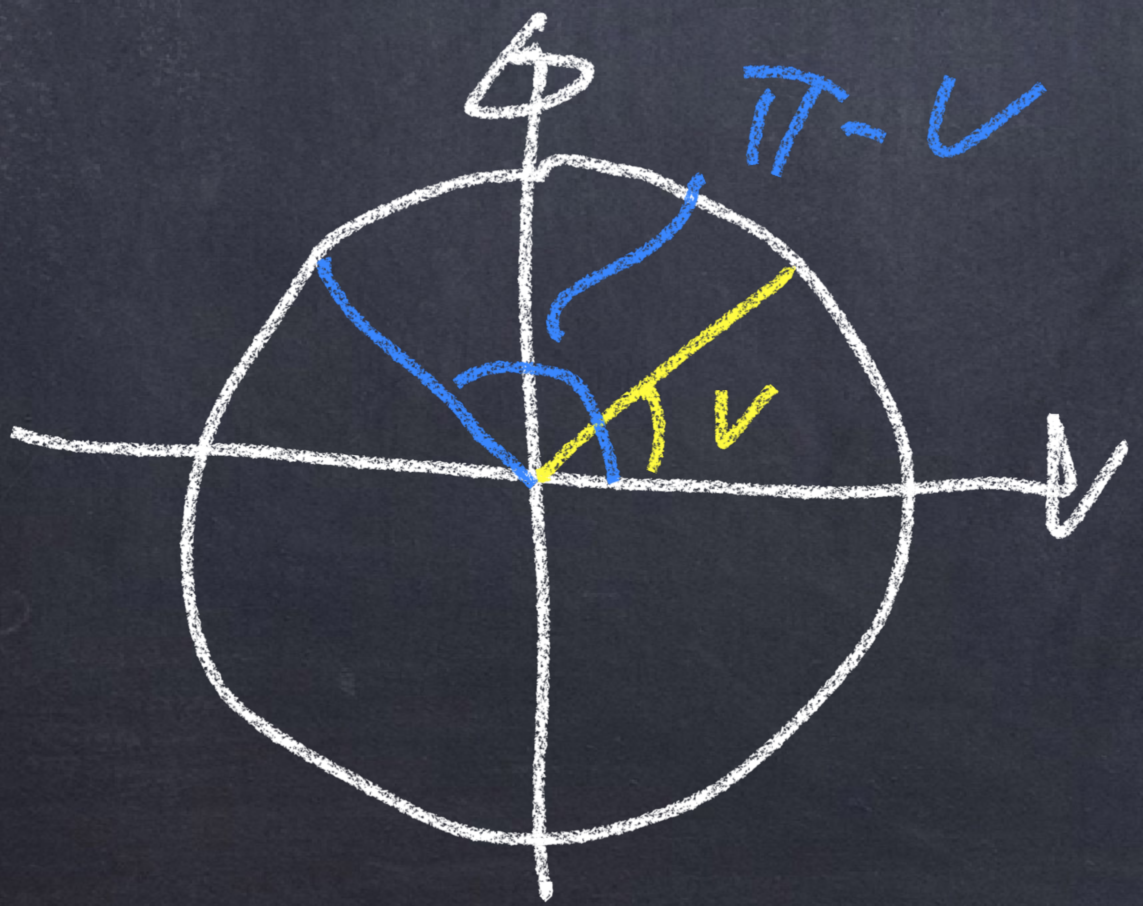
$\Updownarrow$

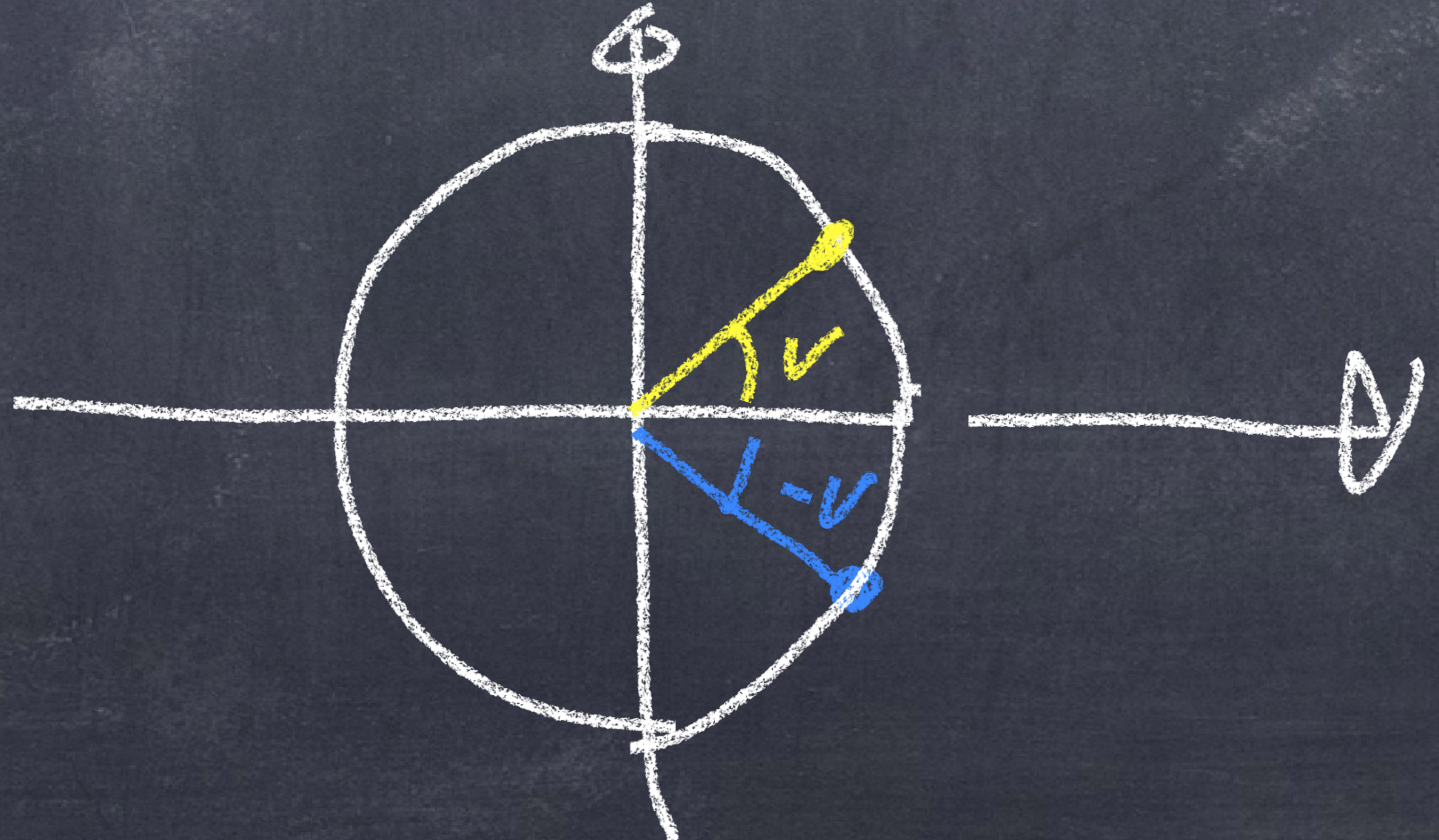
$$x = \begin{cases} v + n \cdot 2\pi \\ -v + n \cdot 2\pi \end{cases}$$

$\tan x = \tan v$

$\Updownarrow$

$$x = v + n \cdot \pi$$





Lös ekvationerna

- $\sin(2x) = \frac{1}{2}$

- $\cos(3x) = 2$

- $\cos(4 \cdot x) = -\frac{\sqrt{3}}{2}$

$$\sin(2x) = \frac{1}{2}$$

$$\Leftrightarrow \quad 2x = \begin{cases} \pi/6 + n \cdot 2\pi \\ \underbrace{(\pi - \pi/6)}_{5\pi/6} + n \cdot 2\pi \end{cases}$$

$$\Leftrightarrow$$

SVAR: $x = \begin{cases} \pi/12 + n \cdot \pi \\ 5\pi/12 + n \cdot \pi \end{cases}, \quad n \in \mathbb{Z}$

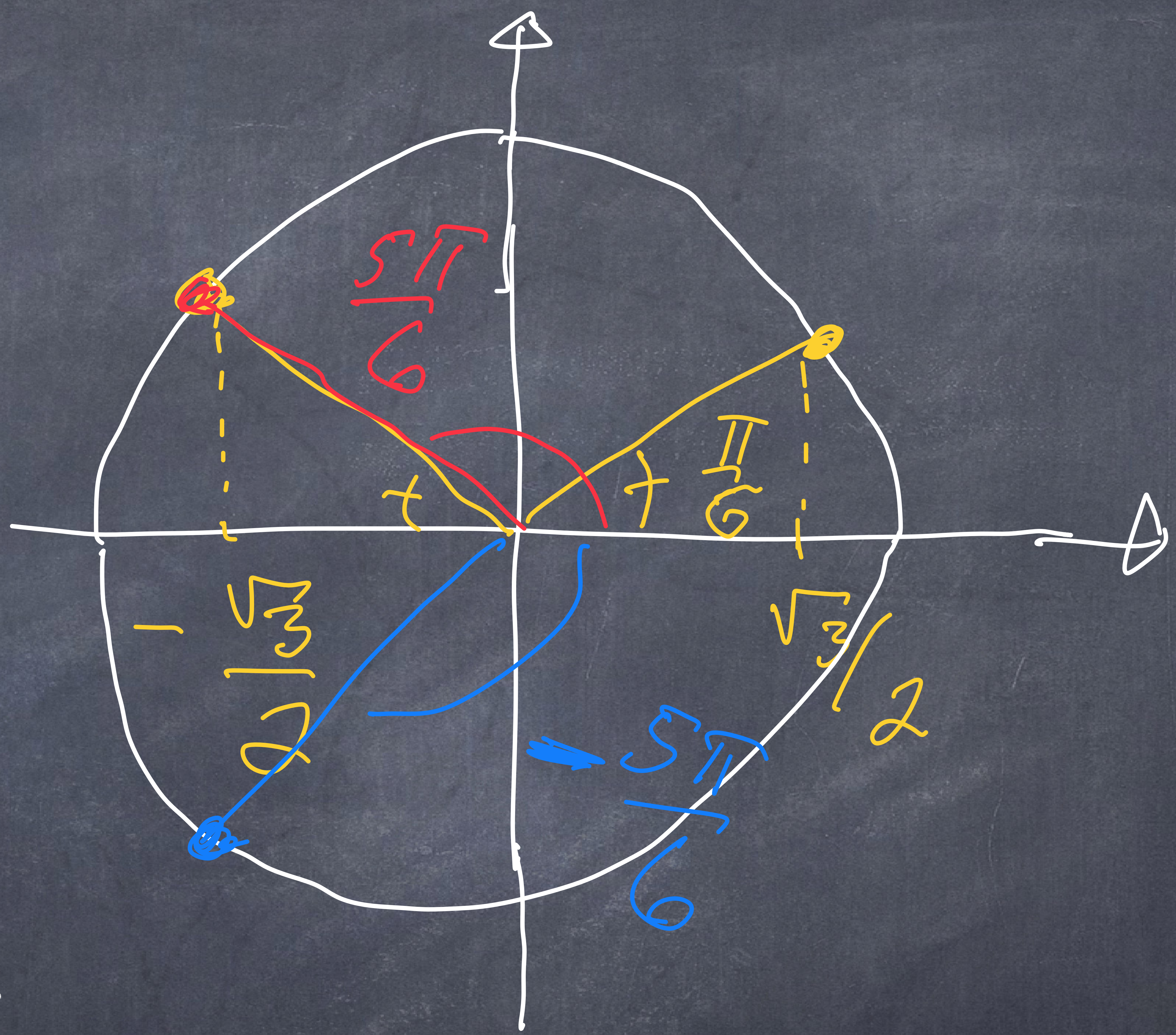
$\cos(3x) = 2$

saknar lösning ty $|\cos v| \leq 1, \forall v.$

$$\cos(4 \cdot x) = -\frac{\sqrt{3}}{2}$$

$$\Leftrightarrow \quad 4x = \begin{cases} \frac{5\pi}{6} + n \cdot 2\pi \\ -\frac{5\pi}{6} + n \cdot 2\pi \end{cases}$$

$$\Leftrightarrow \quad x = \begin{cases} 5\pi/24 + n \cdot \pi/2 \\ -5\pi/24 + n \cdot \pi/2 \end{cases}$$



Lös ekvationen

$\sin 4x = \cos 2x$

Lös ekvationen

$$\sin 4x = \cos 2x \quad \Leftrightarrow$$

$$2\sin 2x \cos 2x = \cos 2x \quad \Leftrightarrow$$

$$\cos 2x = 0 \quad \vee \quad \sin 2x = \frac{1}{2}$$

Lös ekvationen

$\sin 4x = \cos 2x \iff$

$2\sin 2x \cos 2x = \cos 2x \iff$

$\cos 2x = 0 \quad \vee \quad \sin 2x = \frac{1}{2}$

$\cos 2x = 0 \iff$

$2x = \frac{\pi}{2} + n \cdot \pi \iff$

$x = \frac{\pi}{4} + n \cdot \frac{\pi}{2} \quad , \quad n \in \mathbb{Z}$

$\sin 2x = \frac{1}{2}$

$\iff$

$$2x = \begin{cases} \pi/6 + n \cdot 2\pi \\ 5\pi/6 + n \cdot 2\pi \end{cases}$$

$$x = \begin{cases} \pi/12 + n \cdot \pi \\ 5\pi/12 + n \cdot \pi \end{cases}$$

SVAR:

$x = \pi/4 + n \cdot \pi/2$ eller

$x = \pi/12 + n \cdot \pi$ eller

$x = 5\pi/12 + n\pi \quad , \quad n \in \mathbb{Z}$

Lös $\cos^2 v - 3\cos v + 2 = 0$

$t = \cos v$, $-1 \le t \le 1$.

$t^2 - 3t + 2 = 0$

$t = \frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{8}{4}}$

$= \frac{3}{2} \pm \frac{1}{2} = 2 \vee 1$

men $|t| \le 1$ så

$t = 1$ sökt lösning

$\Updownarrow$

$\cos v = 1$

$v = 0 + n \cdot 2\pi$

SVAR:

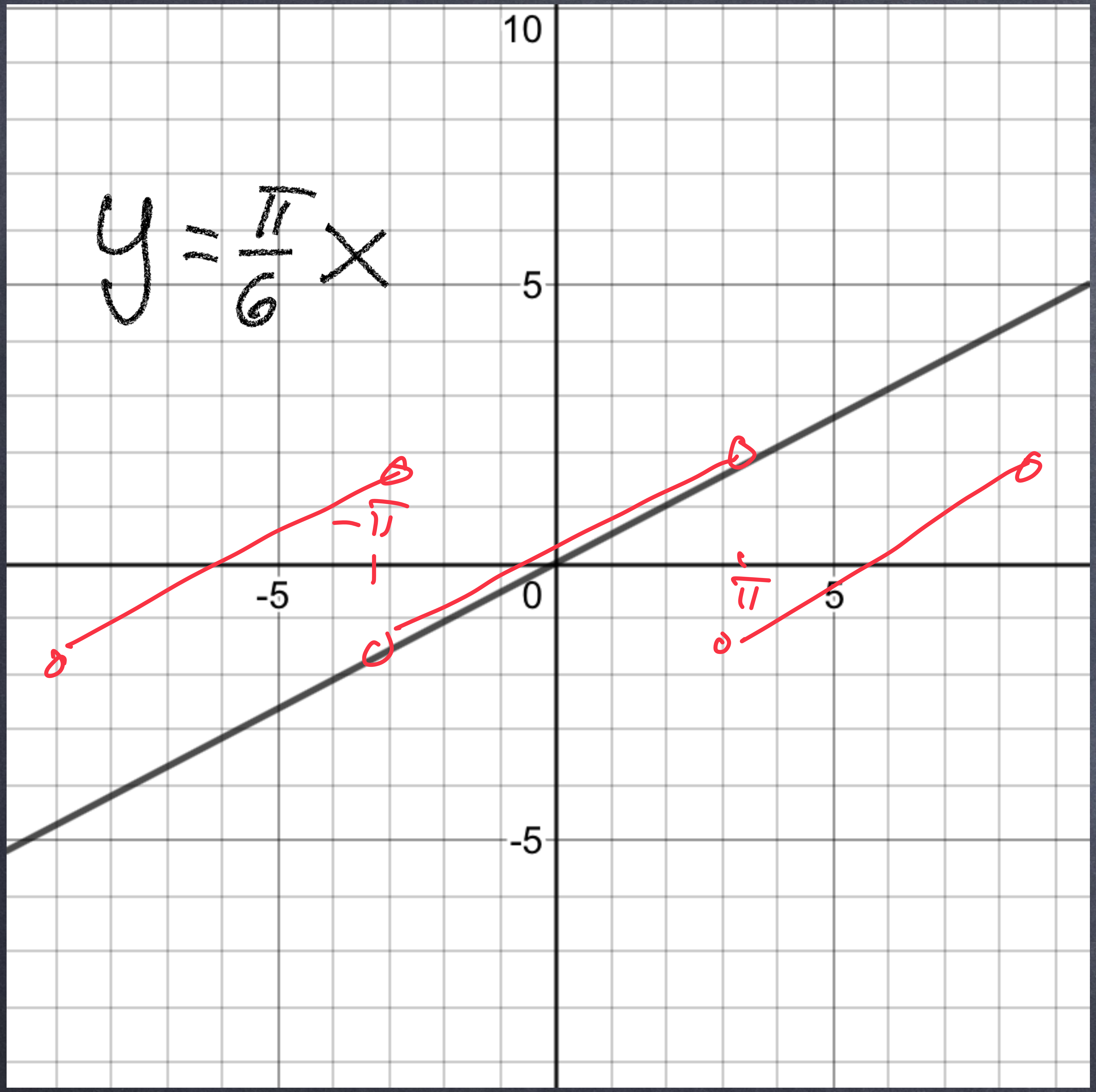
$v = n \cdot 2\pi$, $n \in \mathbb{Z}$

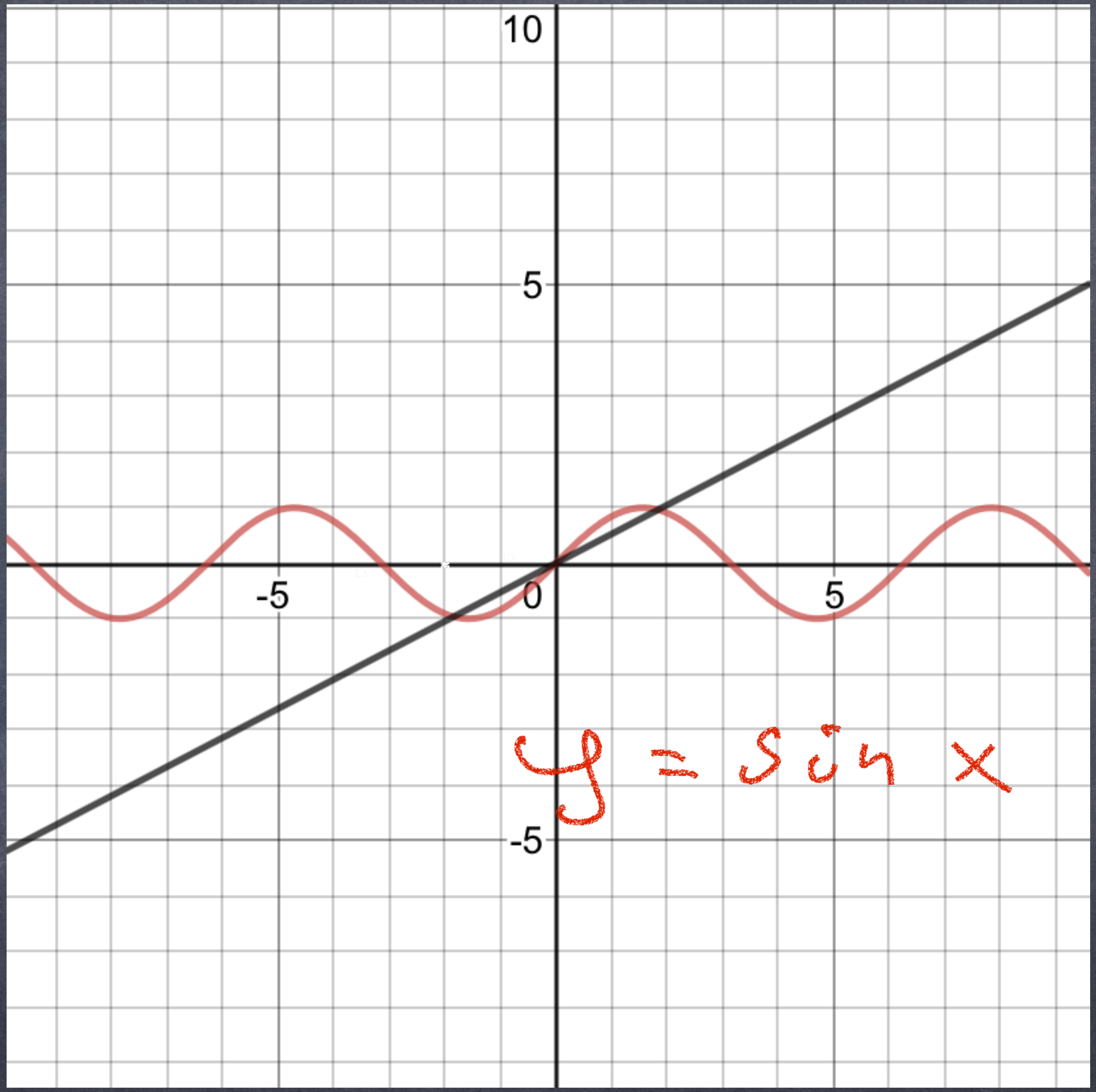
Extra (examineras ej)

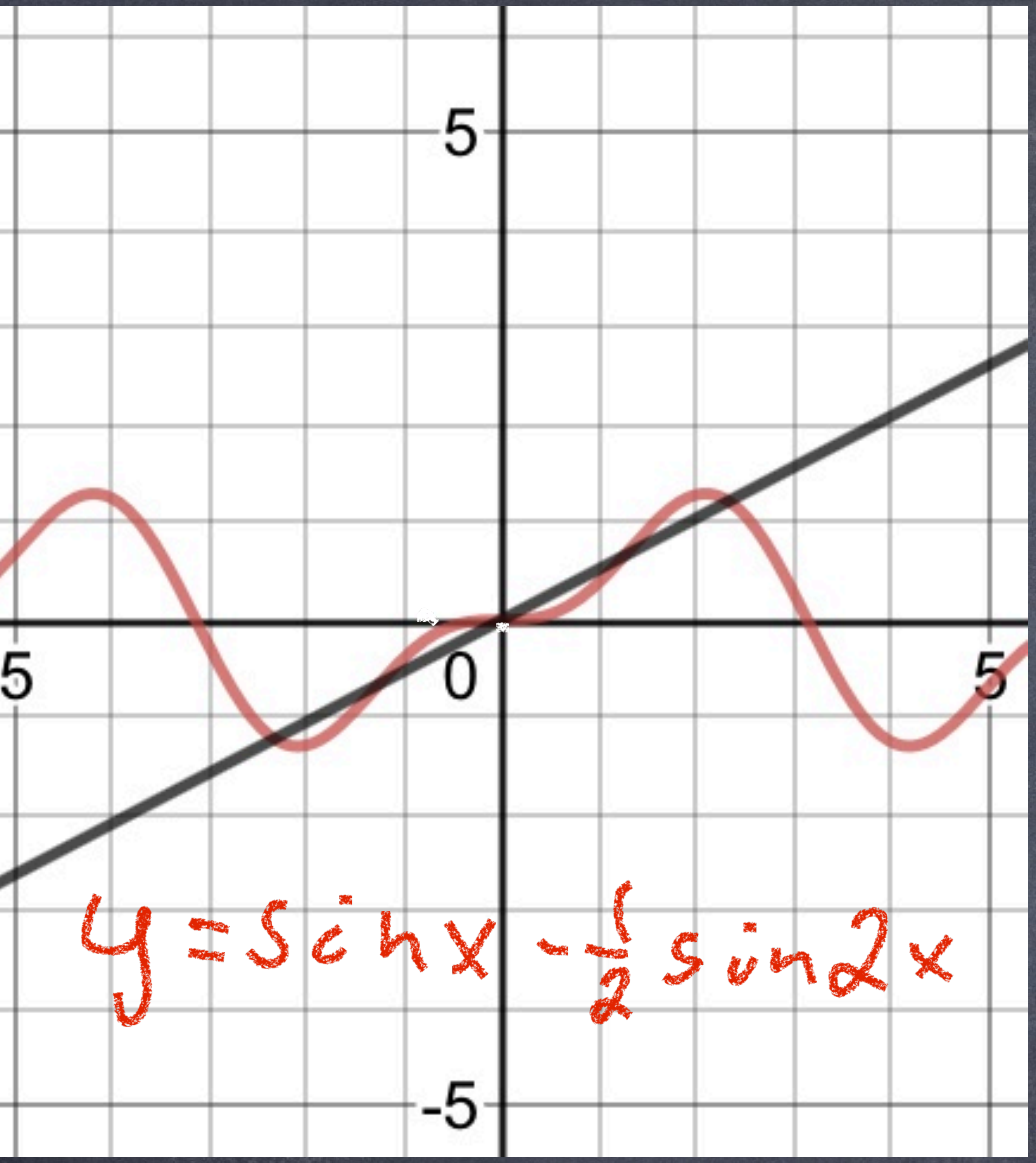
- en glimt av Fourierserier

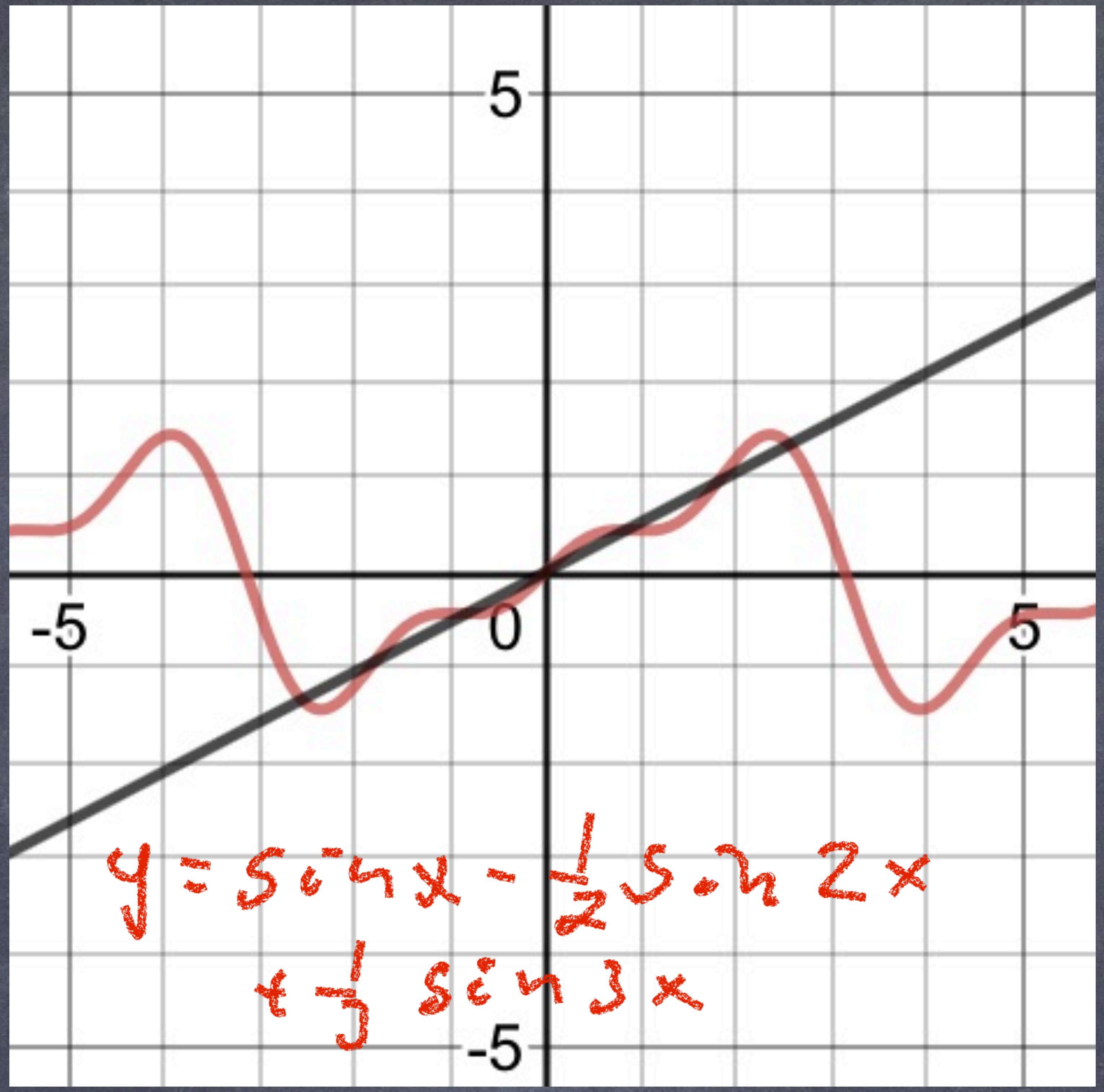
Om f är en periodisk funktion, (period $= 2\pi$) så kan f skrivas

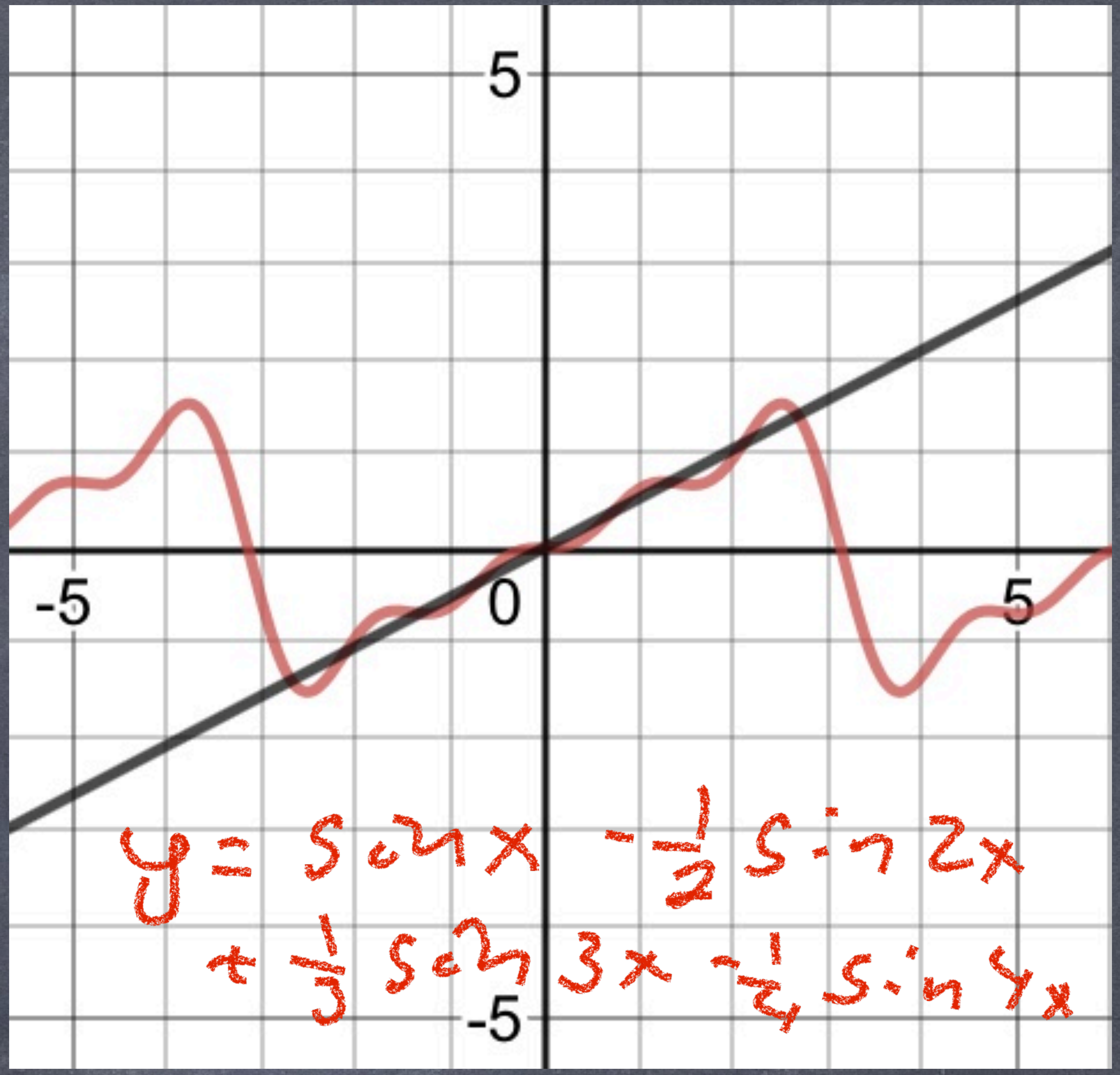
$$f(x) = A_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

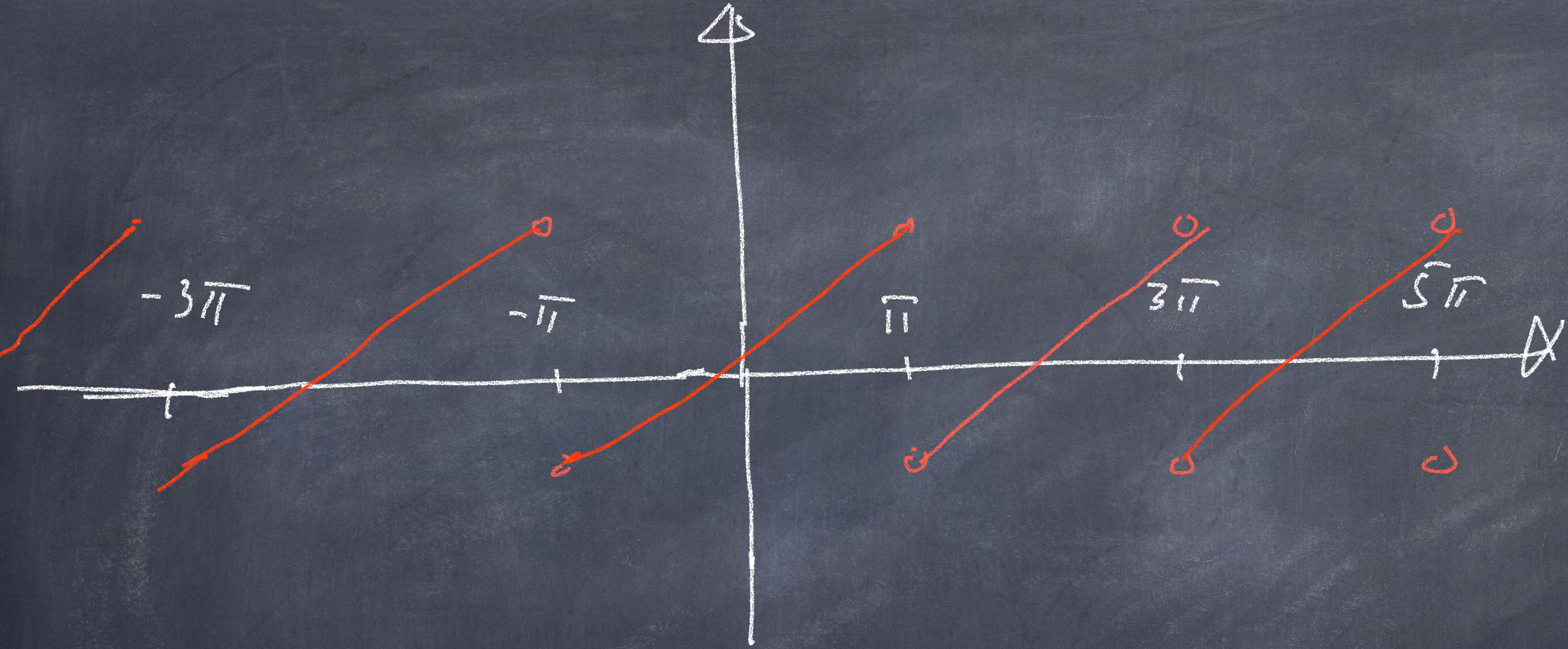

10
5
-5
0
-5
$y = \frac{\pi}{6}x$
$-\pi$
π


10
5
-5
0
5
-5
$y = \sin x$


5
5
0
5
-5
y = sin x − ½ sin 2x


5
-5
0
5
-5
y = sin x − ½ sin 2x + ⅓ sin 3x


5
-5
0
5
-5
y = sin x − ½ sin 2x + ⅓ sin 3x − ¼ sin 4x



$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} \sin kx = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \dots$$