Visualization, DD2257

Prof. Dr. Tino Weinkauf

Topology-based Visualization

- motivational slides:
  - geometric shapes
  - landscapes with one and two mountains
  - vector fields with one and two sources; could be gradient of landscape

Limit Sets

stream lines

forward/backward integration

infinite amount of time

α set

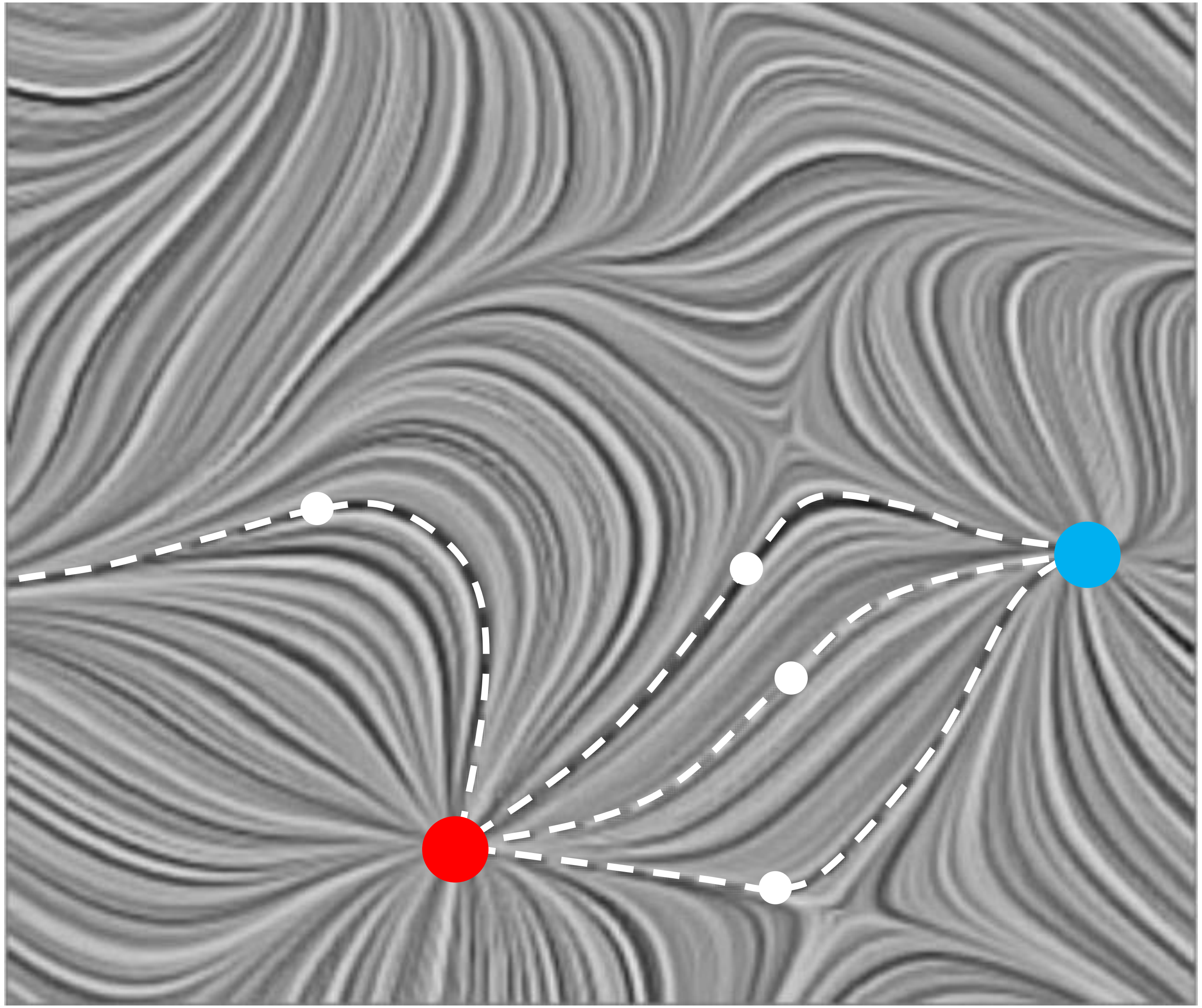
areas where stream lines start

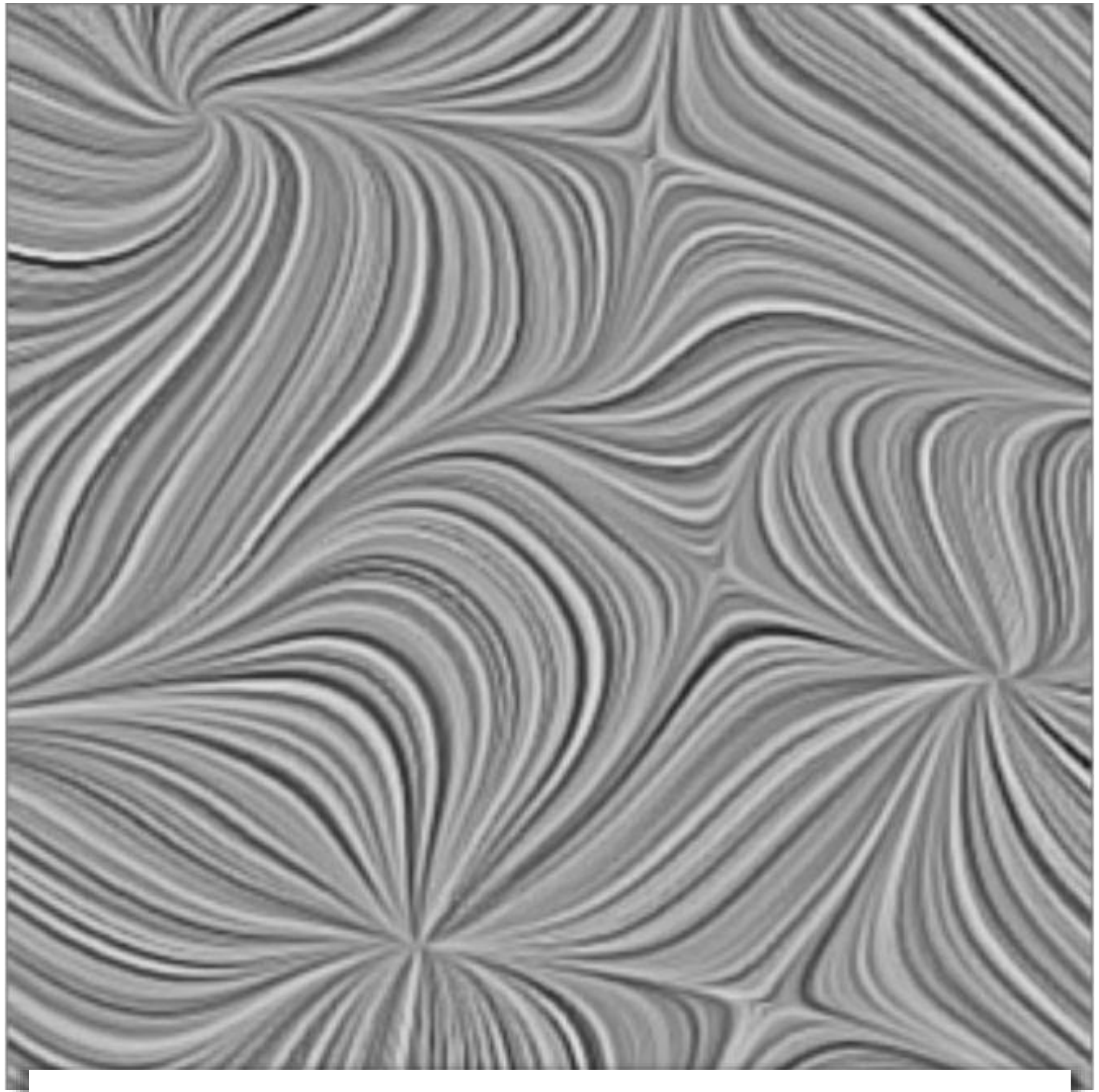
ω set

areas where stream lines end

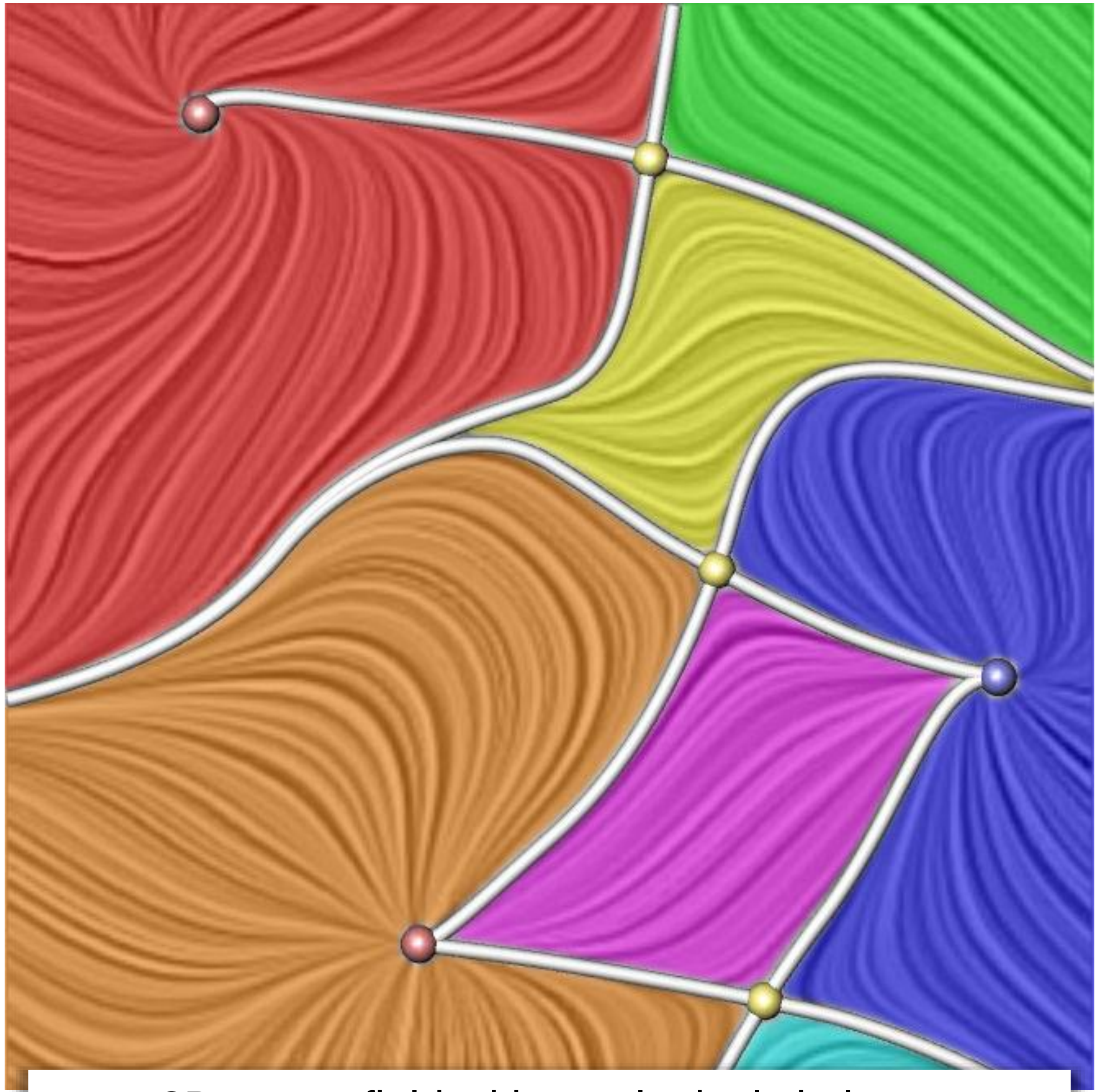
(α, ω) pairs

clusters of stream lines having the same start and end





2D vector field

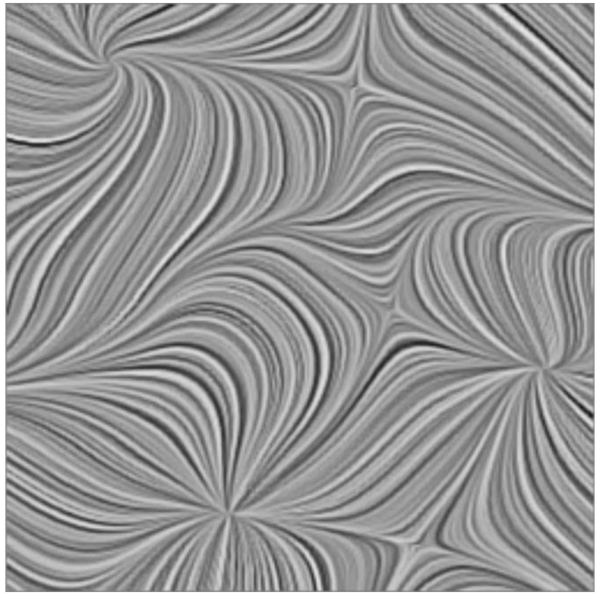


2D vector field with topological skeleton highlighting the sectors of different flow behavior

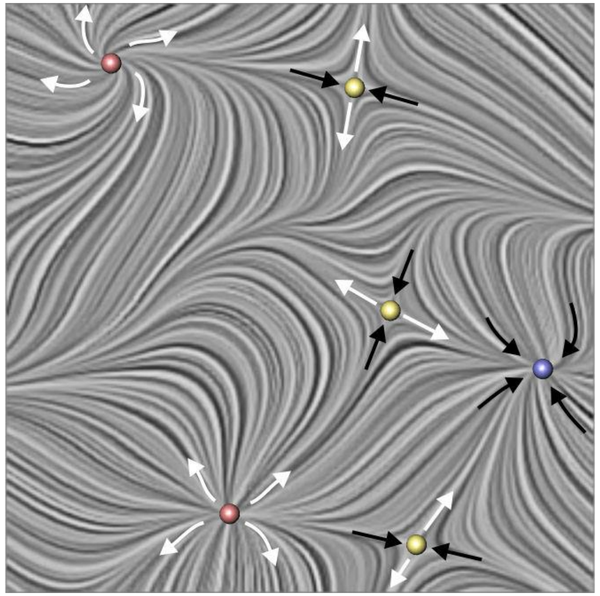
Topological Structures

steady 2D vector field

$$\mathbf{v} = \begin{pmatrix} u \\ v \end{pmatrix}$$

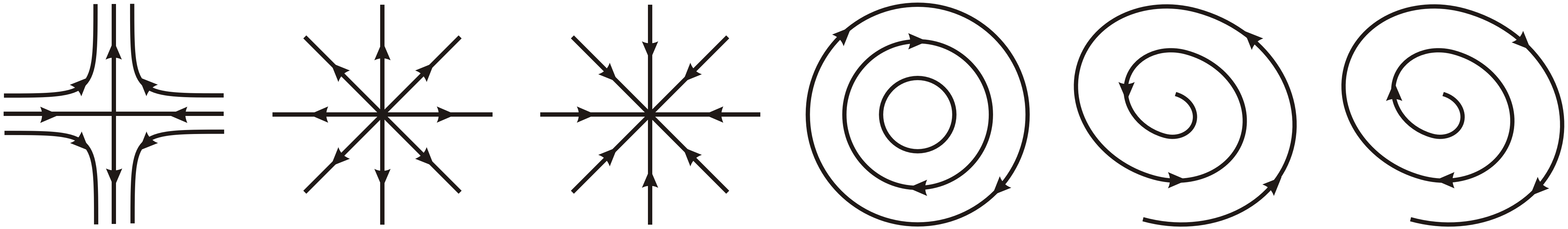


tangent curves



critical points

$$\mathbf{v}(\mathbf{x}_0) = \mathbf{0} \quad \text{with} \quad \mathbf{v}(\mathbf{x}_0 \pm \boldsymbol{\epsilon}) \neq \mathbf{0}$$



Saddle point	Repelling node	Attracting node	Center	Attracting focus	Repelling focus
inflow outflow	outflow	inflow	closed stream lines	inflow	outflow
	source	sink		sink	source

very important

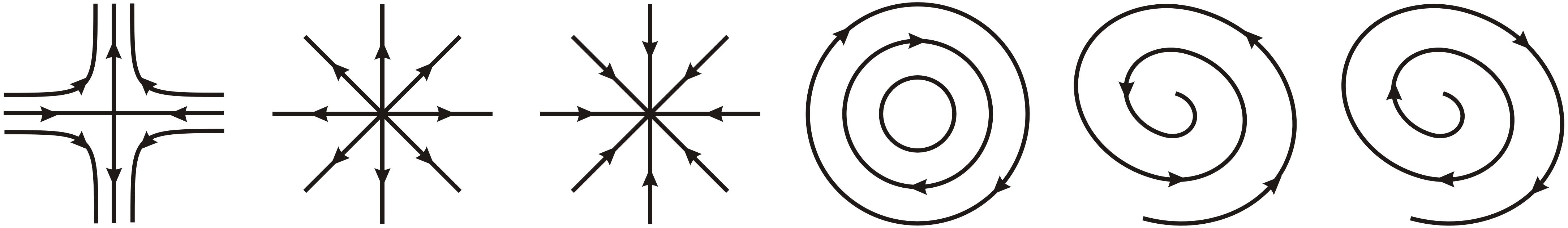
First-order critical points (2D)

The point $\mathbf{x}$ is a *first-order* critical point of the vector field $\mathbf{v}$ if

- $\mathbf{x}$ is an isolated critical point, and
- $\det(\nabla \mathbf{v}(\mathbf{x})) \neq 0$
  (determinant of Jacobian does not vanish)

The different types of critical points can be classified by the eigenvalues of the Jacobian:

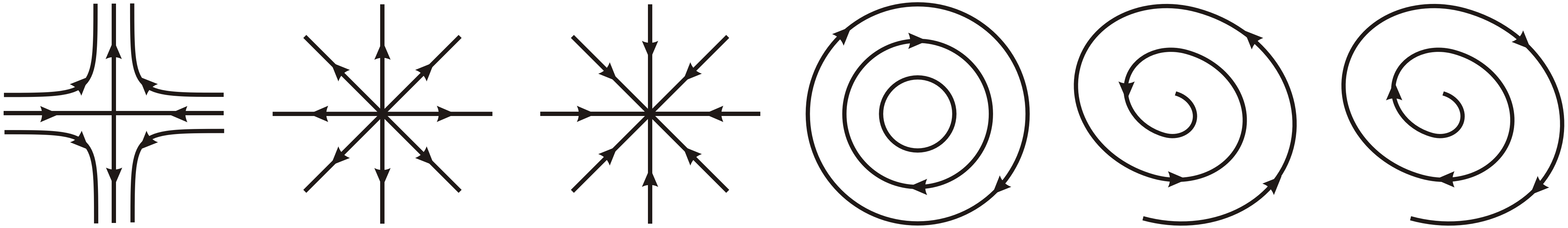
- R_1, R_2 ➔ real part of the eigenvalue
  I_1, I_2 ➔ imaginary part of the eigenvalue



Saddle point | Repelling node | Attracting node | Center | Attracting focus | Repelling focus

source | sink | sink | source

Eigenvalues and Determinant in 2D



Saddle point
$R_1 < 0, R_2 > 0$
$I_1 = I_2 = 0$

Repelling node
$R_1, R_2 > 0$
$I_1 = I_2 = 0$

Attracting node
$R_1, R_2 < 0$
$I_1 = I_2 = 0$

Center
$R_1 = R_2 = 0$
$I_1 = -I_2 \neq 0$

Attracting focus
$R_1 = R_2 < 0$
$I_1 = -I_2 \neq 0$

Repelling focus
$R_1 = R_2 > 0$
$I_1 = -I_2 \neq 0$

$$\mathbf{v}(\mathbf{x}_0) = \mathbf{0} \quad \text{with} \quad \mathbf{v}(\mathbf{x}_0 \pm \boldsymbol{\epsilon}) \neq \mathbf{0}$$

a critical point is an isolated zero

$$\mathbf{x_0} = (x, y)$$

position of the critical point

$$\nabla \mathbf{v}(x, y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

first derivative of the vector field at the critical point

$$e_{1,2} = \frac{(u_x + v_y)}{2} \pm \sqrt{\frac{(u_x + v_y)^2}{4} - (u_x v_y - u_y v_x)}$$

$$e_{1,2} = \frac{T}{2} \pm \sqrt{\frac{T^2}{4} - D}$$

$T = u_x + v_y$ trace of Jacobian

$D = u_x v_y - u_y v_x$ determinant of Jacobian

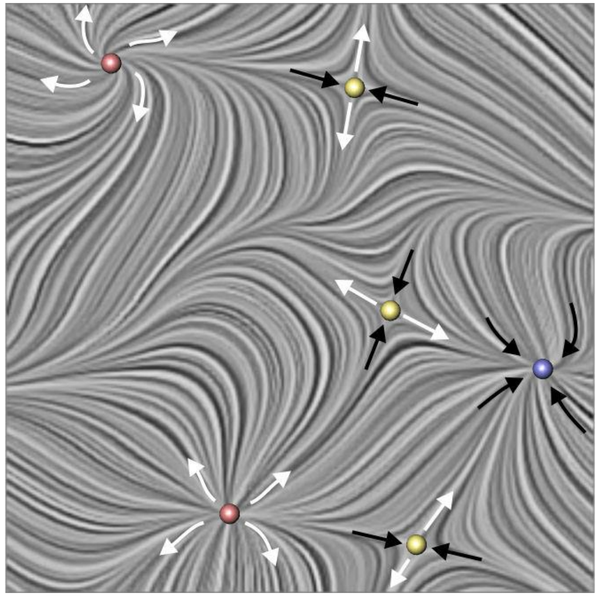
needs to be non-zero!

- Video mit VF wo sich ein CP schoen aendert.
    - position
    - form

Finding Zeros

- **Isolated** zeros

- Direct solution of linear equations
- Newton, ...
- Hodge decomposition
- Domain decomposition & change-of-sign test



critical points

$$\mathbf{v}(\mathbf{x}_0) = \mathbf{0} \quad \text{with} \quad \mathbf{v}(\mathbf{x}_0 \pm \boldsymbol{\epsilon}) \neq \mathbf{0}$$

$$\mathbf{v}(x, y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Domain decomposition & change-of-sign test

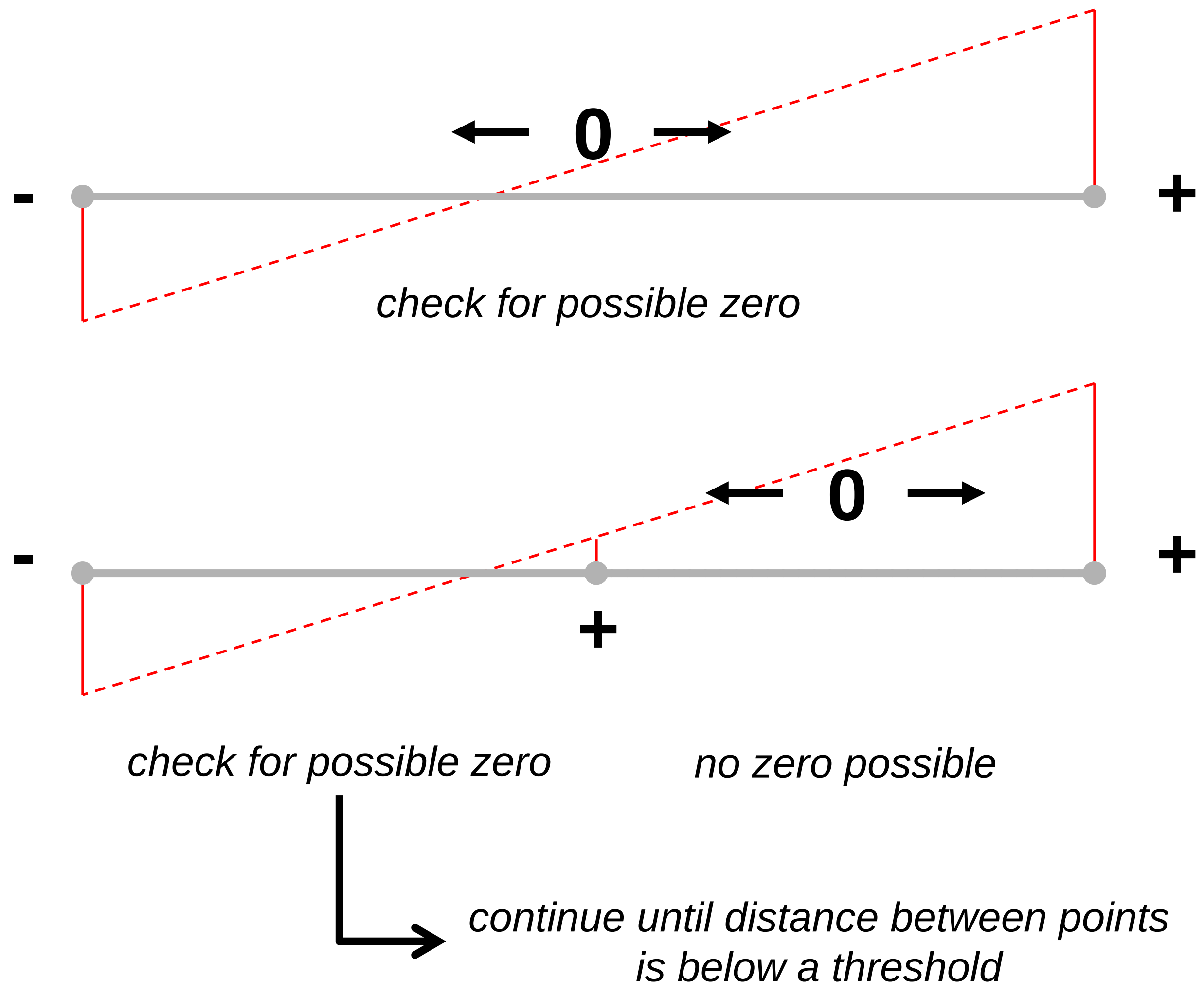
example for 1D scalar field

change-of-sign test

a zero must exist if the signs on either side are different

domain decomposition

divide domain into halves



Domain decomposition & change-of-sign test

generic version

many dimensions & components

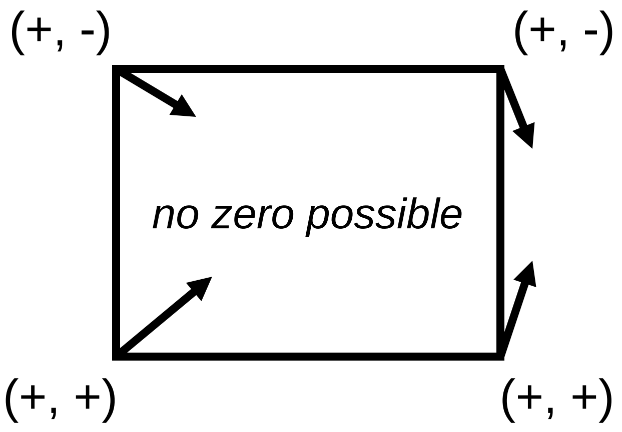
change-of-sign test

*a zero **cannot** exist in a data cell if, for any component, the signs of the sample points are the same*

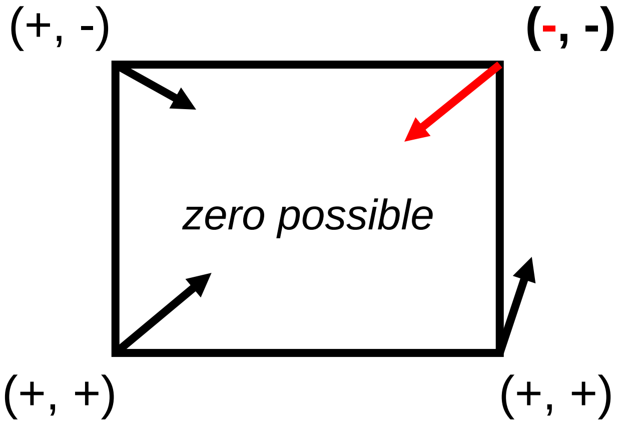
*a zero **may** exist in a data cell if, for all components, the signs of the sample points are different*

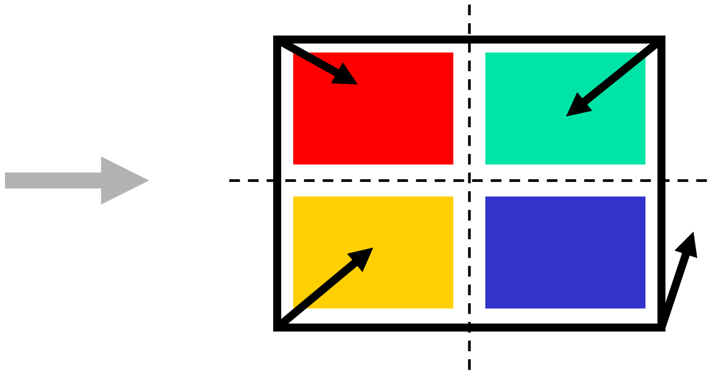
domain decomposition

divide domain into halves along each dimension



- Criteria itself is necessary, but not sufficient
- Very fast & parallelizable
- Easy to implement (C++ Template)
- n dimensions, m components

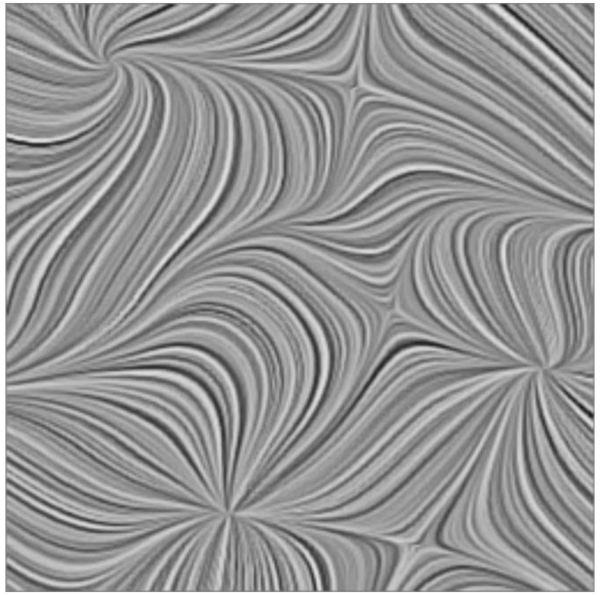




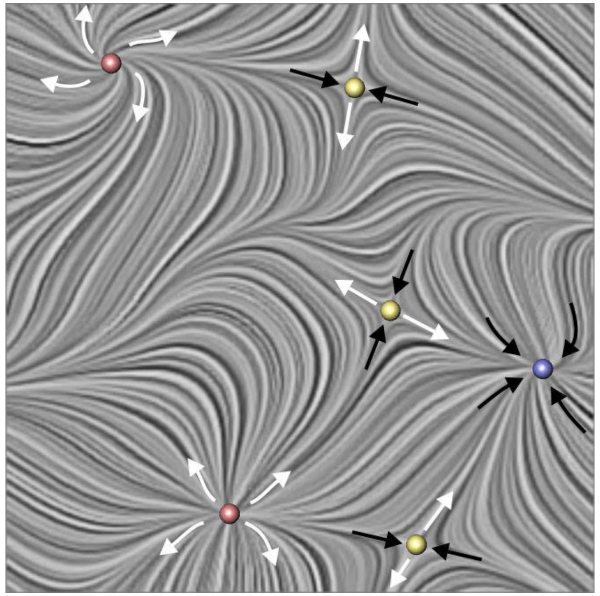
Topological Structures

steady 2D vector field

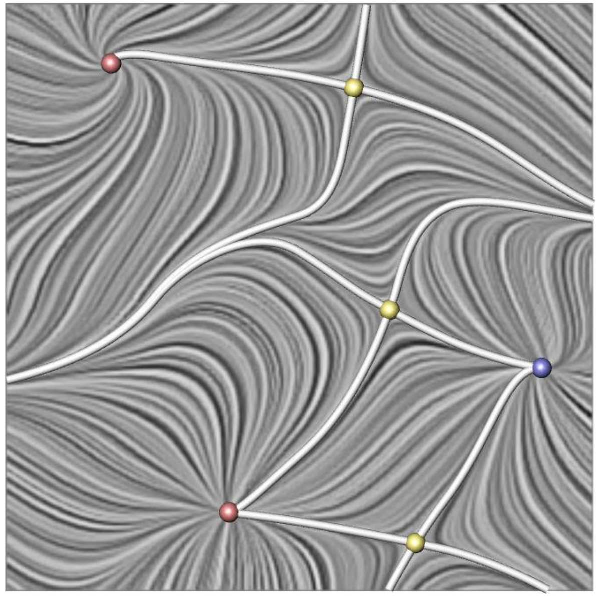
$$\mathbf{v} = \begin{pmatrix} u \\ v \end{pmatrix}$$



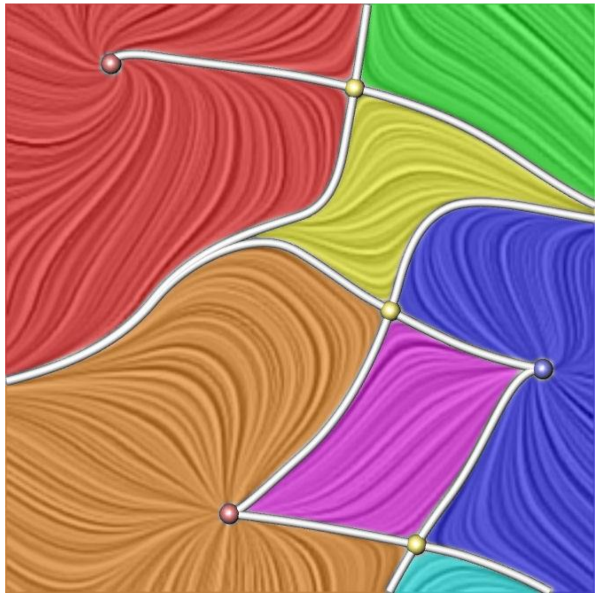
tangent curves



critical points



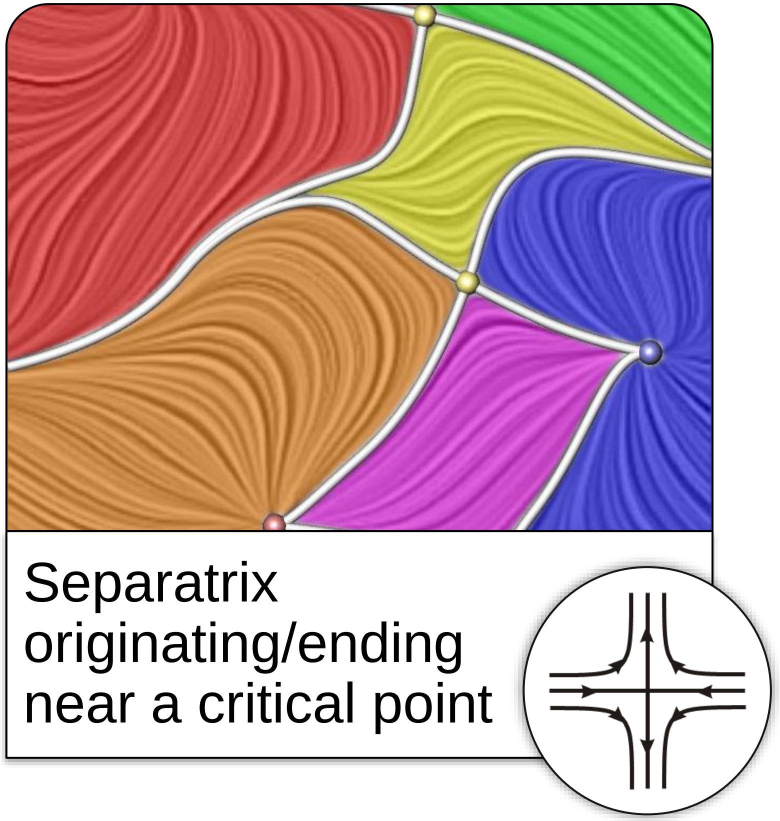
separation lines



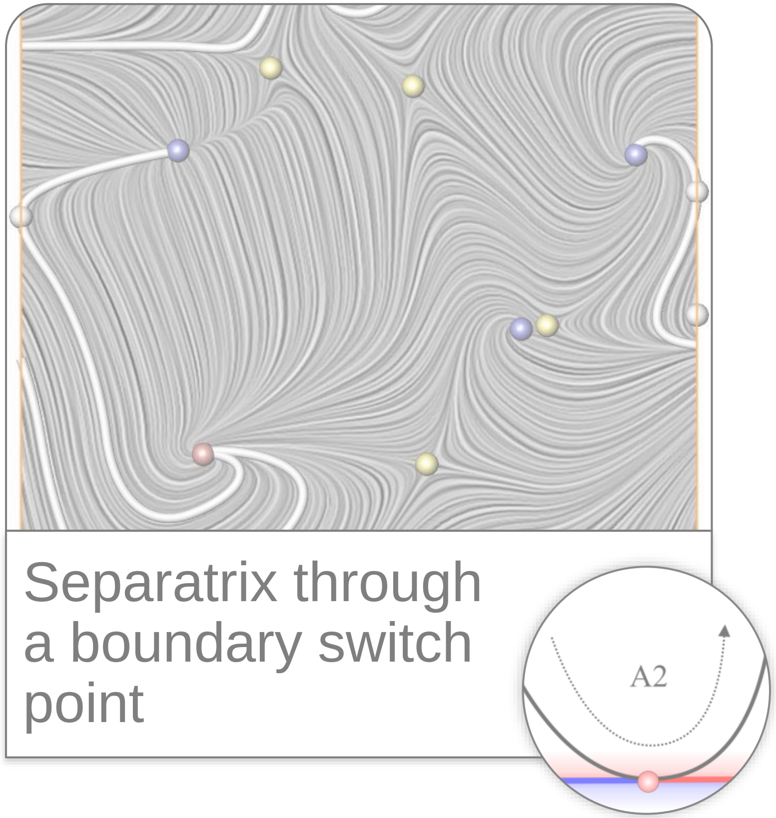
sectors of different flow behavior

Separatrices (2D)

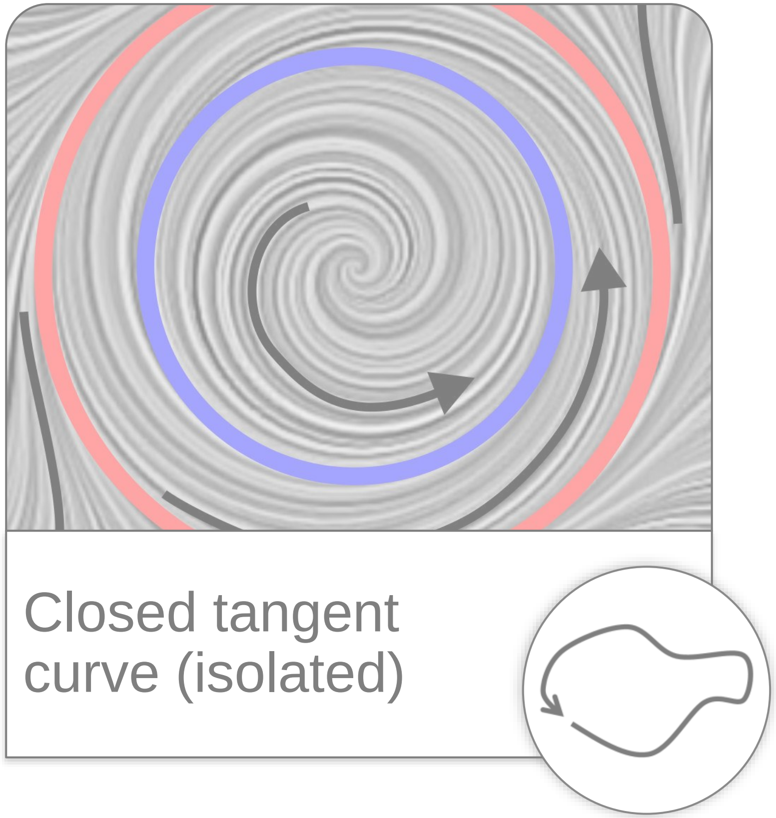
Definition *Separatrix*:
special tangent curve separating regions of different flow behavior



Separatrix originating/ending near a critical point



Separatrix through a boundary switch point



Closed tangent curve (isolated)

Separatrices from Saddle

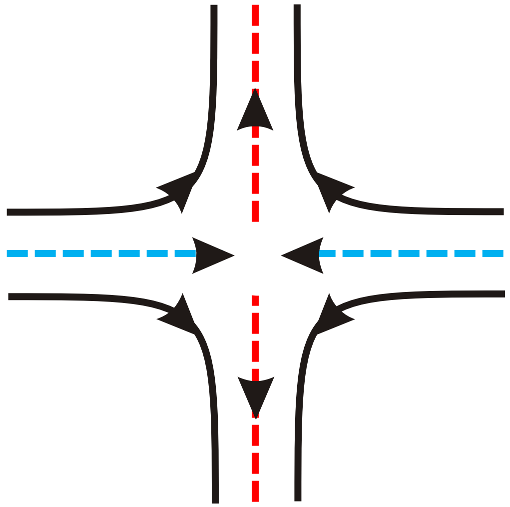
2 Eigenvectors from Jacobian

direction, no orientation

4 oriented vectors around saddle

negative Eigenvalue: points to saddle

positive Eigenvalue: points away from saddle



Saddle point

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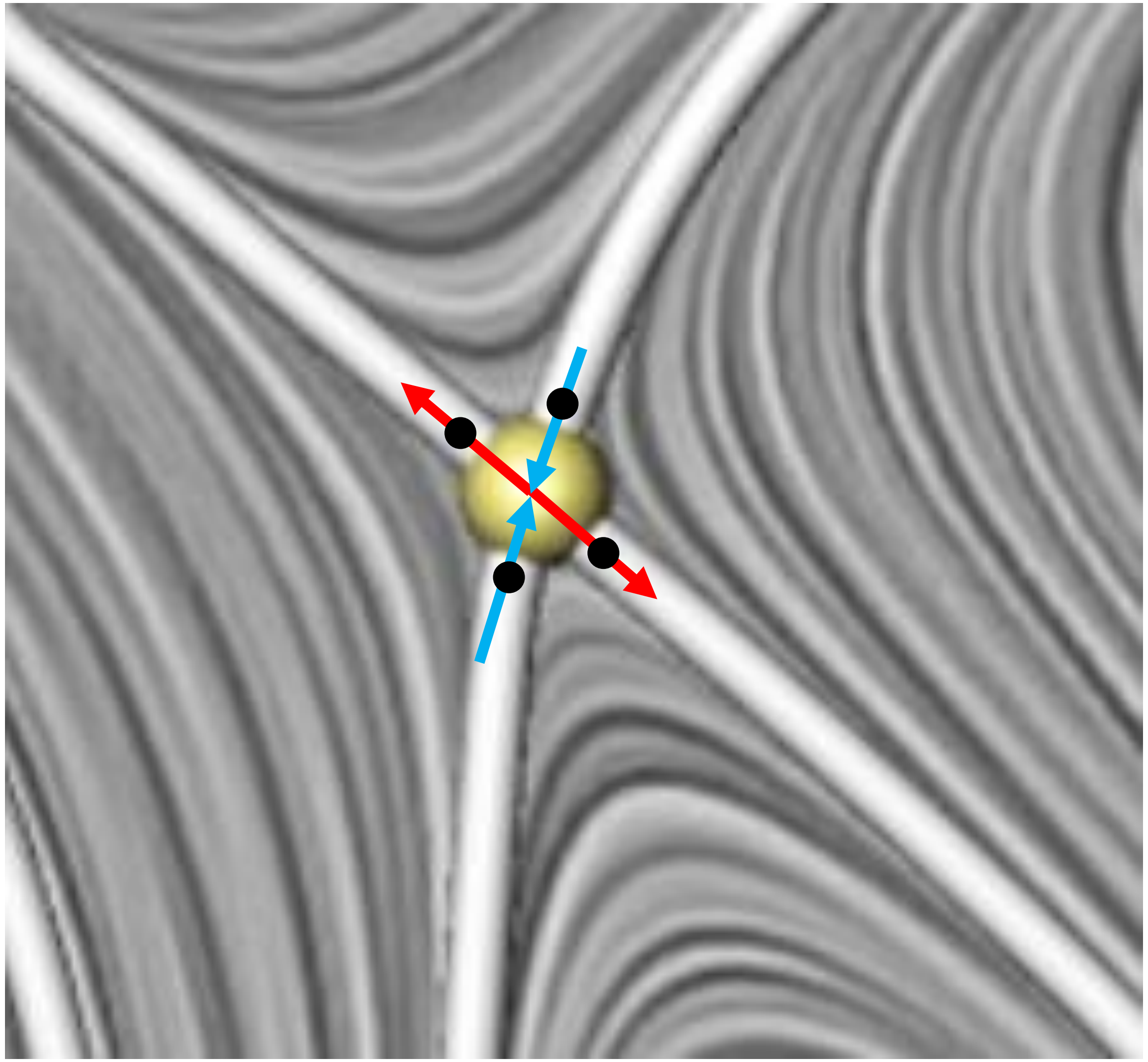

Eigenvectors in 2D

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position of the critical point

$$\nabla\mathbf{v}(x, y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

first derivative of the vector field
at the critical point

$$e_{1,2} = \frac{(u_x + v_y)}{2} \pm \sqrt{\frac{(u_x + v_y)^2}{4} - (u_x v_y - u_y v_x)}$$

$$e_{1,2} = \frac{T}{2} \pm \sqrt{\frac{T^2}{4} - D}$$

$T = u_x + v_y$ trace of Jacobian

$D = u_x v_y - u_y v_x$ determinant of Jacobian

needs to be non-zero!

$$\mathbf{e}_1 = \begin{pmatrix} e_1 - v_y \\ v_x \end{pmatrix} \quad \mathbf{e}_2 = \begin{pmatrix} e_2 - v_y \\ v_x \end{pmatrix}$$

if v_x is not zero

$$\mathbf{e}_1 = \begin{pmatrix} u_y \\ e_1 - u_x \end{pmatrix} \quad \mathbf{e}_2 = \begin{pmatrix} u_y \\ e_2 - u_x \end{pmatrix}$$

if u_y is not zero

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

if both u_y and v_x are zero

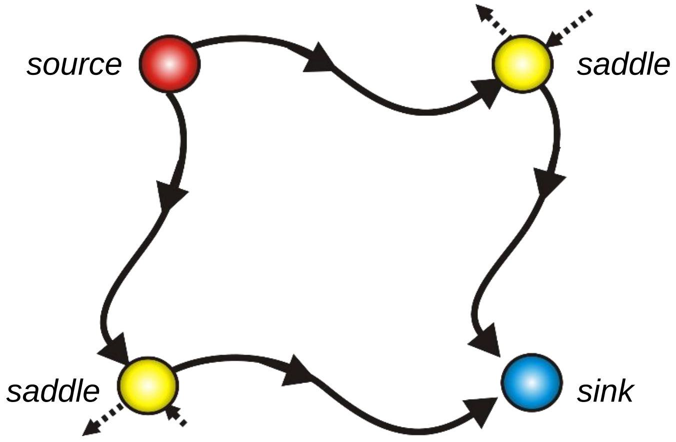
Separatrices connect critical points

- find several pictures

Separatrices connect critical points: substructures

4 critical points

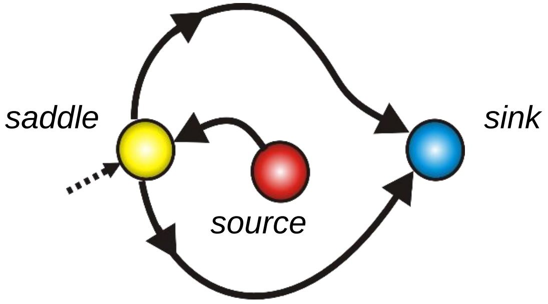
2 saddles
1 source
1 sink



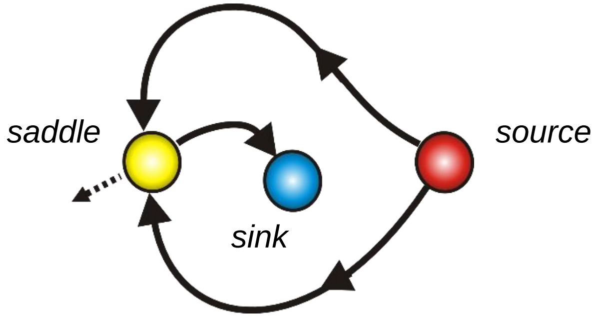
3 critical points

1 saddle connected to same source/sink
1 source
1 sink

source inside



sink inside



Separatrices (2D)

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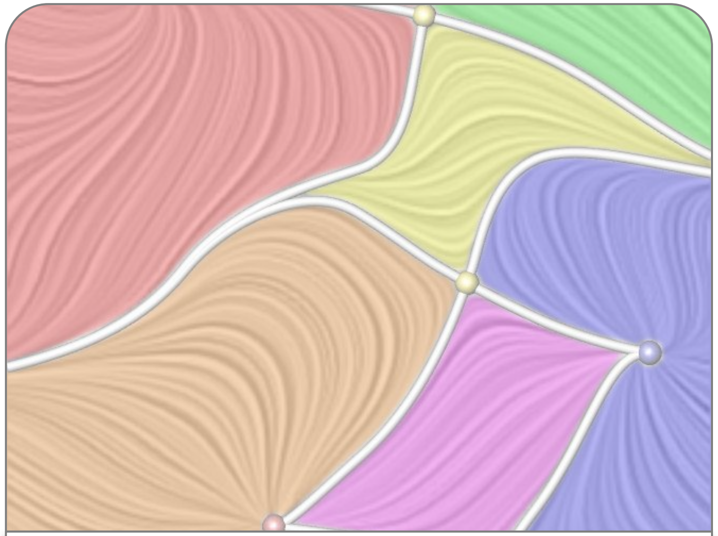
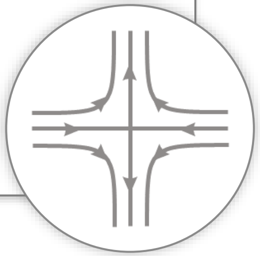
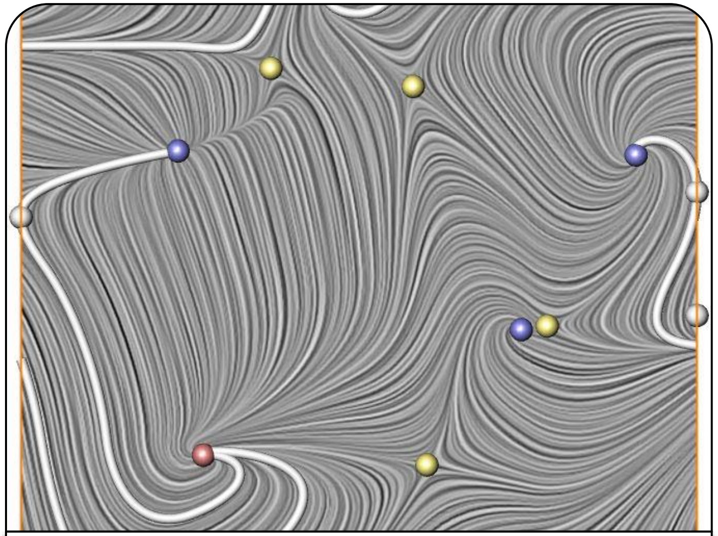
special tangent curve separating regions of different flow behavior

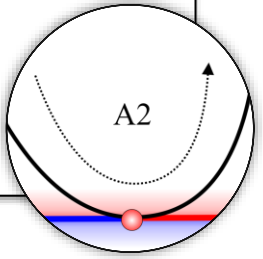


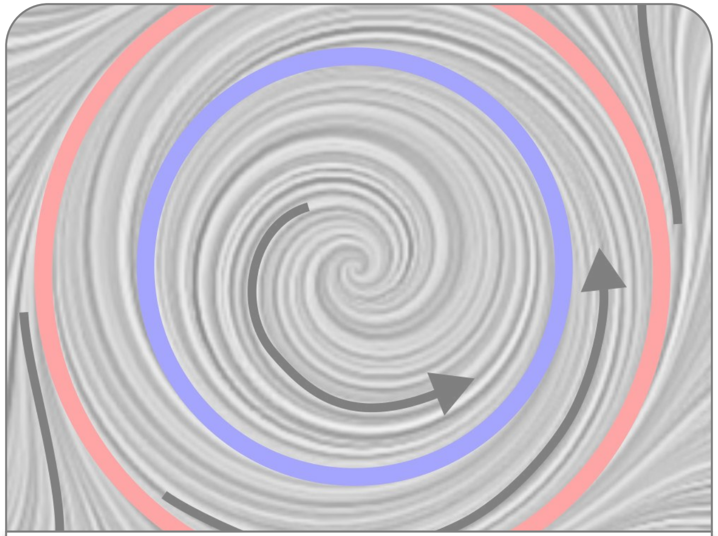
Separatrix originating/ending near a critical point



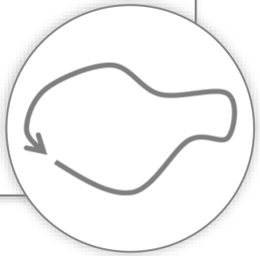


Separatrix through a boundary switch point





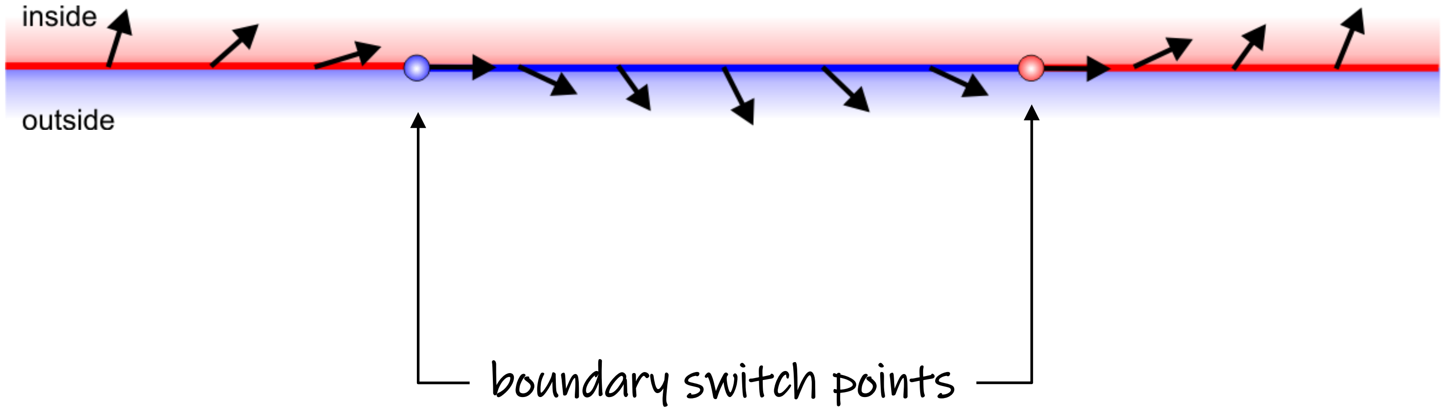
Closed tangent curve (isolated)



Separatrices from Boundary Switch Points

change inflow ⇔ outflow at boundary of the domain

boundary switch point $\mathbf{x}$: $\mathbf{v}(\mathbf{x})$ parallel to tangent of boundary curve at $\mathbf{x}$



Separatrices from Boundary Switch Points

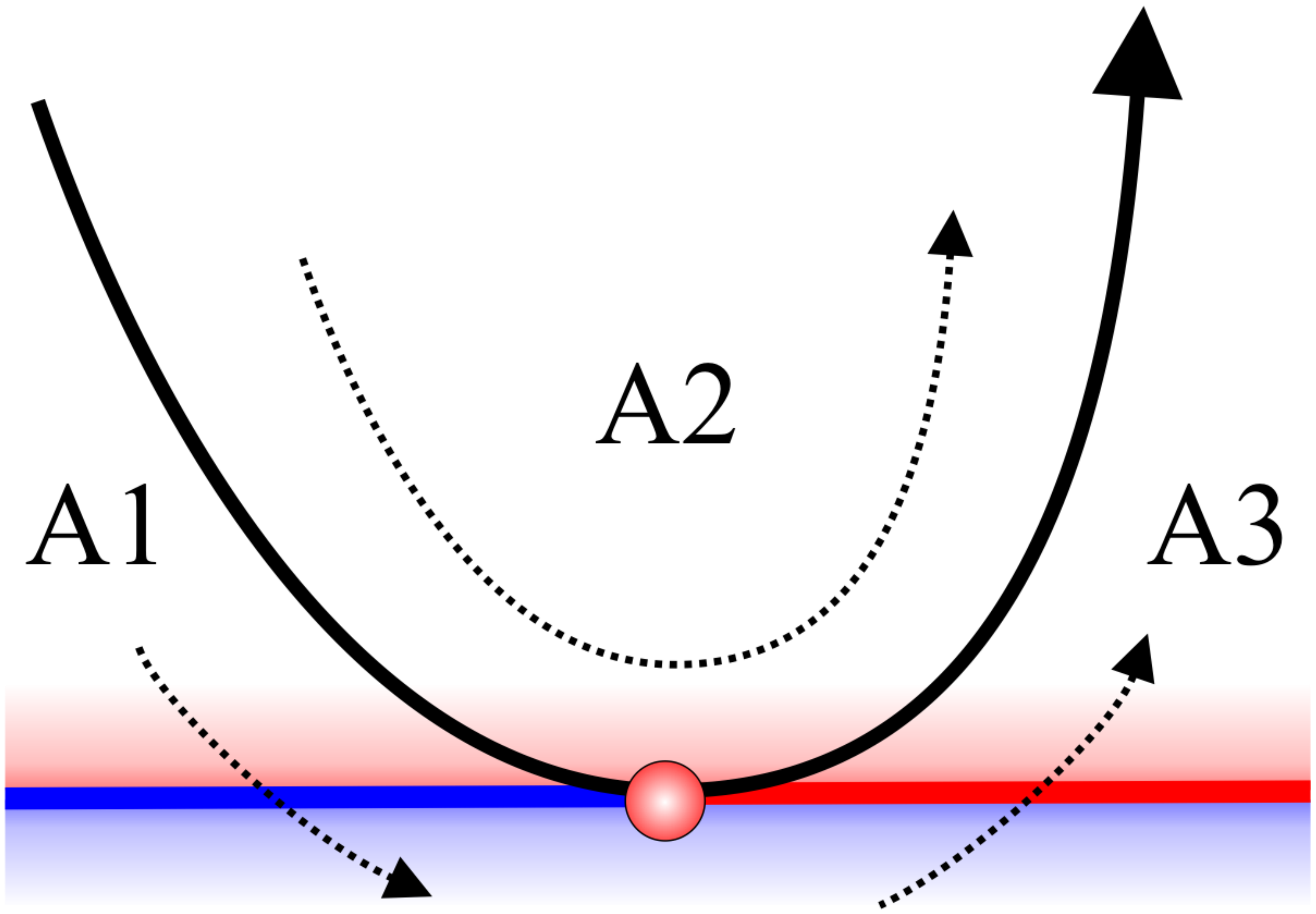
separatrix:
tangent curve started from boundary switch point

separates three areas

A1: outflow area

A2: stay-inside area

A3: inflow area



Separatrices from Boundary Switch Points

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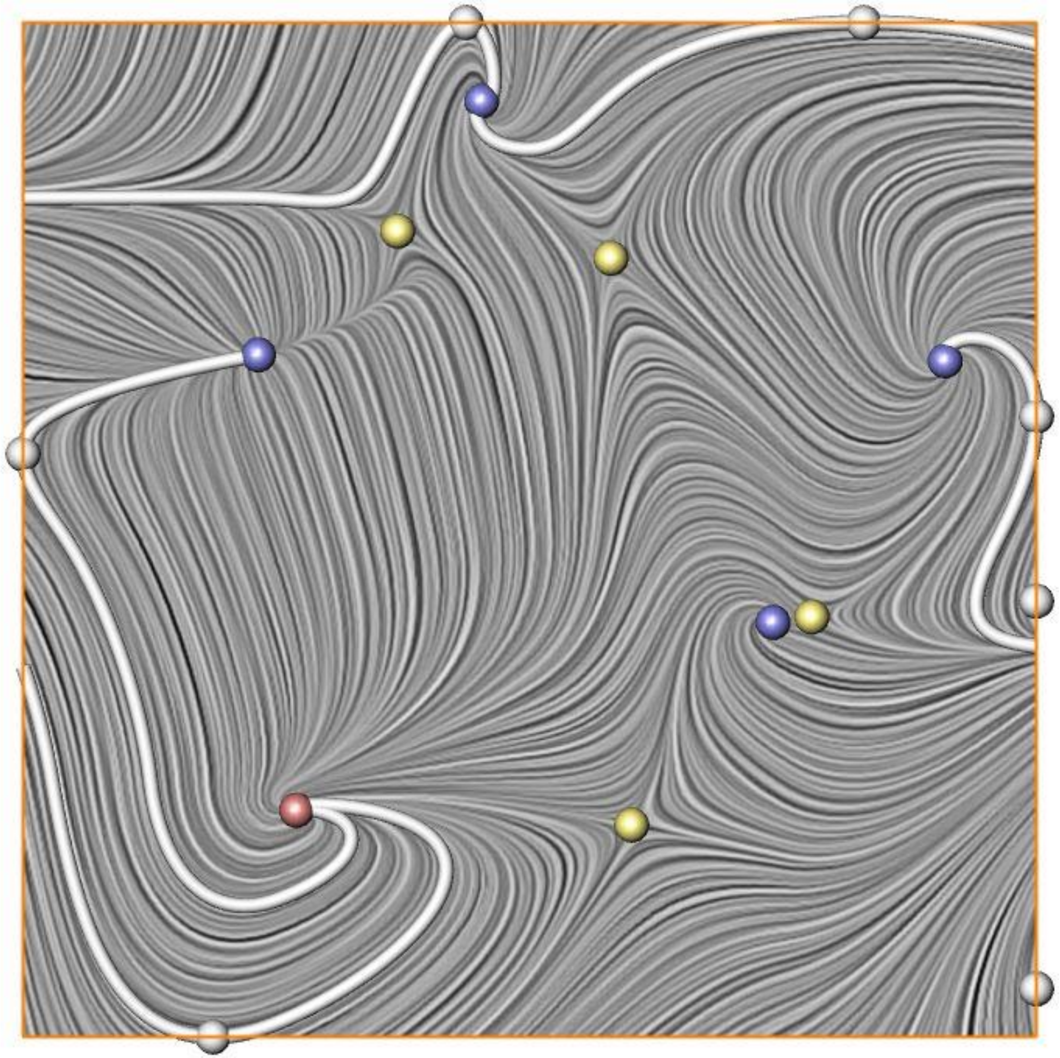

separates three areas

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Separatrices (2D)

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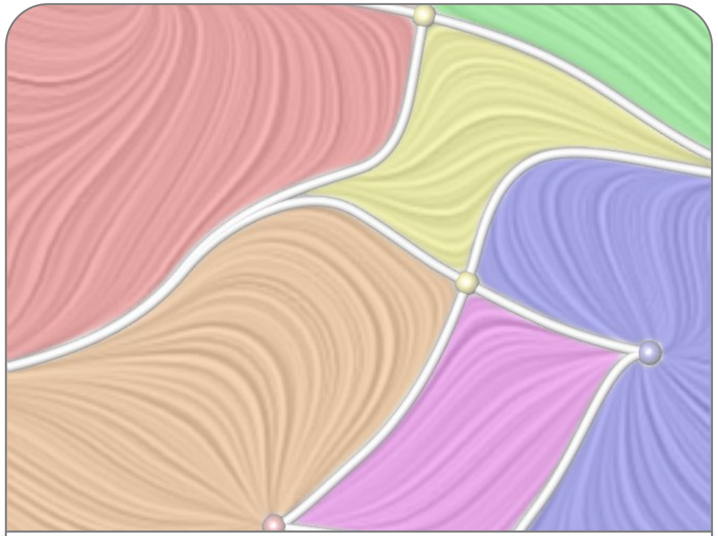
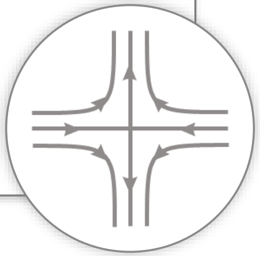
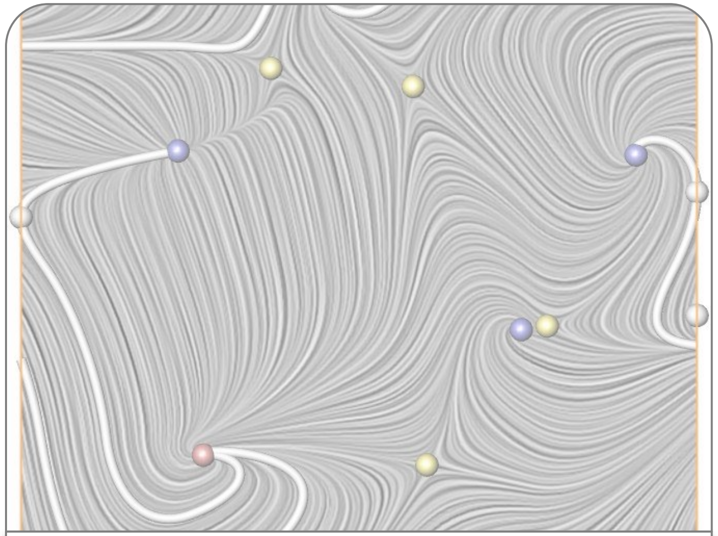
special tangent curve separating regions of different flow behavior

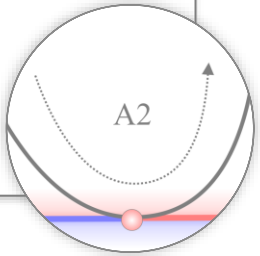


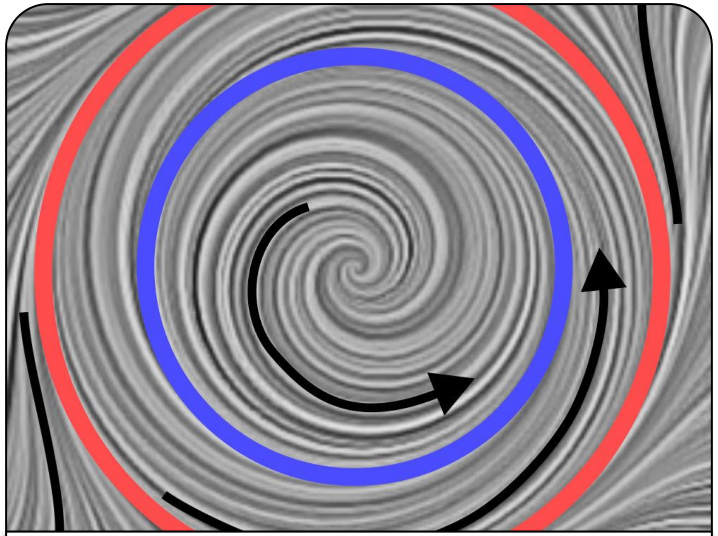
Separatrix originating/ending near a critical point



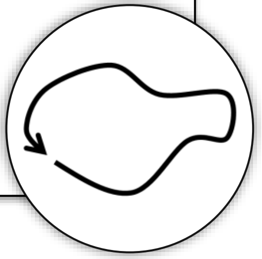


Separatrix through a boundary switch point





Closed tangent curve (isolated)



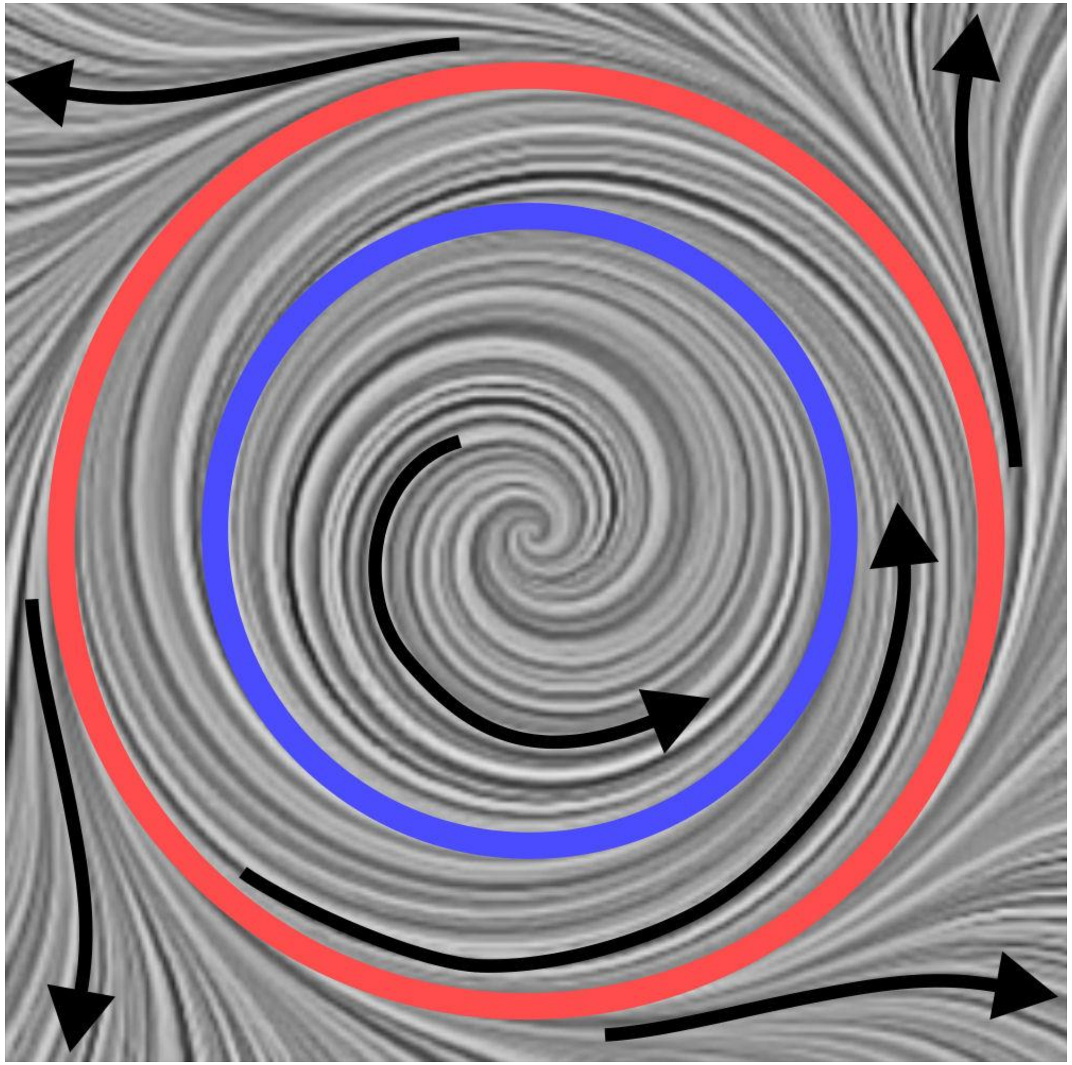
Isolated Closed Tangent Curves

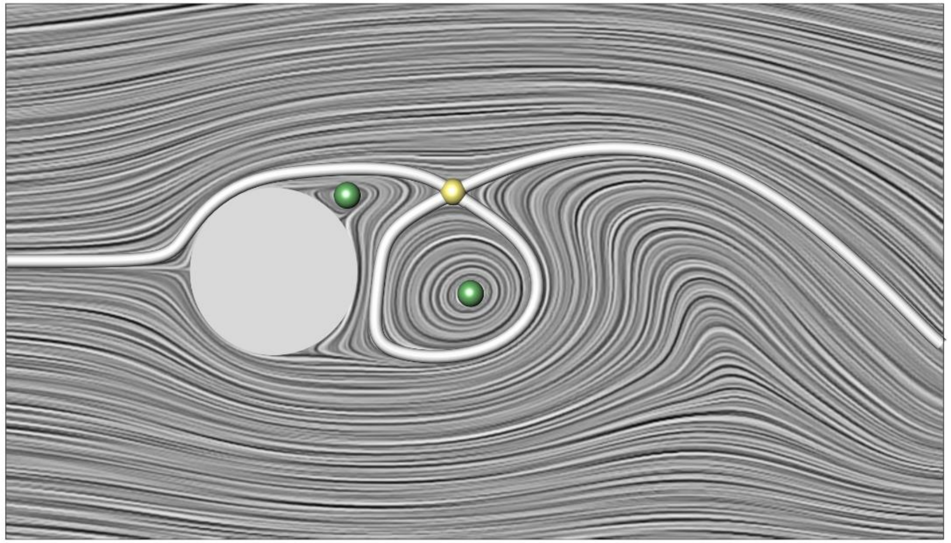
closed tangent curve

returns to its seed point

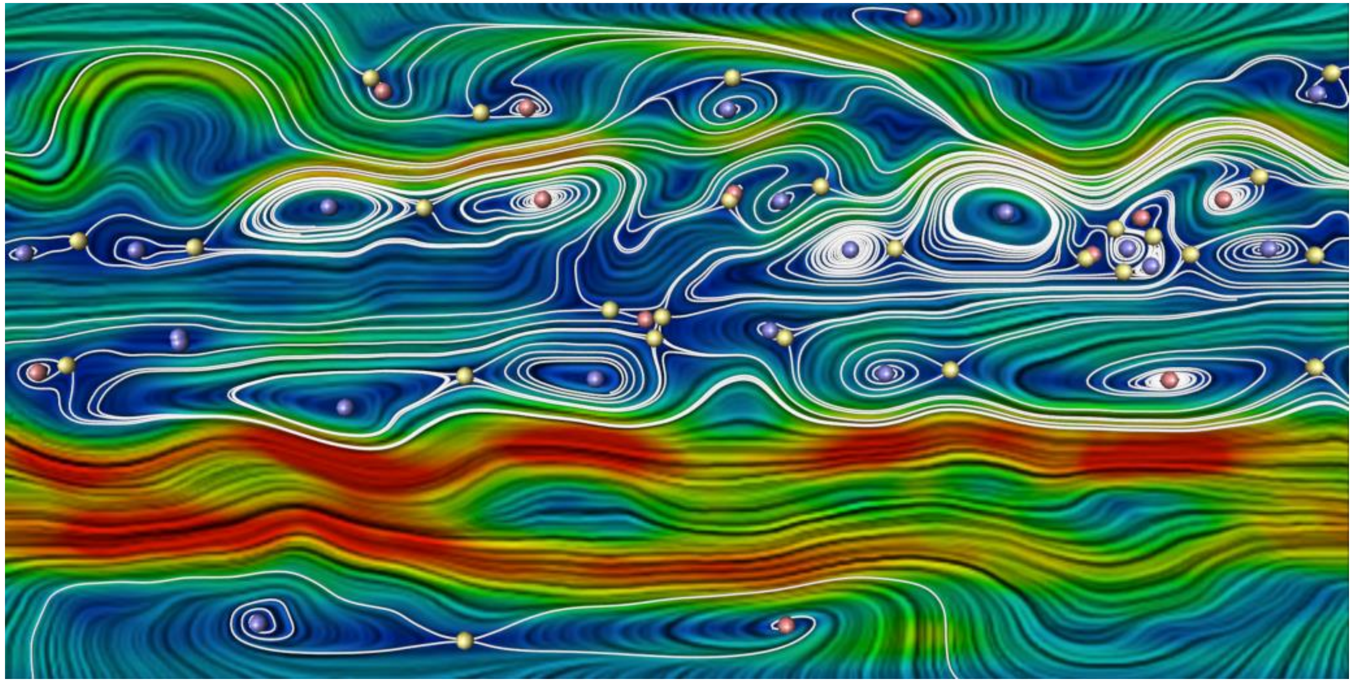
no other closed tangent curve
in its vicinity

acts as source or sink

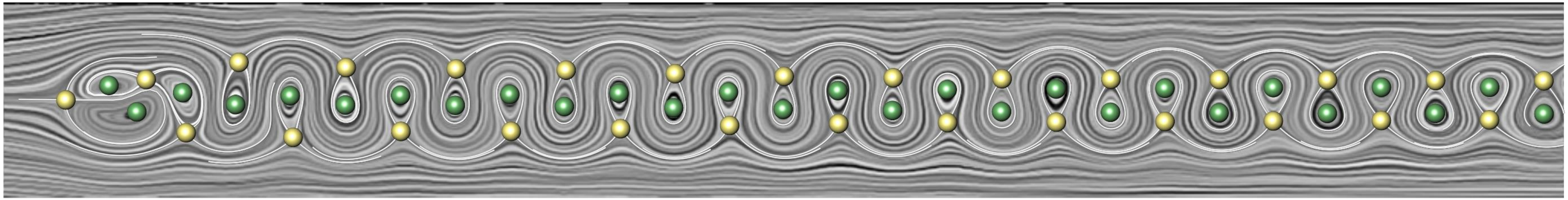




flow around cylinder
(in close vicinity to the cylinder)



wind in Earth's atmosphere (wind speed colored)

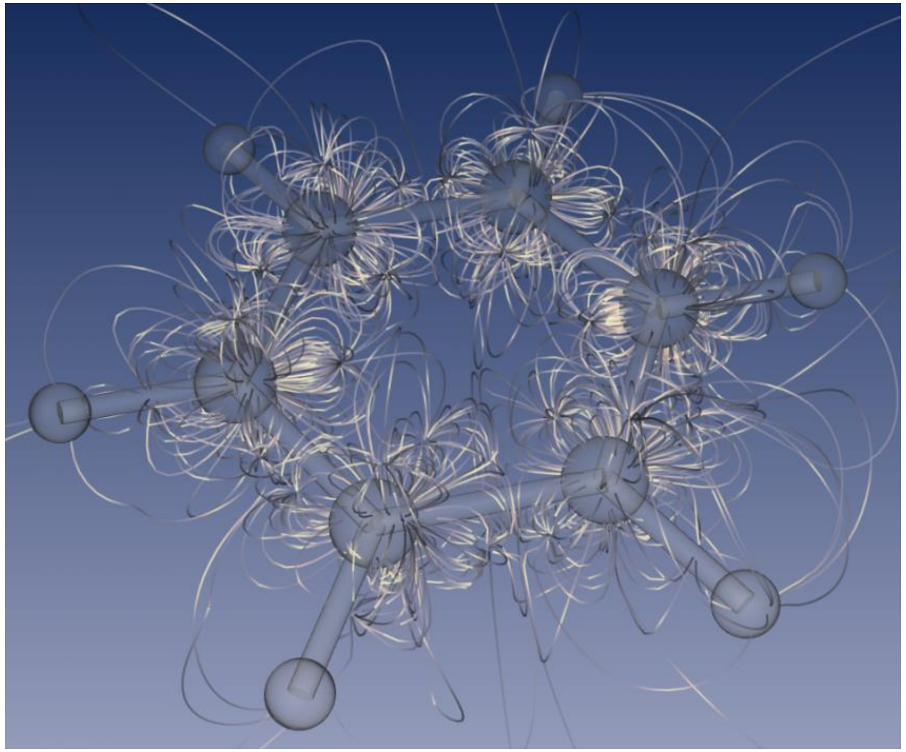


flow around cylinder (entire simulation domain)

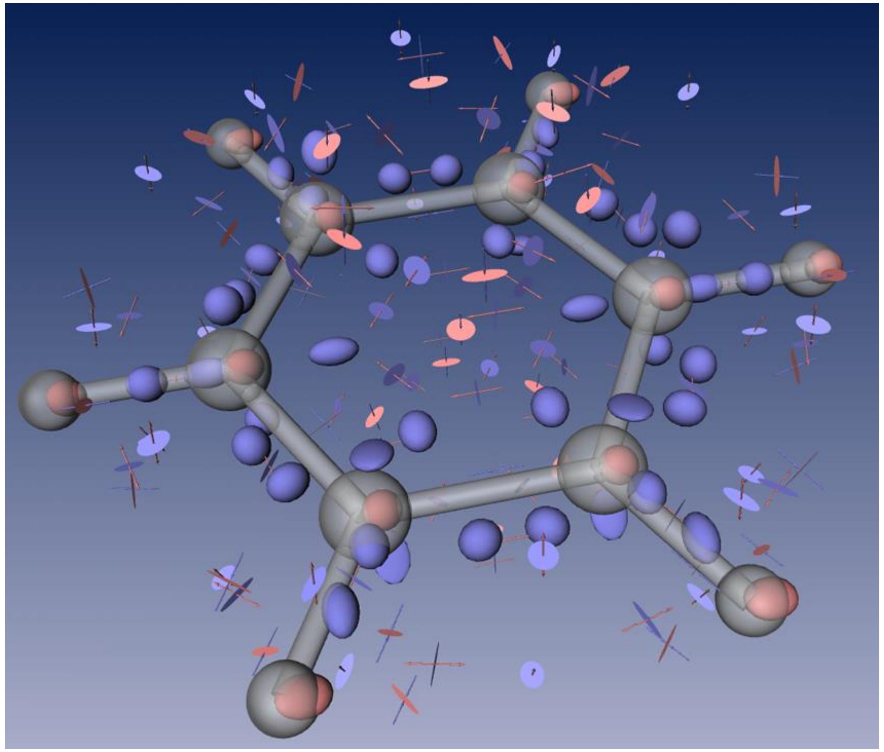
Topological Structures

steady 3D vector field

$$\mathbf{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

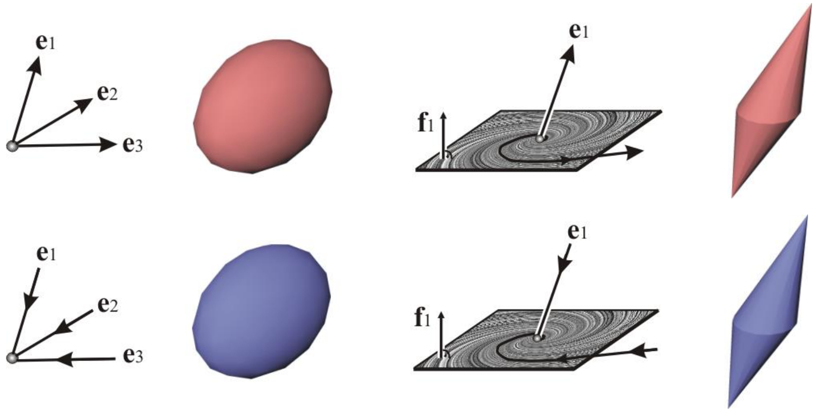


stream lines

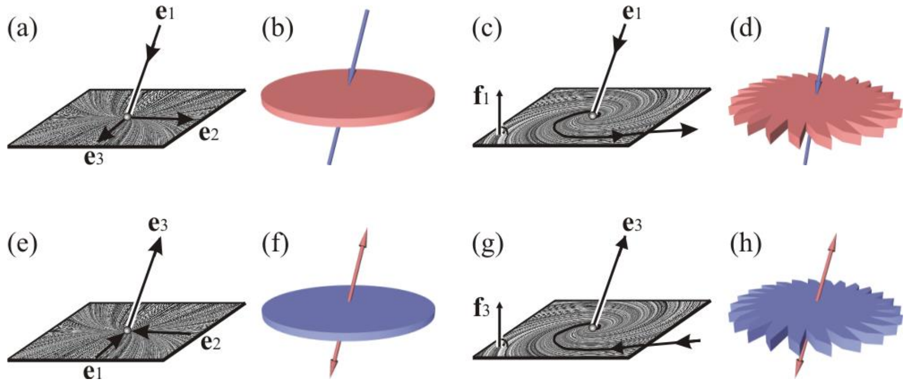


critical points

$$\mathbf{v}(\mathbf{x}_0) = \mathbf{0} \quad \text{with} \quad \mathbf{v}(\mathbf{x}_0 \pm \boldsymbol{\epsilon}) \neq \mathbf{0}$$



sources and sinks



repelling and attracting saddles

First-order critical points (3D)

Classification by an eigenvalue/eigenvector analysis of Jacobian matrix

- Outflow / Inflow behavior

Sources:	$0 < Re(\lambda_1) \le Re(\lambda_2) \le Re(\lambda_3)$
Repelling saddles:	$Re(\lambda_1) < 0 < Re(\lambda_2) \le Re(\lambda_3)$
Attracting saddles:	$Re(\lambda_1) \le Re(\lambda_2) < 0 < Re(\lambda_3)$
Sinks:	$Re(\lambda_1) \le Re(\lambda_2) \le Re(\lambda_3) < 0$

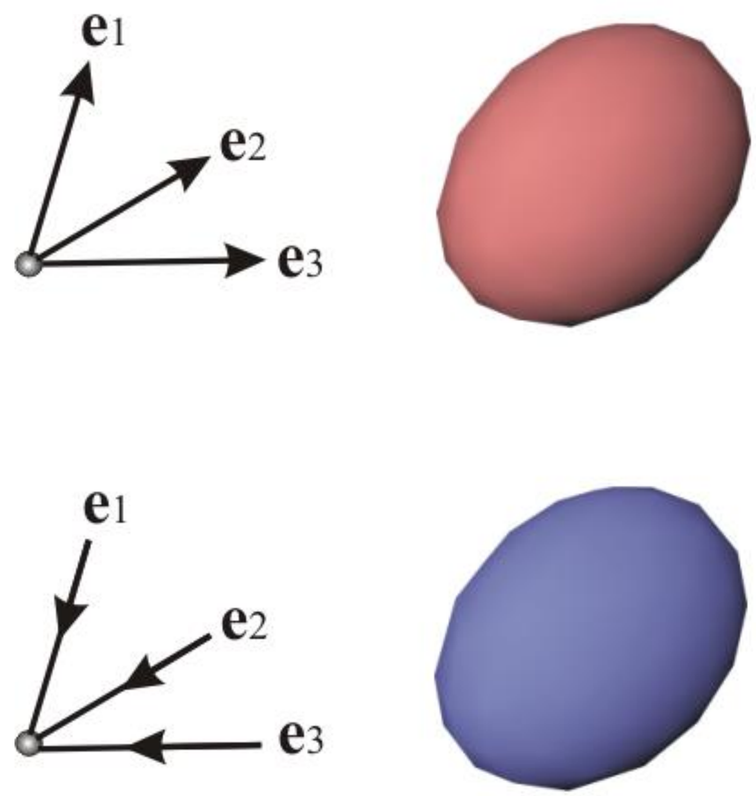
Repelling Parts (Outflow)

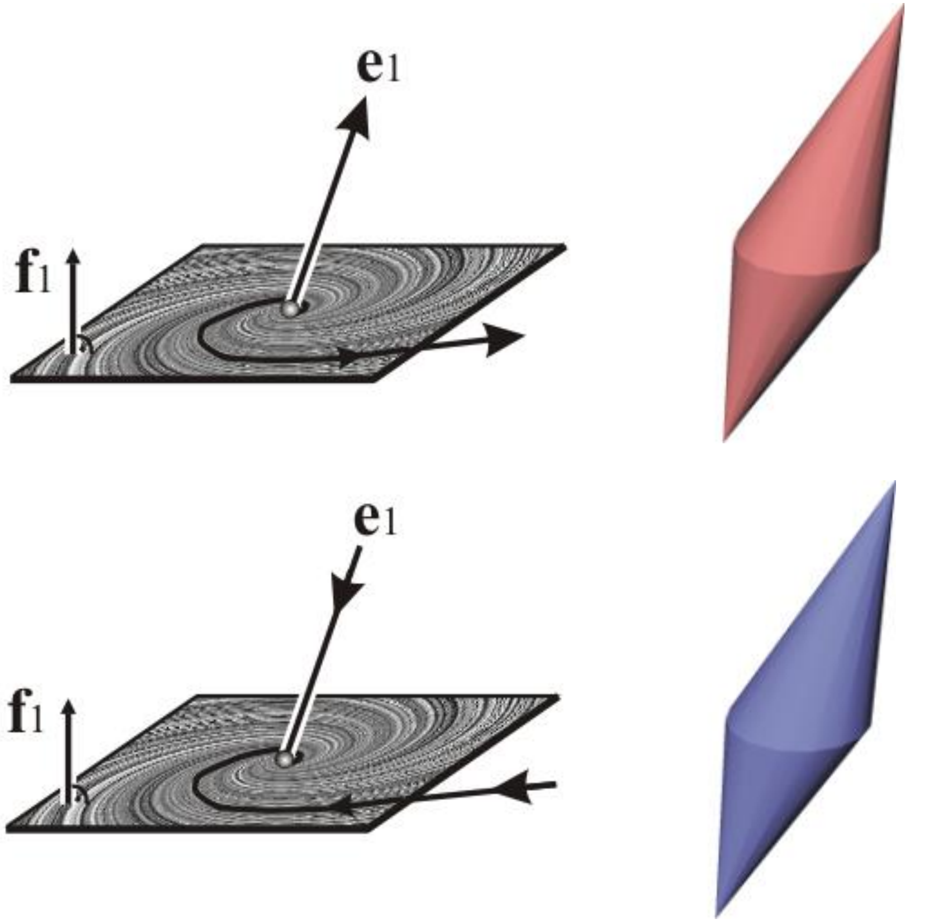
(Inflow) Attracting Parts

- Focus (Swirling) / Node behavior

Foci:	$Im(\lambda_1) = 0$ and $Im(\lambda_2) = -Im(\lambda_3) \neq 0$
Nodes:	$Im(\lambda_1) = Im(\lambda_2) = Im(\lambda_3) = 0$

Sources and Sinks





Sources: $0 < Re(\lambda_1) \leq Re(\lambda_2) \leq Re(\lambda_3)$

Repelling saddles: $Re(\lambda_1) < 0 < Re(\lambda_2) \leq Re(\lambda_3)$

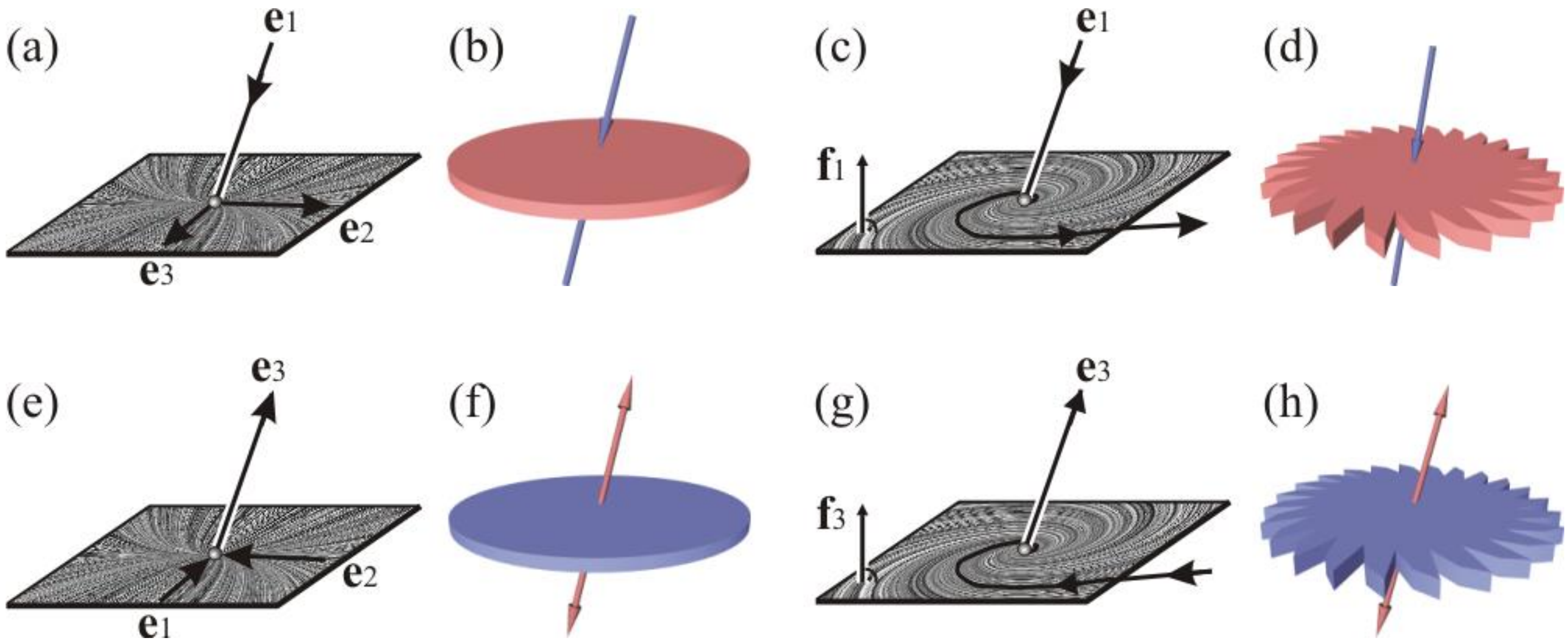
Attracting saddles: $Re(\lambda_1) \leq Re(\lambda_2) < 0 < Re(\lambda_3)$

Sinks: $Re(\lambda_1) \leq Re(\lambda_2) \leq Re(\lambda_3) < 0$

Foci: $Im(\lambda_1) = 0$ and $Im(\lambda_2) = -Im(\lambda_3) \neq 0$

Nodes: $Im(\lambda_1) = Im(\lambda_2) = Im(\lambda_3) = 0$

Repelling and Attracting Saddles



Sources: $0 < Re(\lambda_1) \leq Re(\lambda_2) \leq Re(\lambda_3)$

Repelling saddles: $Re(\lambda_1) < 0 < Re(\lambda_2) \leq Re(\lambda_3)$

Attracting saddles: $Re(\lambda_1) \leq Re(\lambda_2) < 0 < Re(\lambda_3)$

Sinks: $Re(\lambda_1) \leq Re(\lambda_2) \leq Re(\lambda_3) < 0$

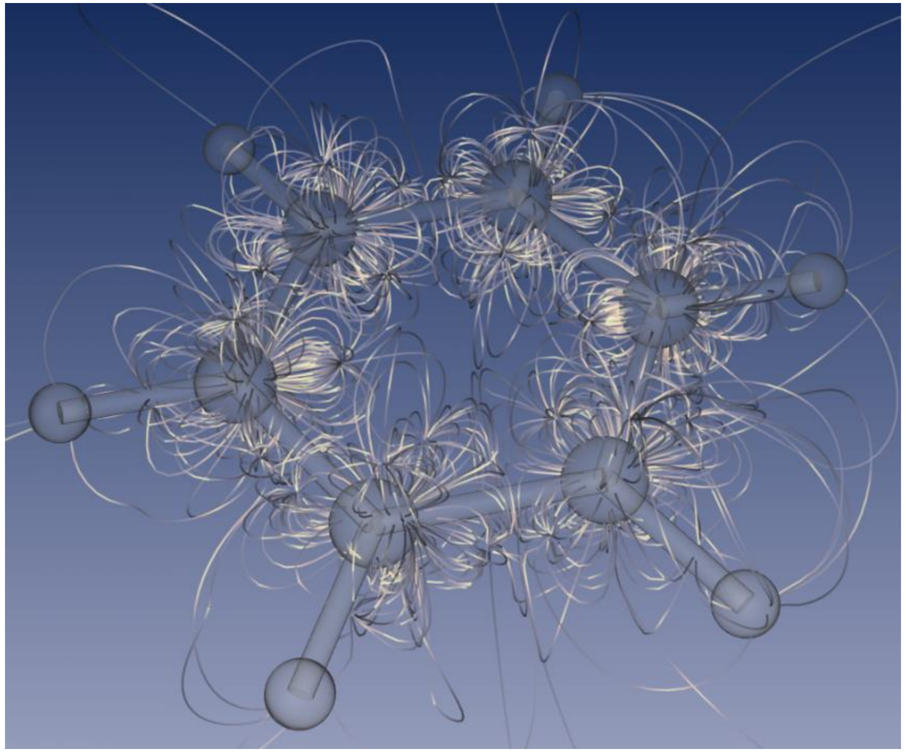
Foci: $Im(\lambda_1) = 0$ and $Im(\lambda_2) = -Im(\lambda_3) \neq 0$

Nodes: $Im(\lambda_1) = Im(\lambda_2) = Im(\lambda_3) = 0$

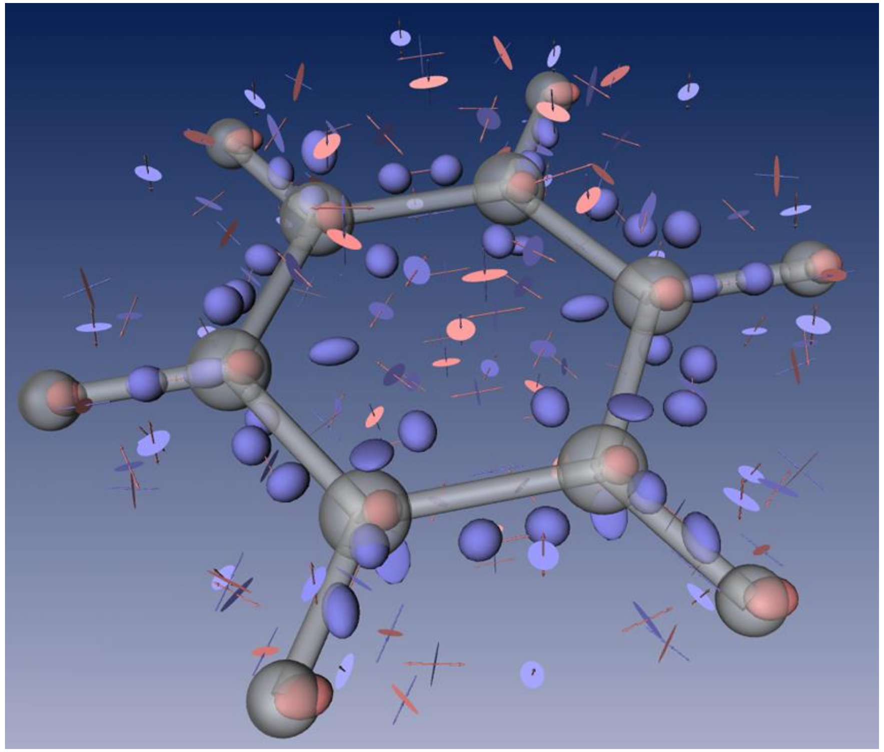
Topological Structures

steady 3D vector field

$$\mathbf{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$



stream lines

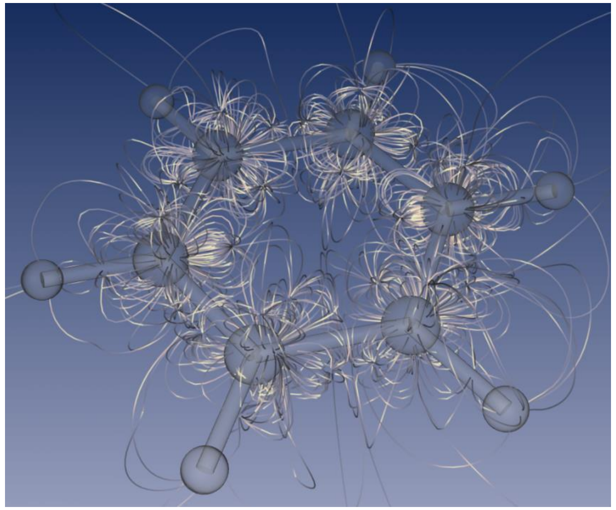


critical points

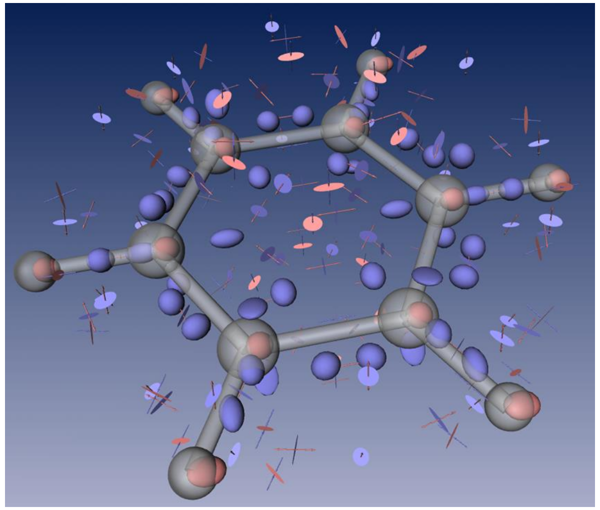
Topological Structures

steady 3D vector field

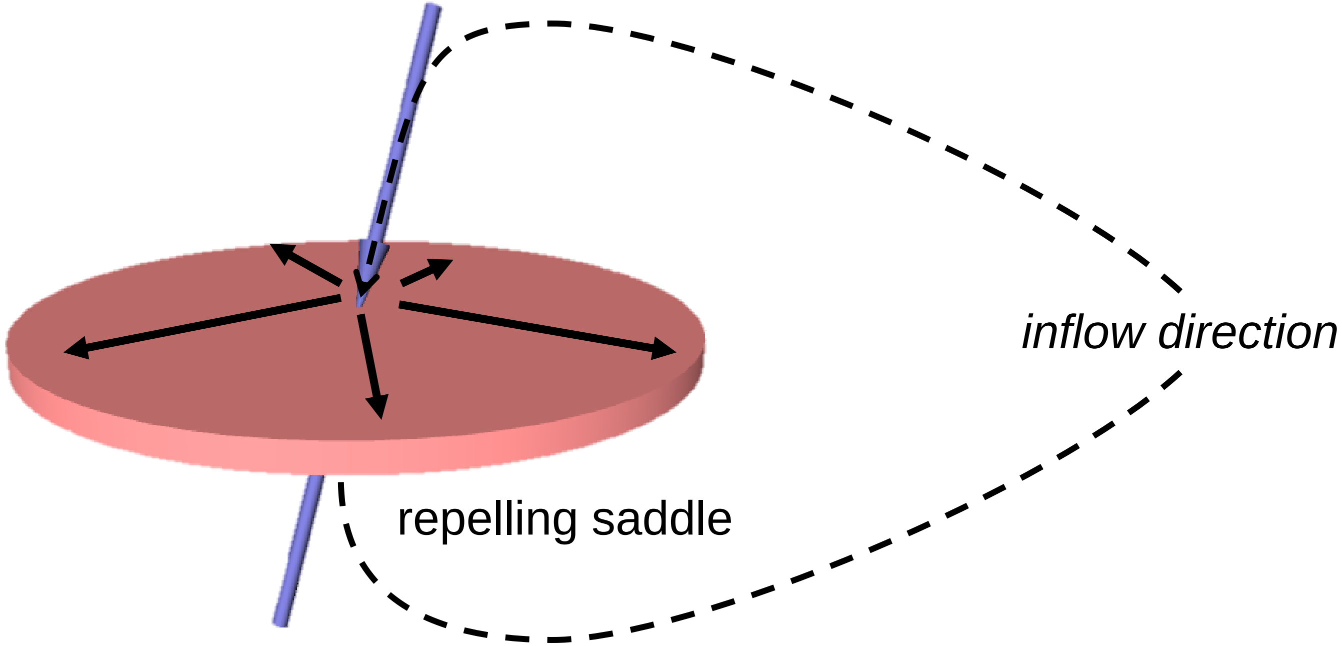
$$\mathbf{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$



stream lines



critical points



Topological Structures

steady 3D vector field

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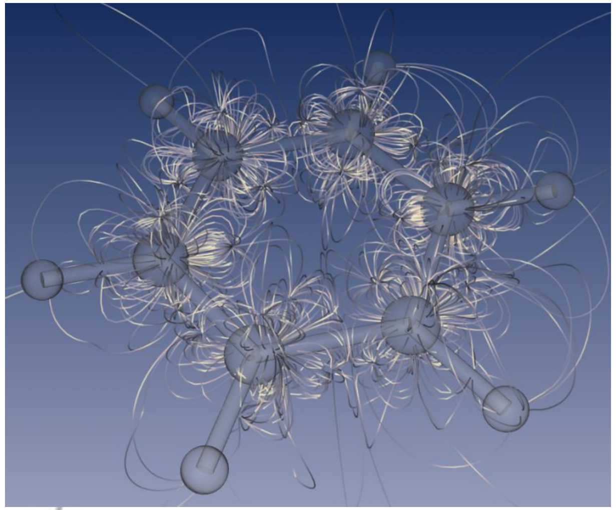
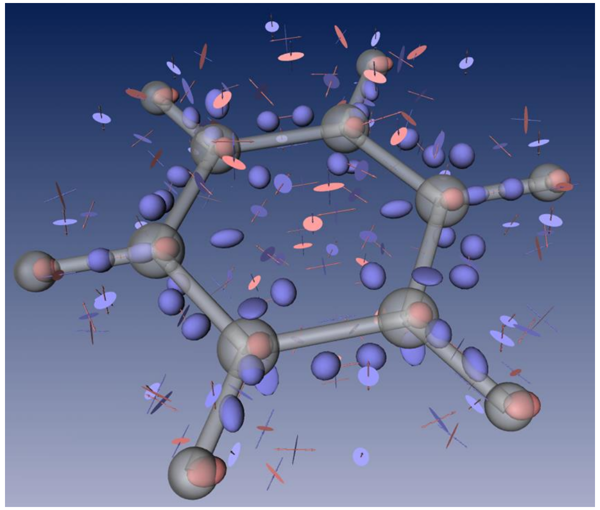
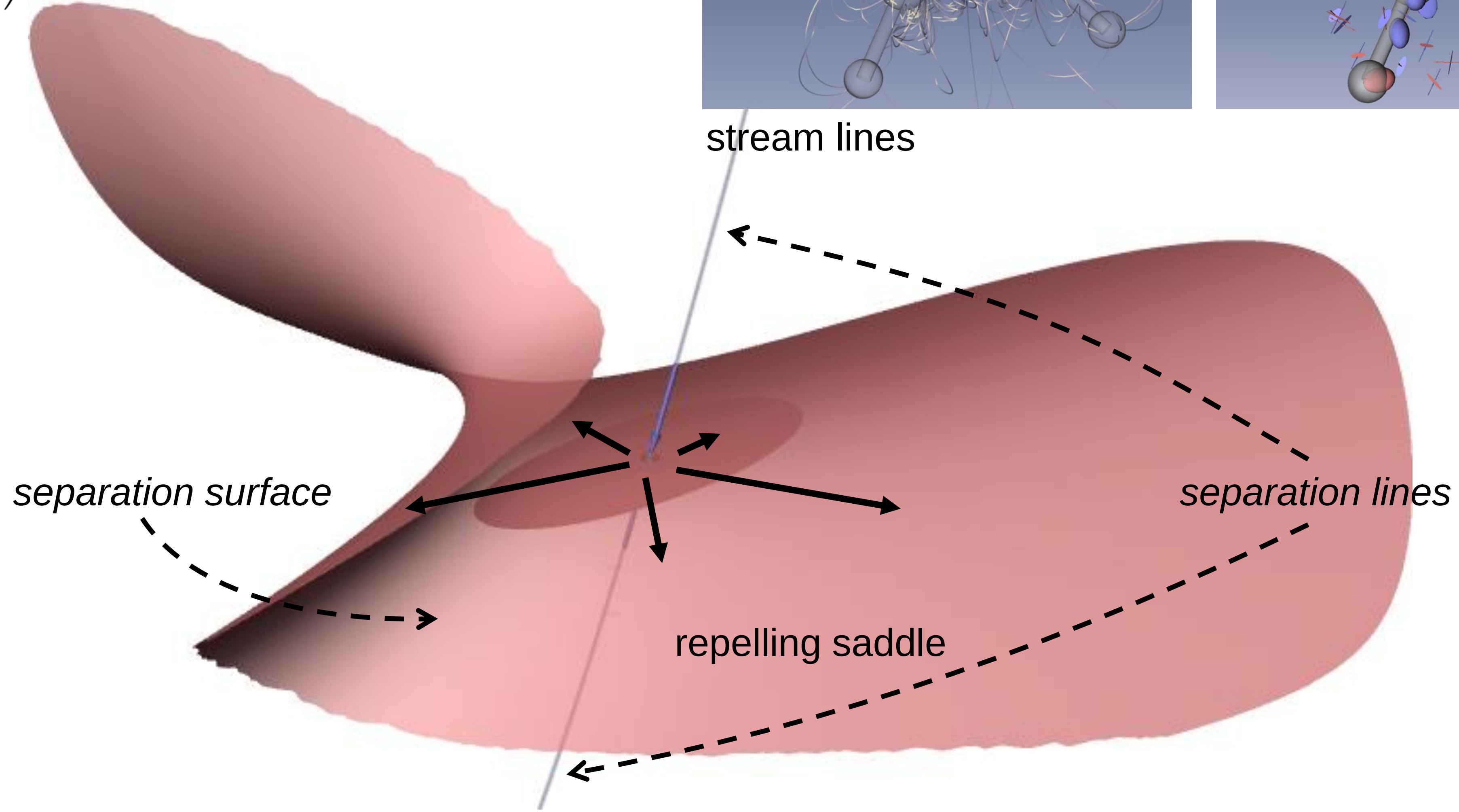


stream lines



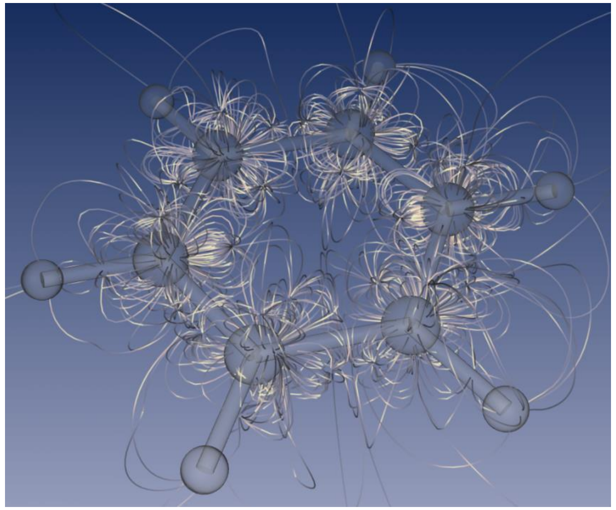
critical points



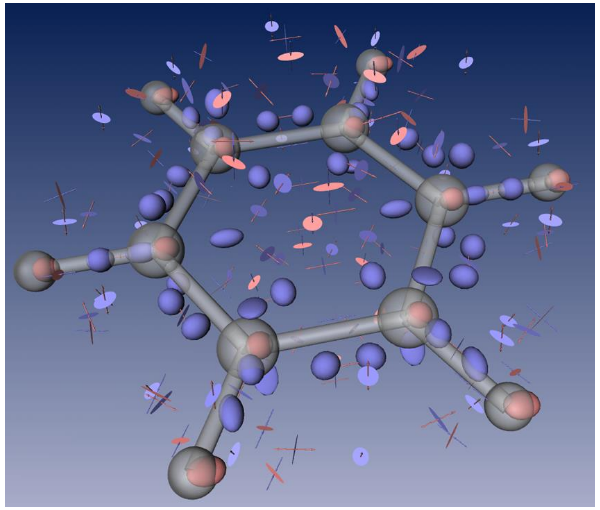
Topological Structures

steady 3D vector field

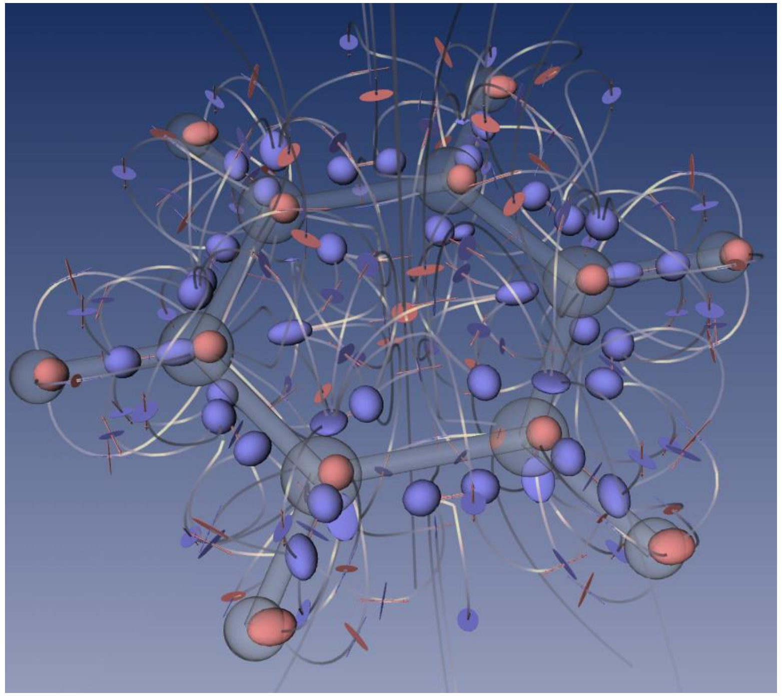
$$\mathbf{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$



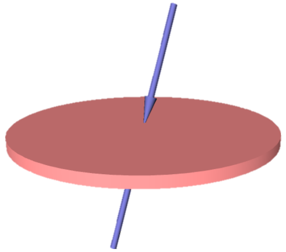
stream lines



critical points

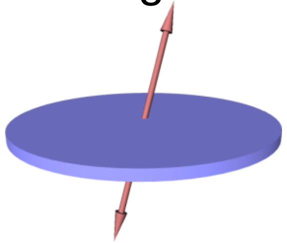


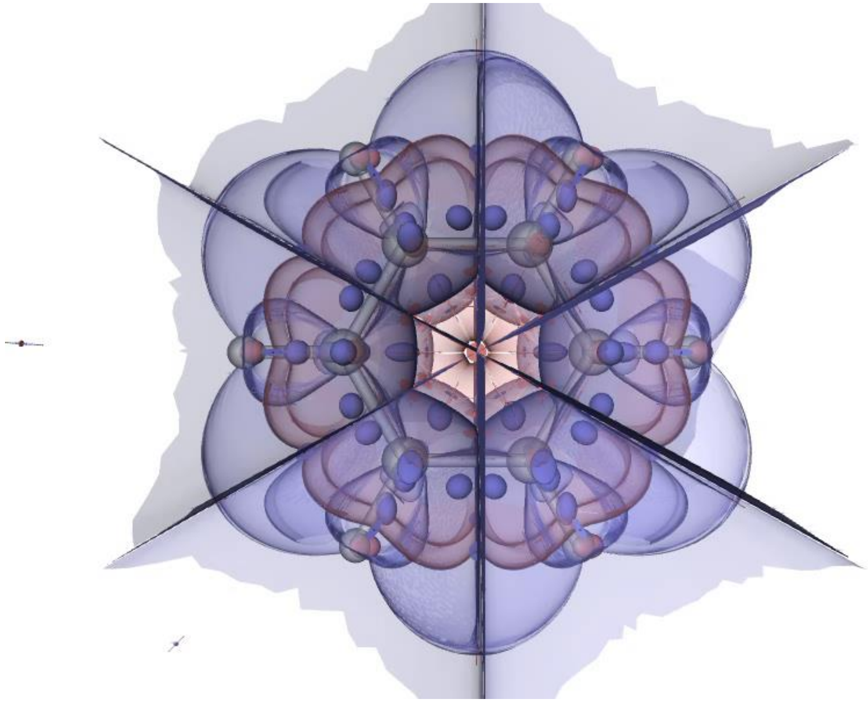
separation lines



repelling saddle

attracting saddle

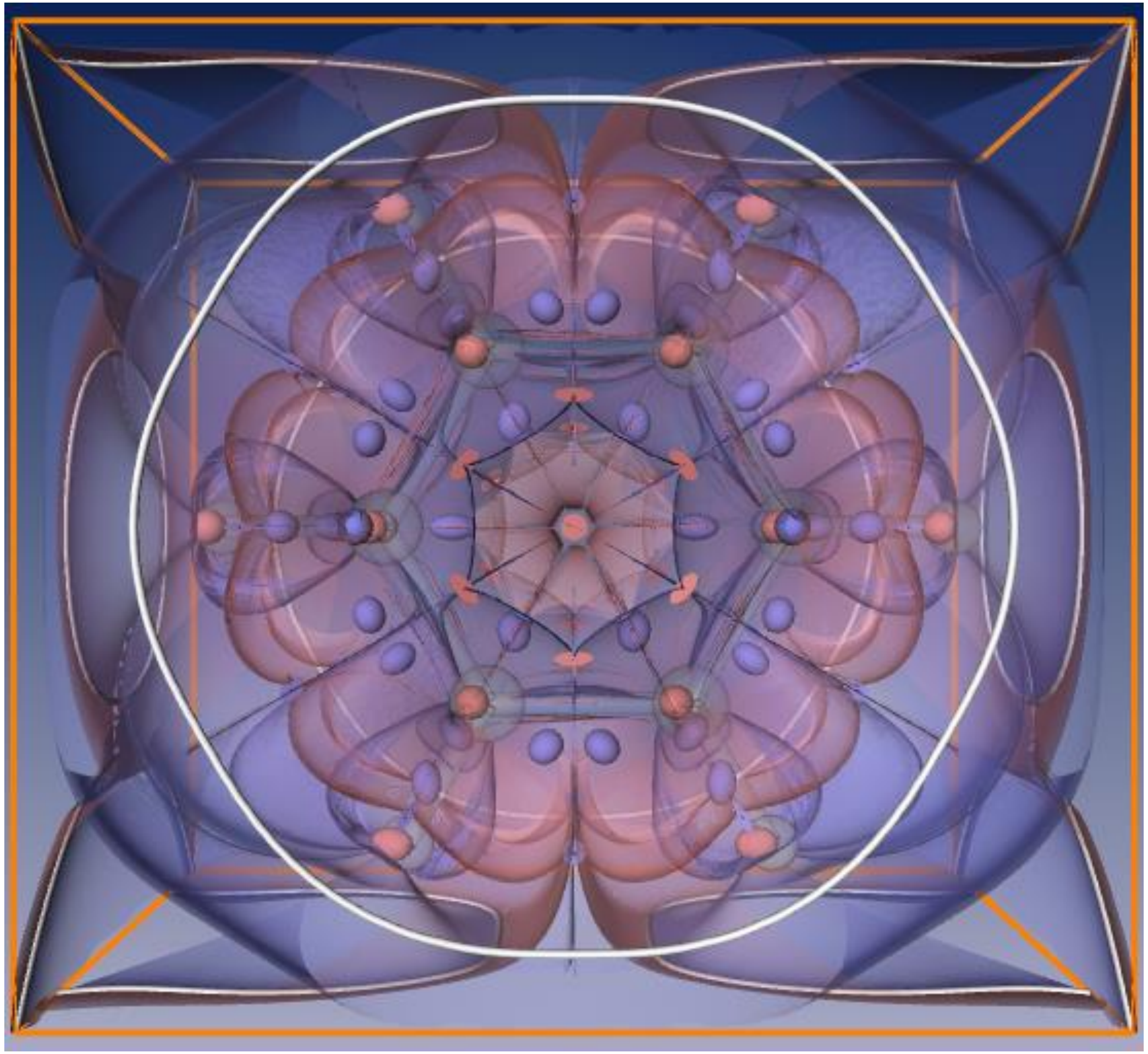




separation surfaces

Separatrices

- stream lines and stream surfaces separating areas of different flow behavior
- starting from saddle points:
  - stream surface starting in the inflow/outflow plane
  - stream lines starting in the inflow/outflow directions
- starting from inbound sectors of boundary switch curves



Saddle Connectors

repelling separation surface

red

attracting separation surface

blue

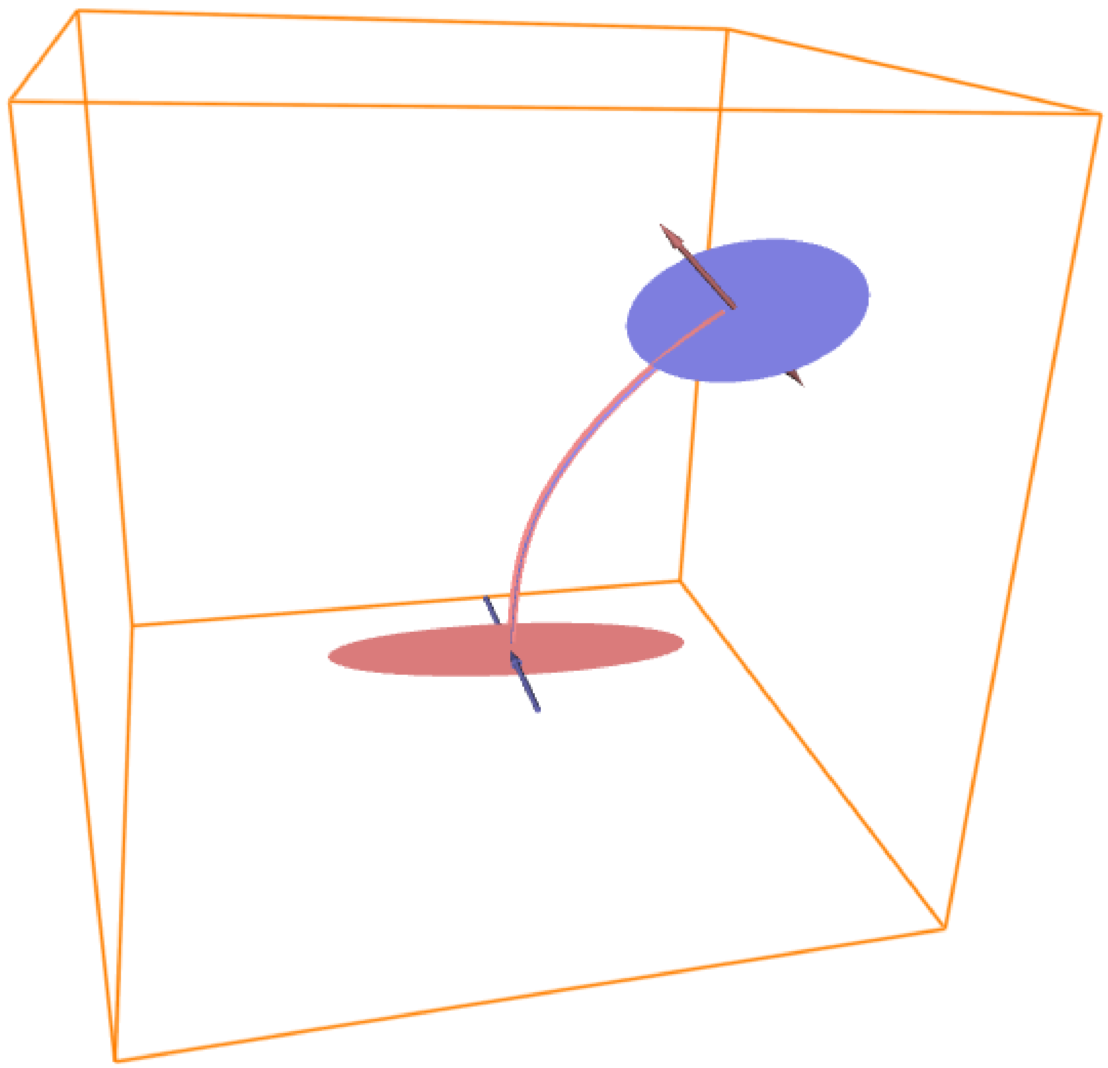
saddle connector

intersection of these surfaces

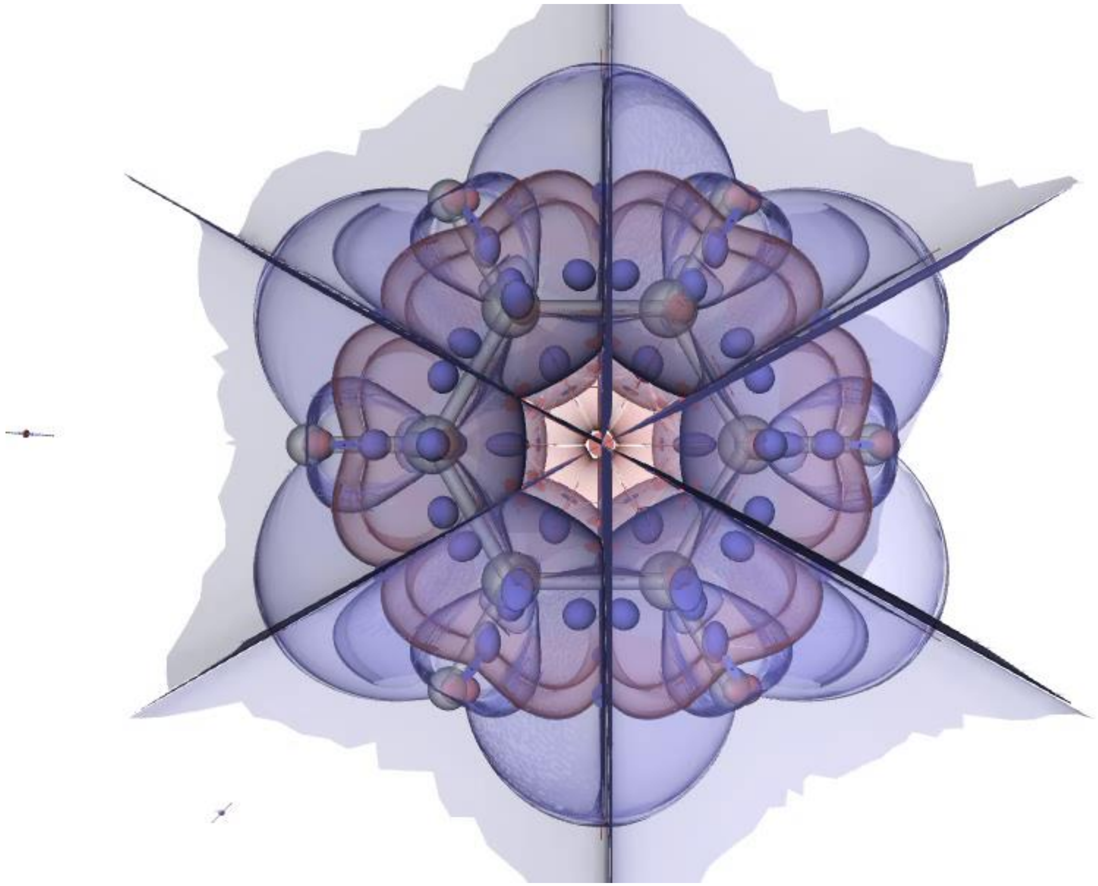
stream line

lies in both surfaces

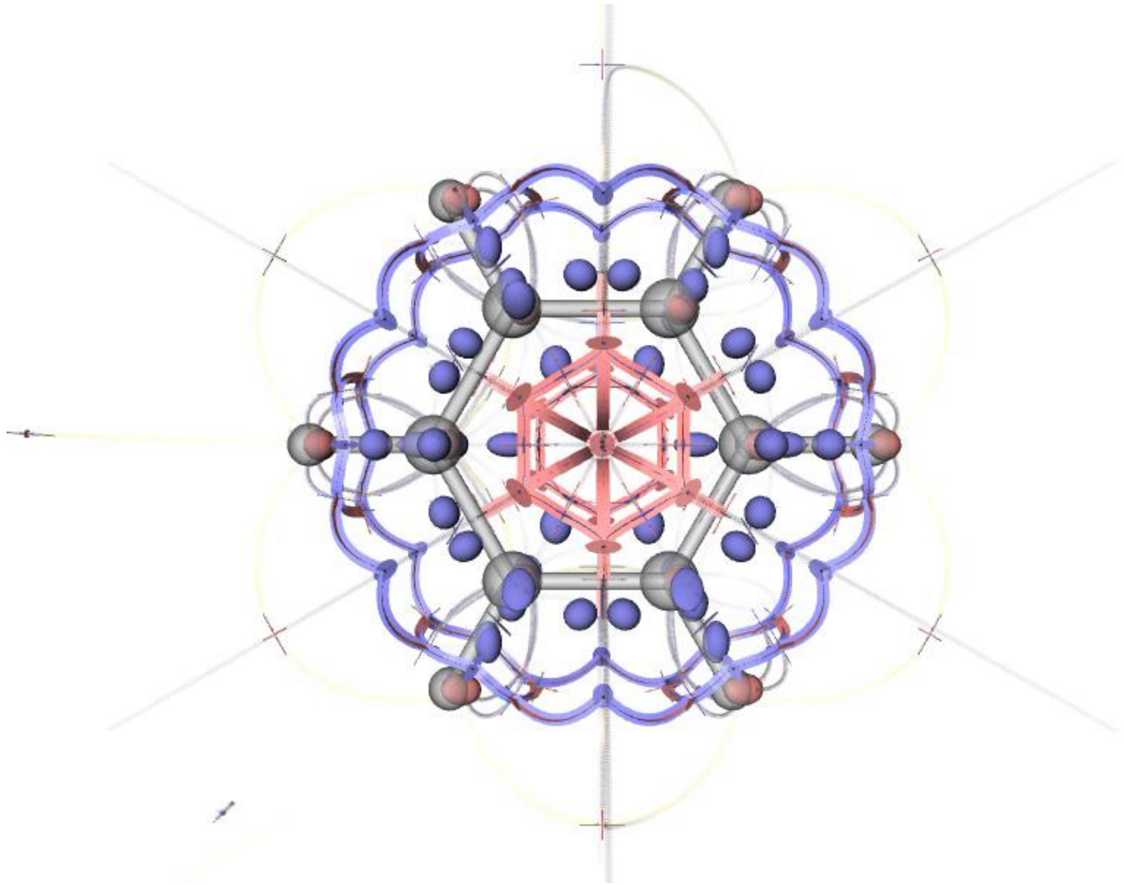
stable structure

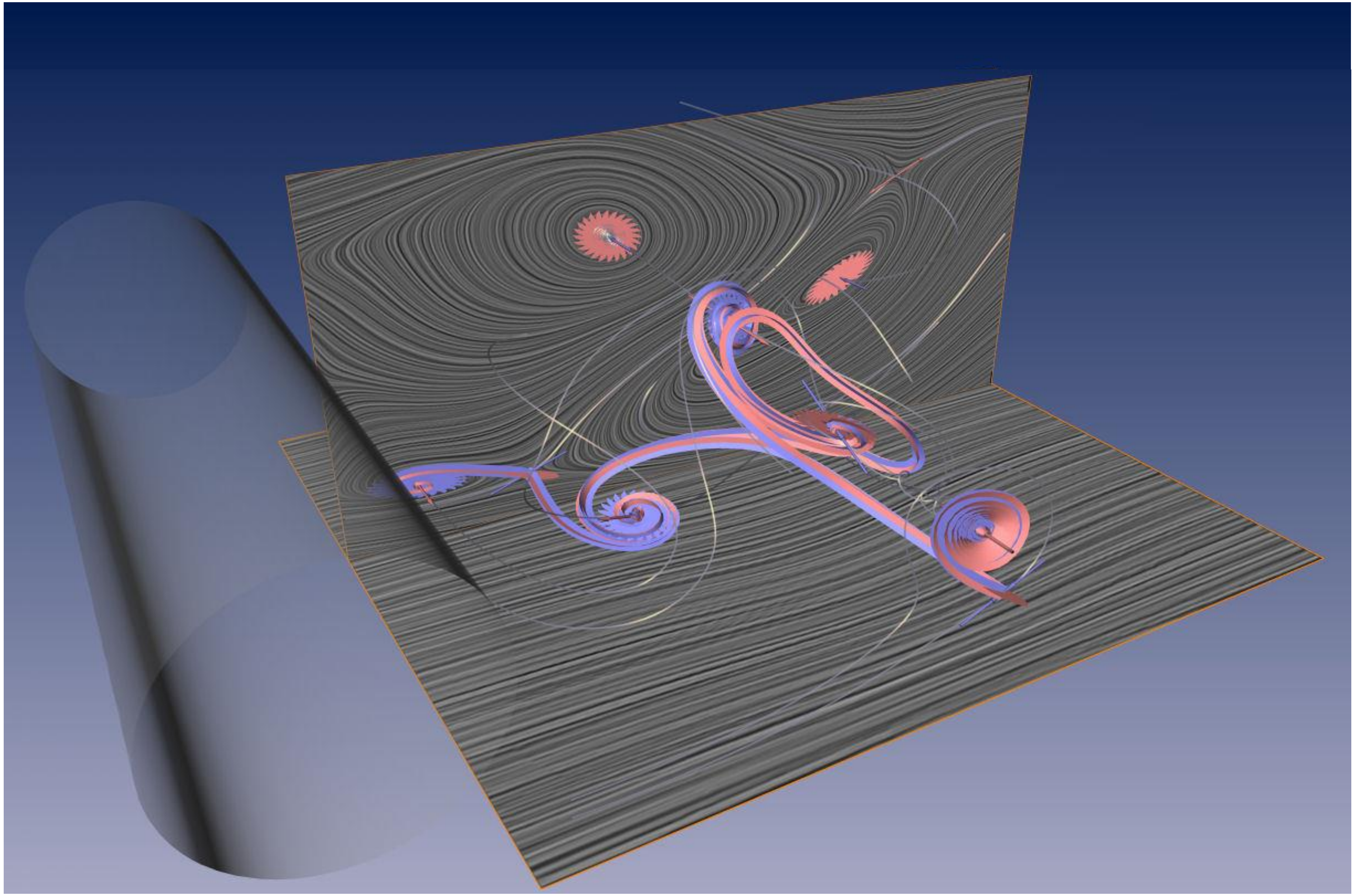


Separation Surfaces



Saddle Connectors





Flow behind a cylinder. Data courtesy of Gerd Mutschke (FZ Rossendorf) and Bernd R. Noack (TU Berlin).

very important

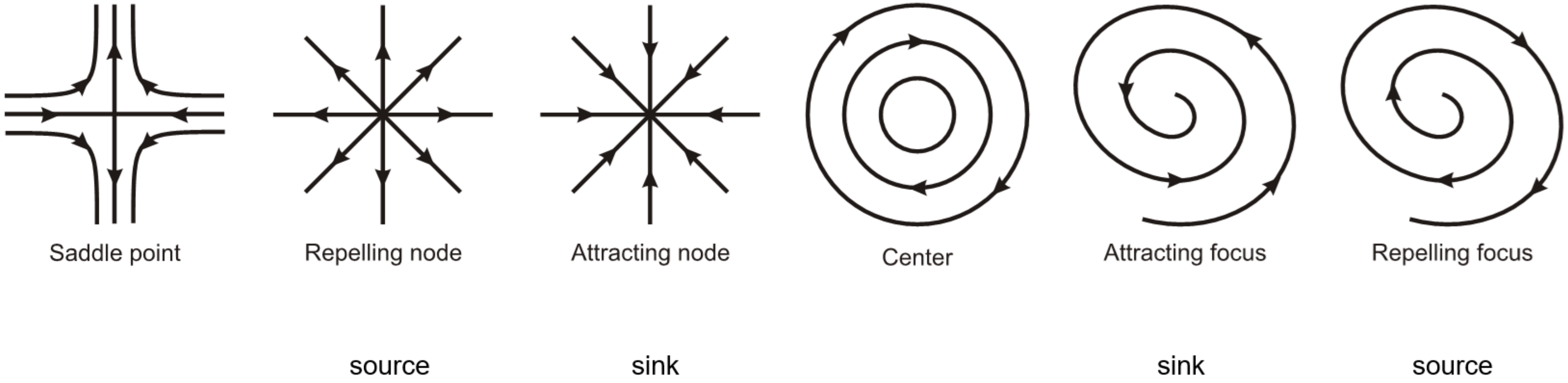
First-order critical points (2D)

The point $\mathbf{x}$ is a *first-order* critical point of the vector field $\mathbf{v}$ if

- $\mathbf{x}$ is an isolated critical point, and
- $\det(\nabla\mathbf{v}(\mathbf{x})) \neq 0$
  (determinant of Jacobian does not vanish)

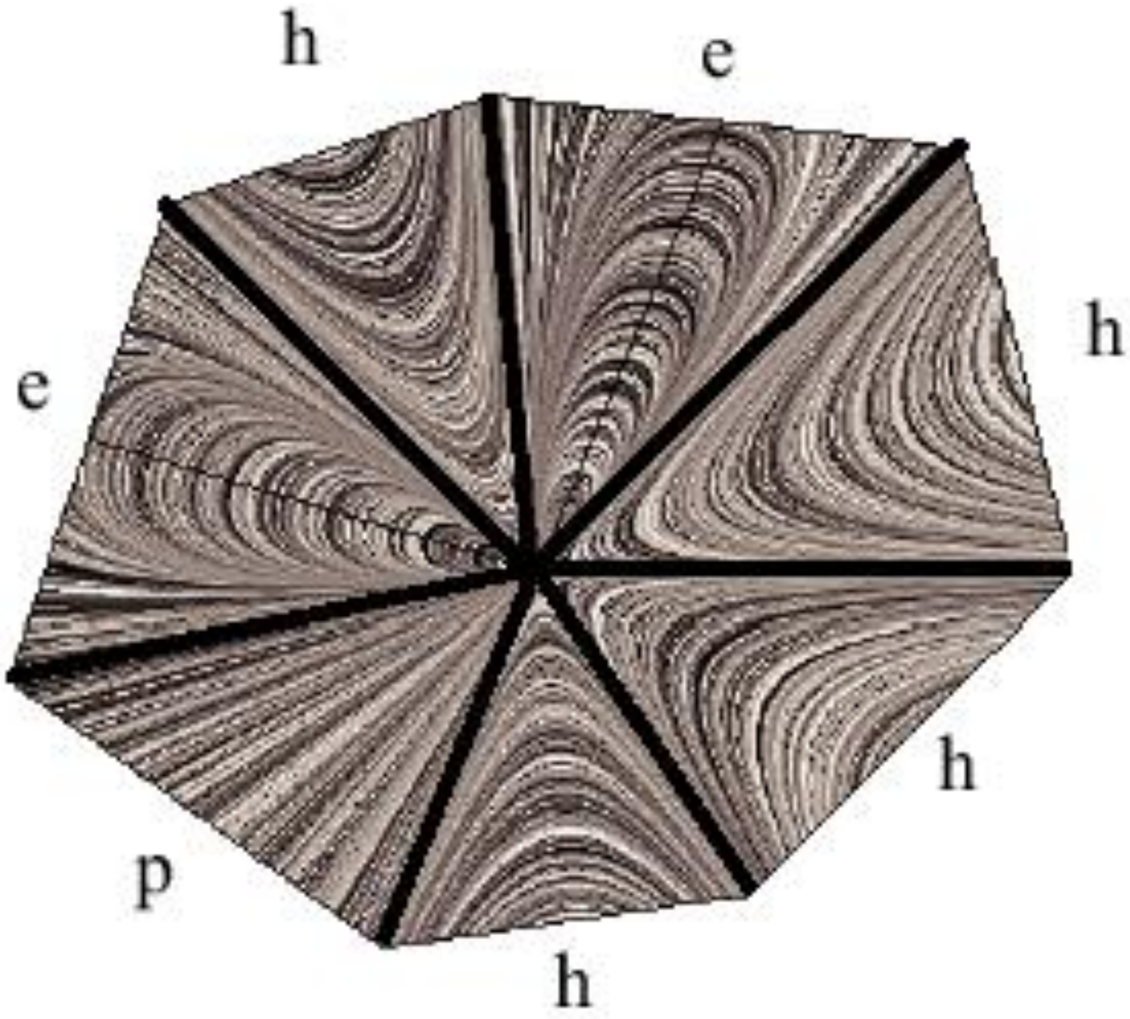
The different types of critical points can be classified by the eigenvalues of the Jacobian:

- R_1, R_2 ➔ real part of the eigenvalue
  I_1, I_2 ➔ imaginary part of the eigenvalue

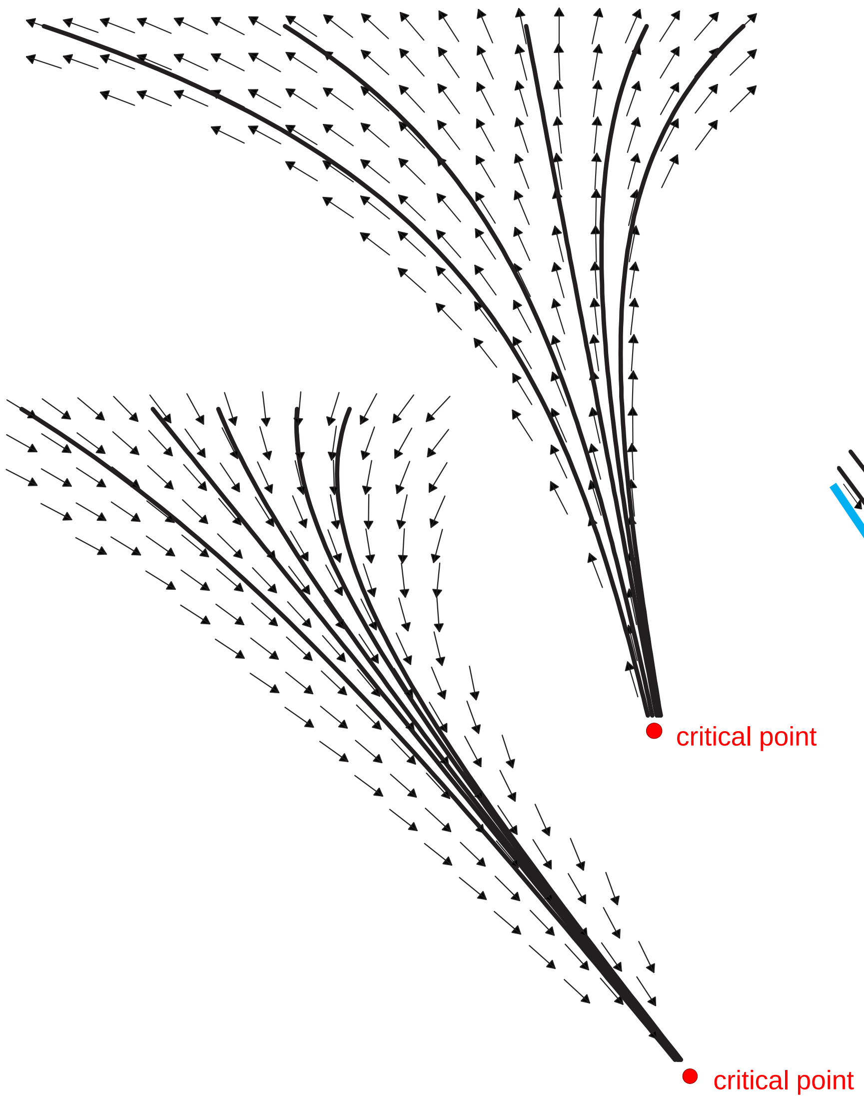


General classification of 2D critical points

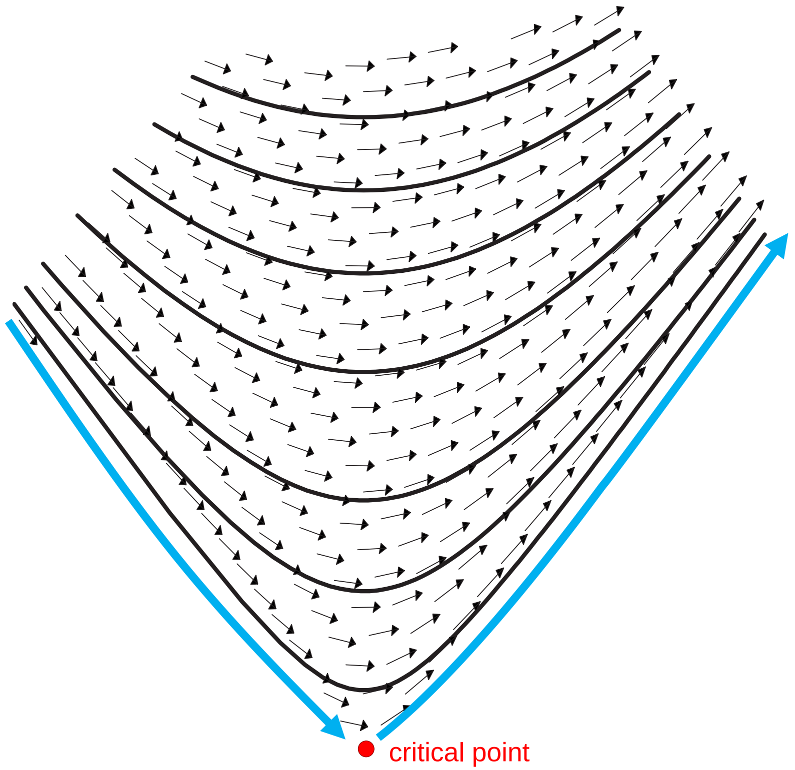
Distinguish regions of different flow behavior around a critical point



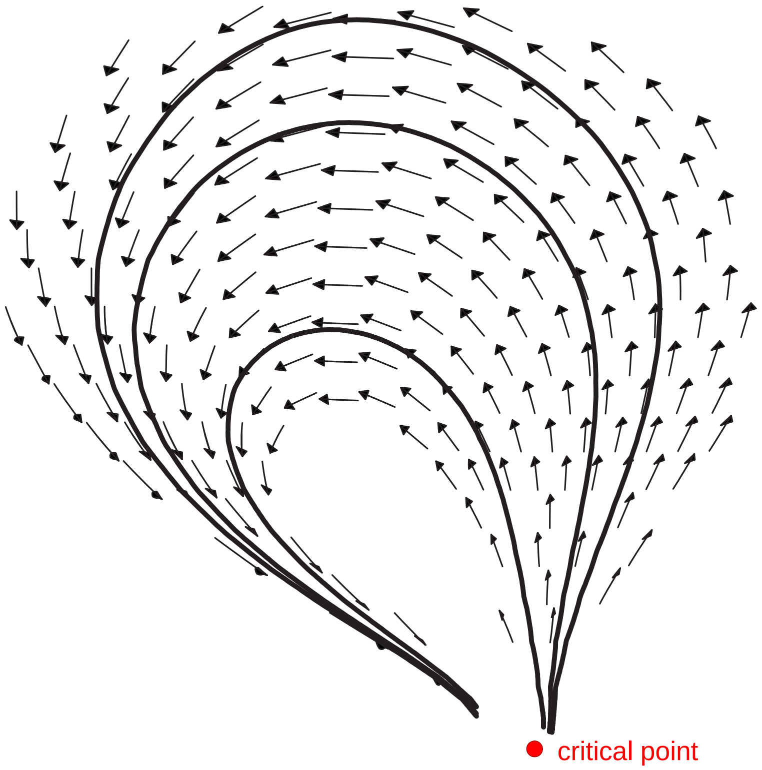
critical point consisting of 7 sectors



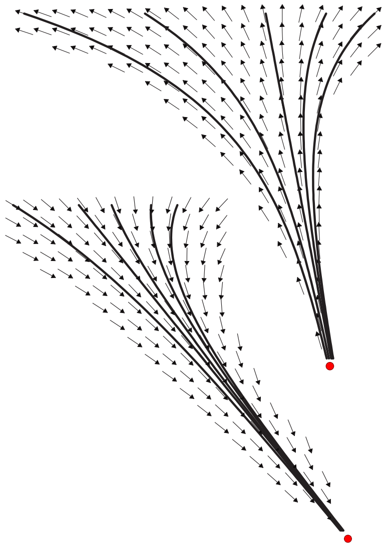
parabolic sectors



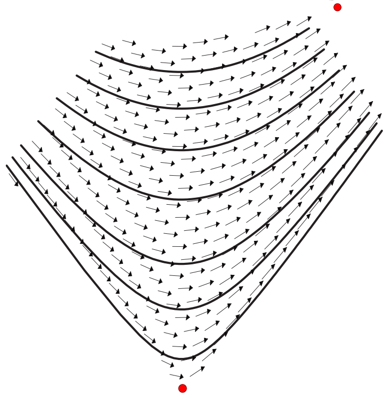
hyperbolic sector



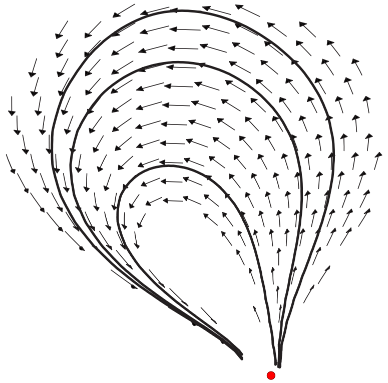
elliptic sector



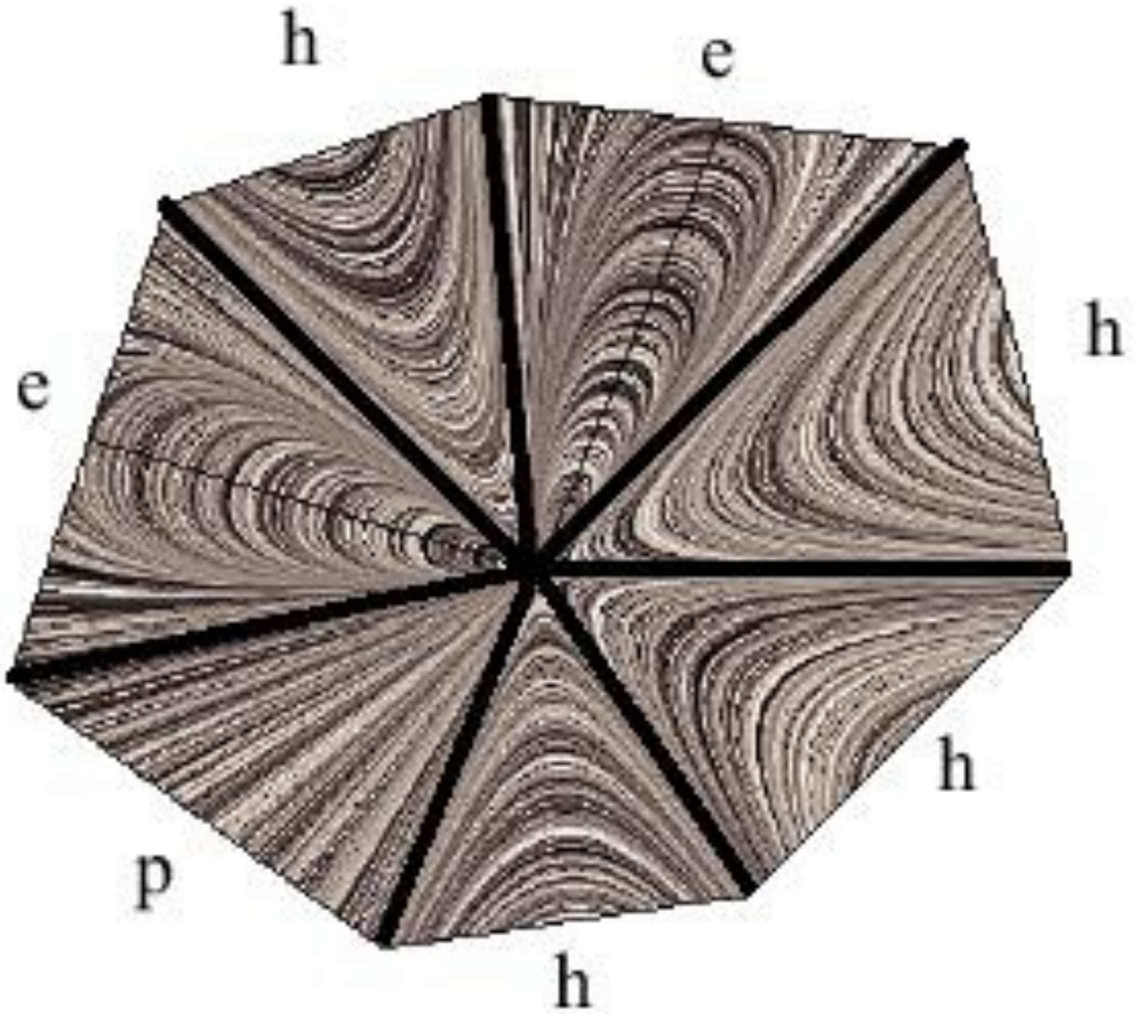
parabolic sectors



hyperbolic sector



elliptic sector



critical point consisting of 7 sectors

General classification of 2D critical points

Distinguish regions of different flow behavior around a critical point.

3 cases are possible:

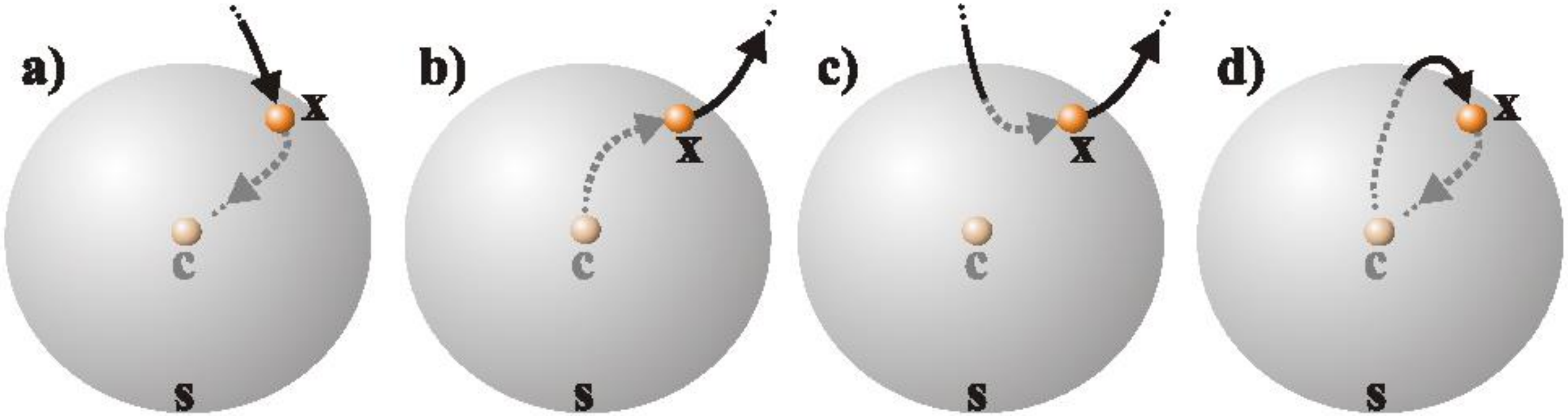
- In a ***parabolic sector*** either all tangent curves end, or all tangent curves originate, near the critical point.
- In a ***hyperbolic sector*** all tangent curves pass by the critical point, except for two tangent curves making the boundaries of the sector. One of these two tangent curves ends near the critical point while the other one originates near it.
- In an ***elliptic sector*** all tangent curves originate and end near the critical point.

General classification of 3D critical points

- place small sphere around point
- classify inflow/outflow behavior of each point on the sphere
  ➔ segments of similar behavior

x is part of:

a) inflow sector
b) outflow sector
c) hyperbolic sector
d) elliptic sector



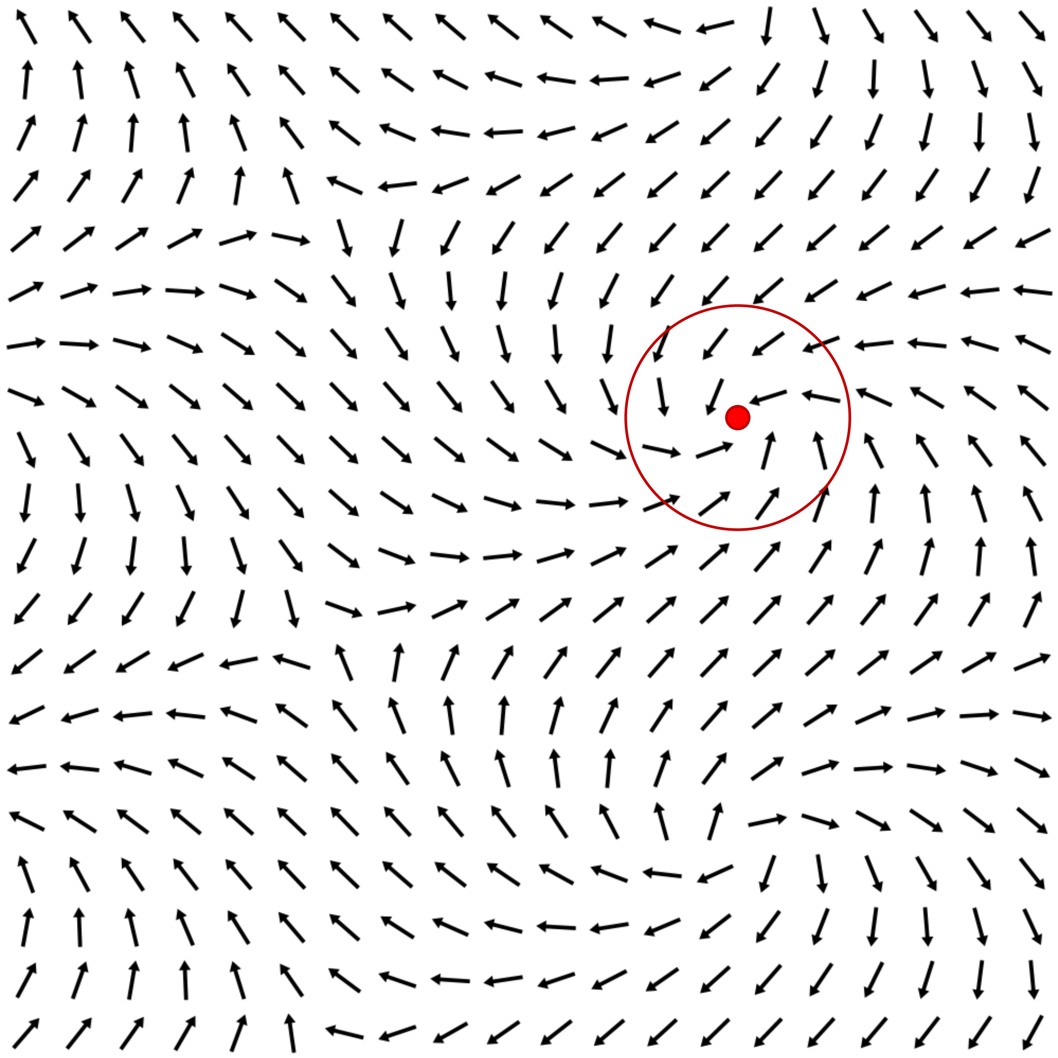
Poincaré-Index of a Critical Point

place small closed curve
around critical point

index: number of counterclockwise rotations of the vectors of **v** while traveling counterclockwise on the closed curve

index is an integer

possibly negative



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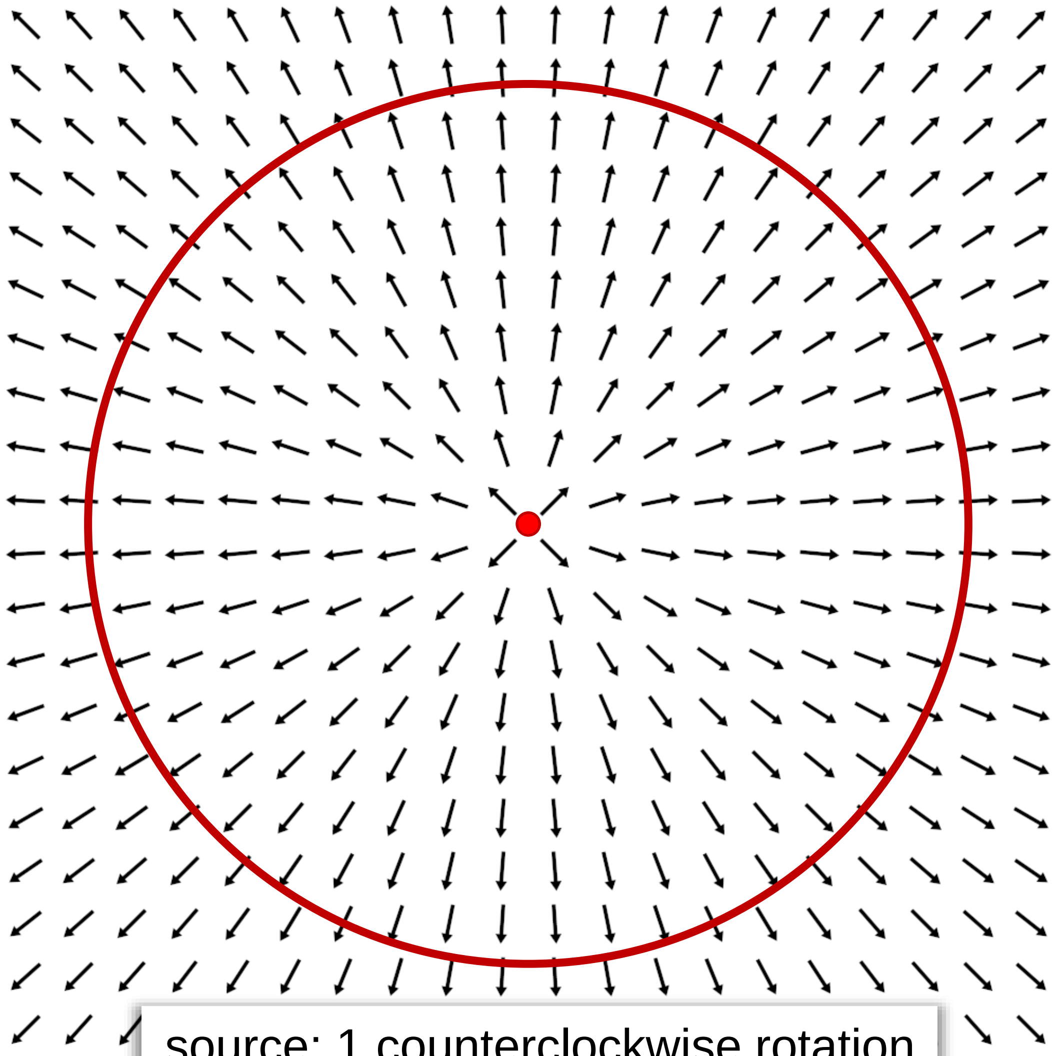


source: 1 counterclockwise rotation
index = +1

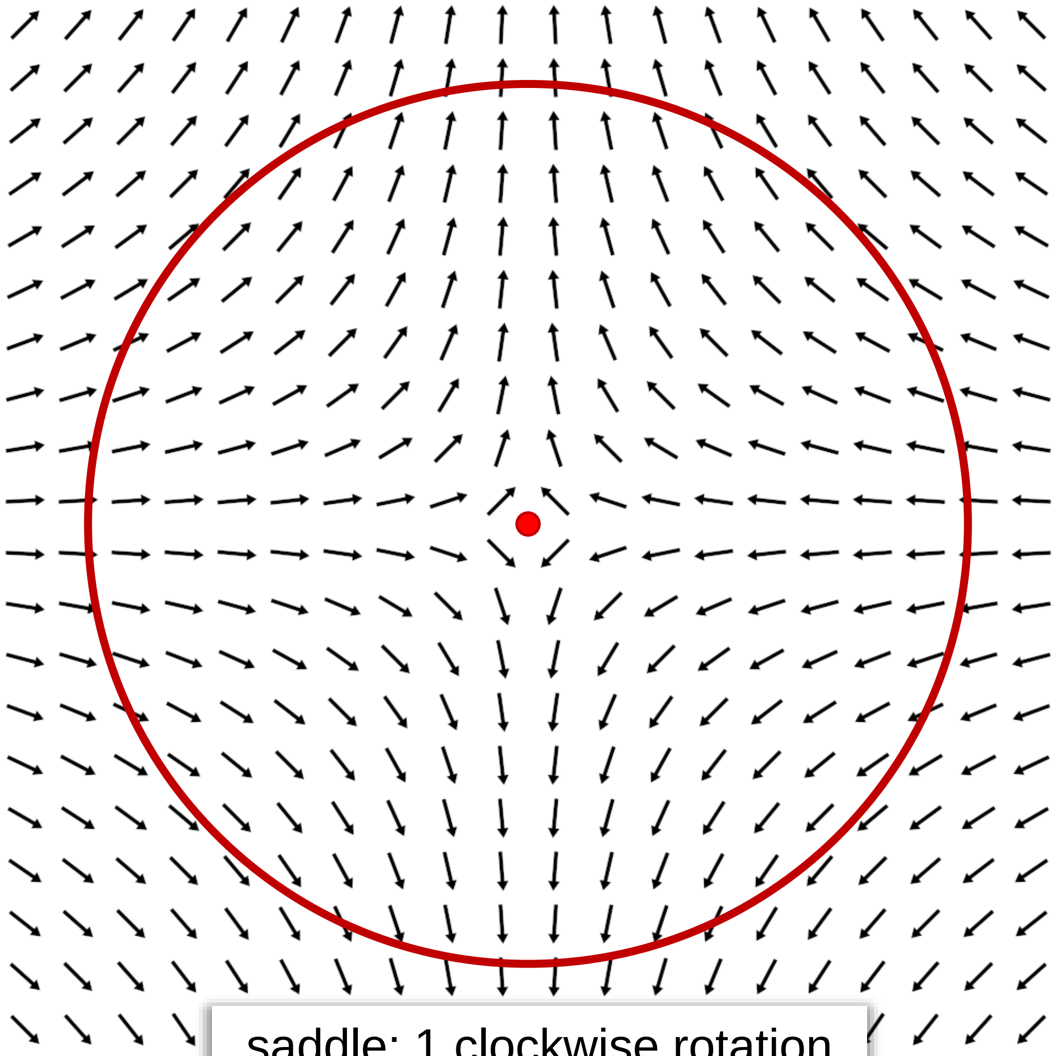
Poincaré-Index of a Critical Point

place small closed curve around critical point

index: number of counterclockwise rotations of the vectors of **v** while traveling counterclockwise on the closed curve

index is an integer

possibly negative



saddle: 1 clockwise rotation
index = -1

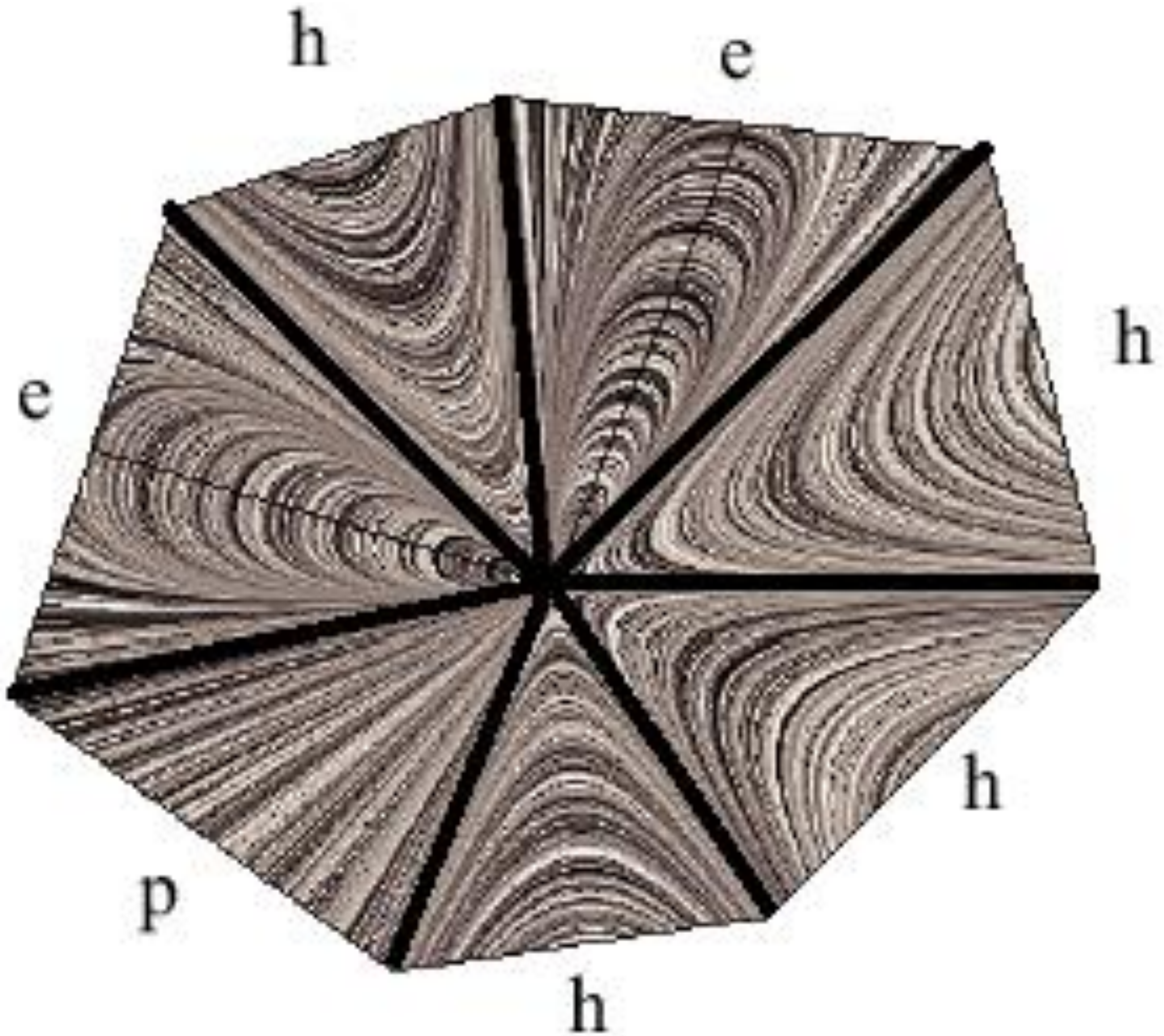
Poincaré-Index of a 2D Critical Point

index can be computed based on sectors around critical point

$$\text{index} = 1 + \frac{n_e - n_h}{2}$$

n_e: number of elliptic sectors

n_h: number of hyperbolic sectors



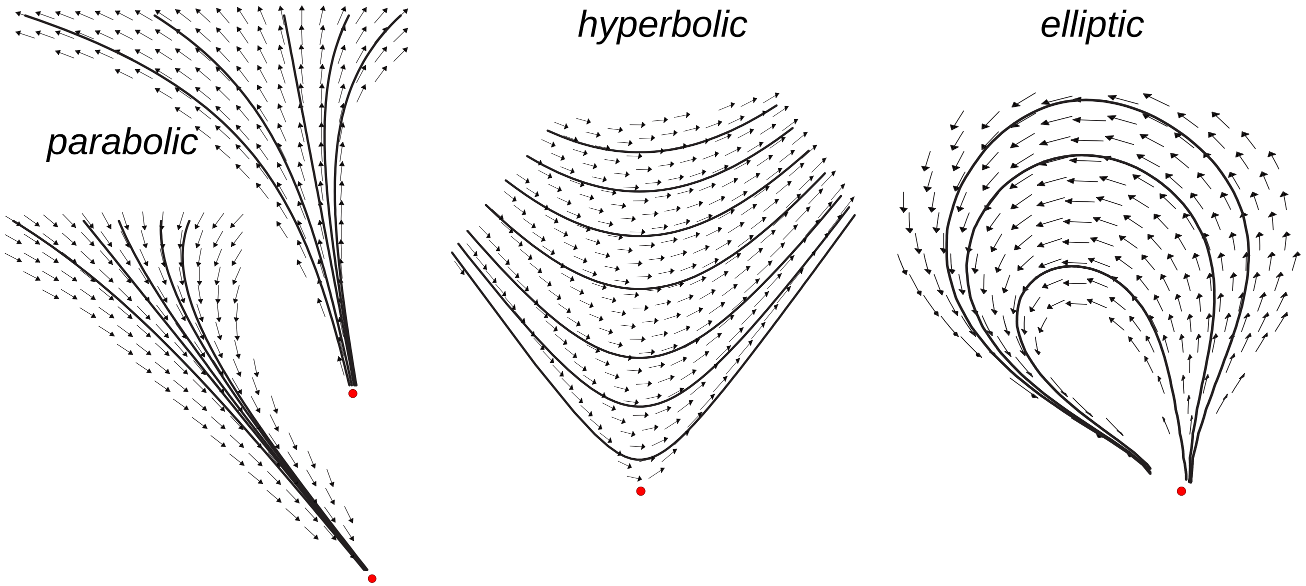
critical point consisting of 7 sectors

Poincaré-Index of a First-Order Critical Point in 2D

$$\text{index} = 1 + \frac{n_e - n_h}{2}$$

n_e: number of elliptic sectors

n_h: number of hyperbolic sectors



saddle

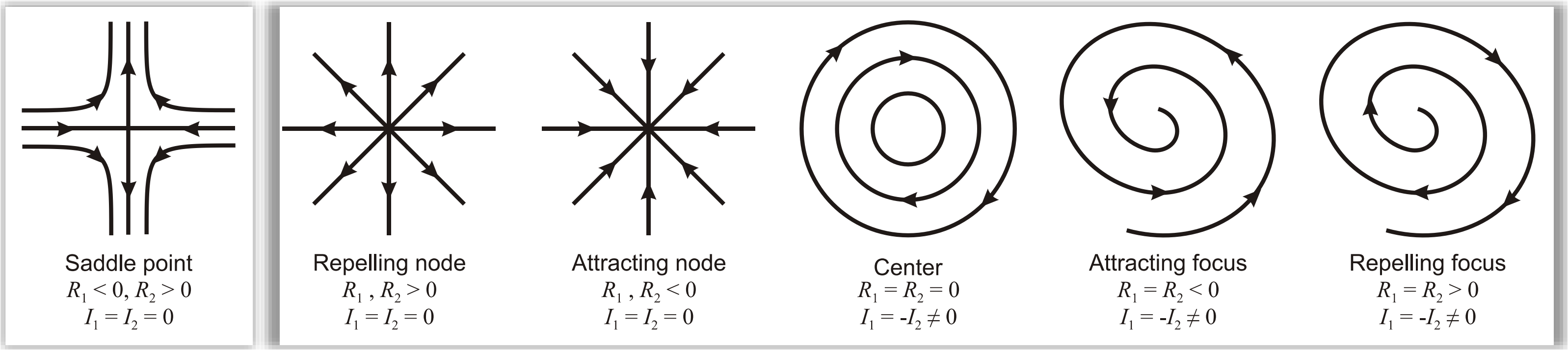
4 hyperbolic sectors

source/sink/center

one parabolic sector

index(saddle) = -1

index(source/sink/center) = +1

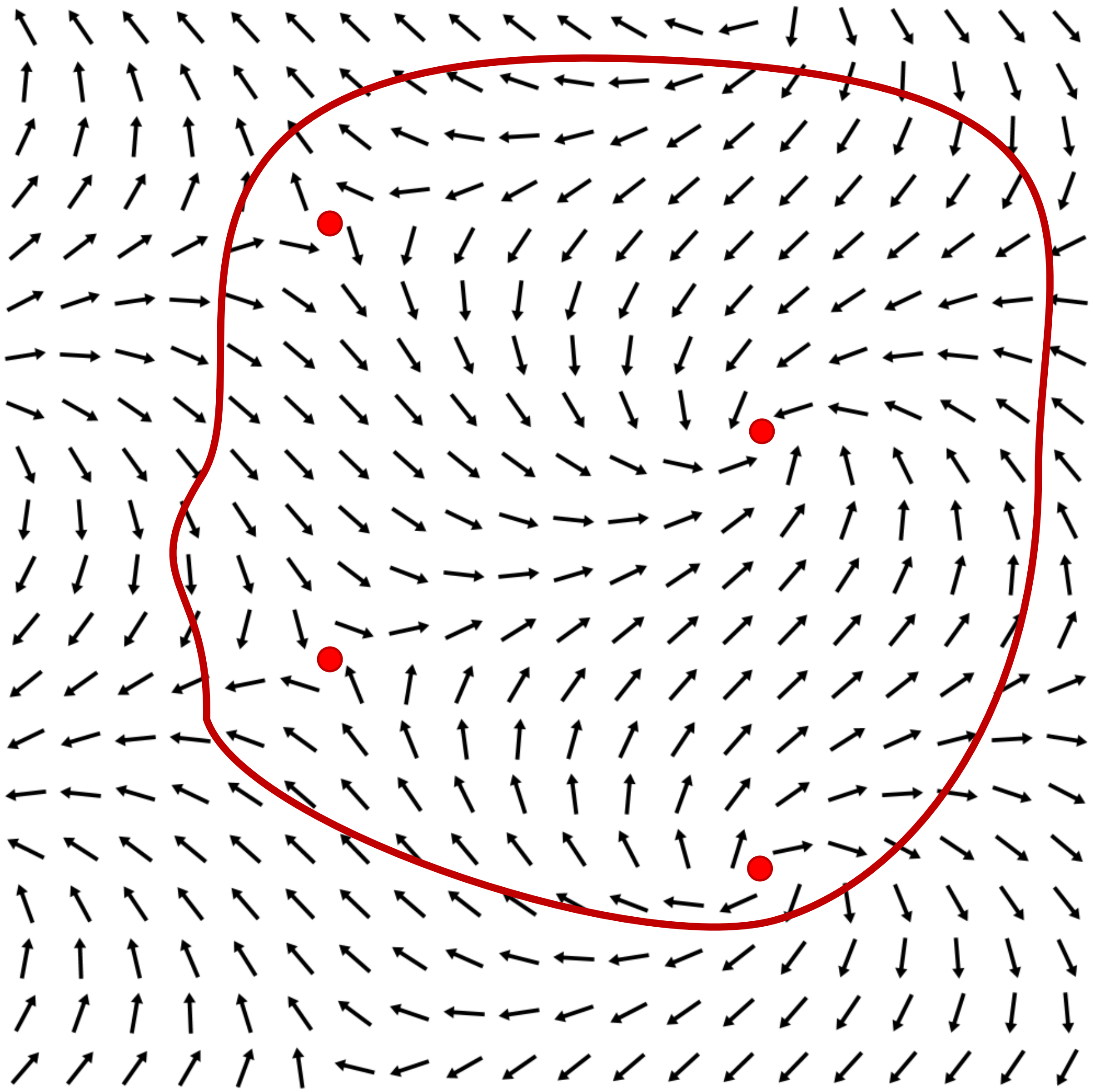


Index Theorem

The Poincaré index of an area equals the sum of the indices of the critical points in the area.

$$\mathrm{index}(\mathrm{area}) = \sum \mathrm{index}(\mathbf{x}_i)$$

$\mathbf{x}_i$: critical points in the area



Poincaré-Index of a First-Order Critical Point in 3D

		Index:
Sources:	$0 < Re(\lambda_1) \leq Re(\lambda_2) \leq Re(\lambda_3)$	1
Repelling saddles:	$Re(\lambda_1) < 0 < Re(\lambda_2) \leq Re(\lambda_3)$	-1
Attracting saddles:	$Re(\lambda_1) \leq Re(\lambda_2) < 0 < Re(\lambda_3)$	1
Sinks:	$Re(\lambda_1) \leq Re(\lambda_2) \leq Re(\lambda_3) < 0$	-1

Scalar Field Topology

2D scalar field

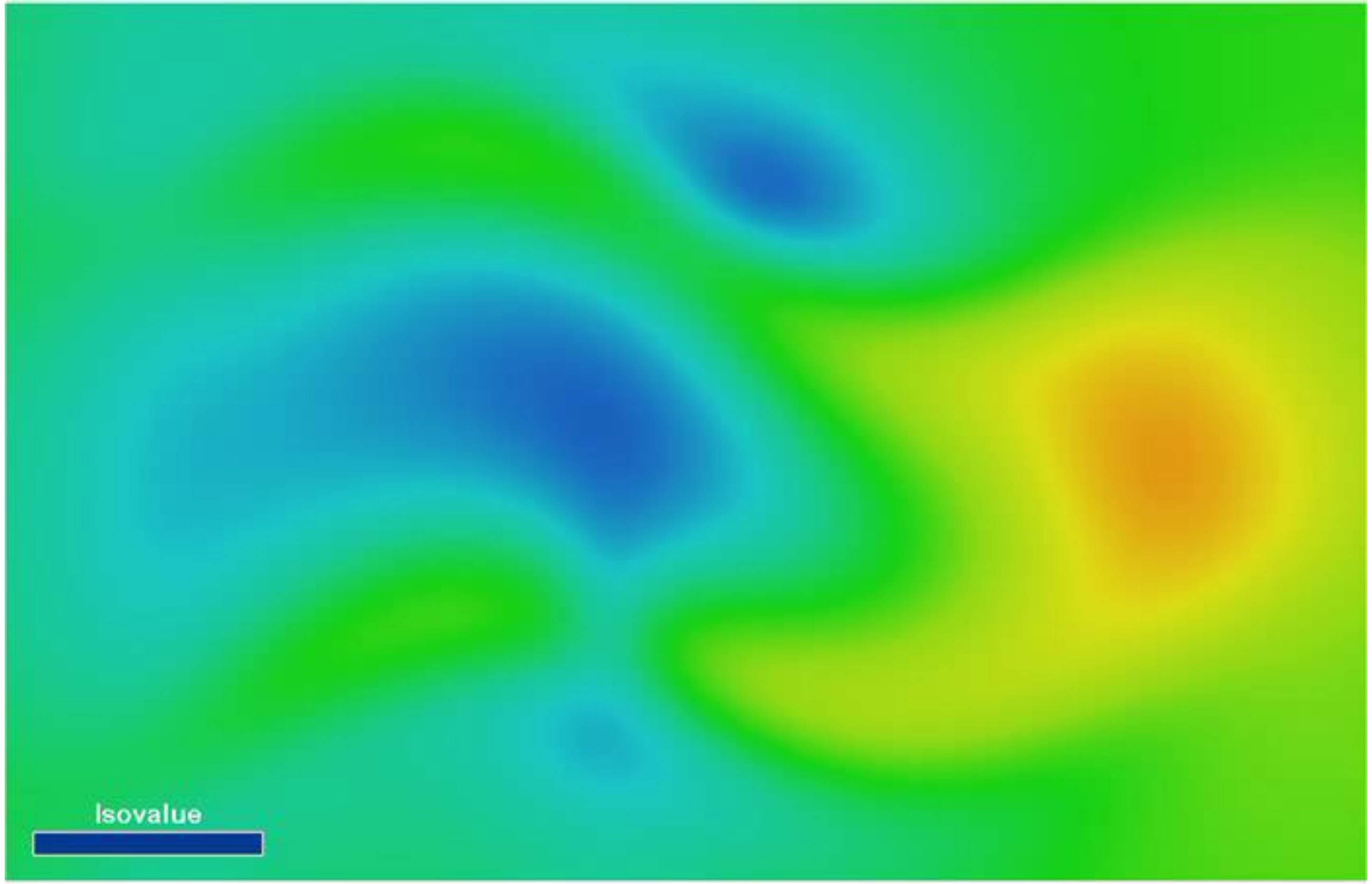
increasing isovalue

events for contours

appear

merge / split

disappear



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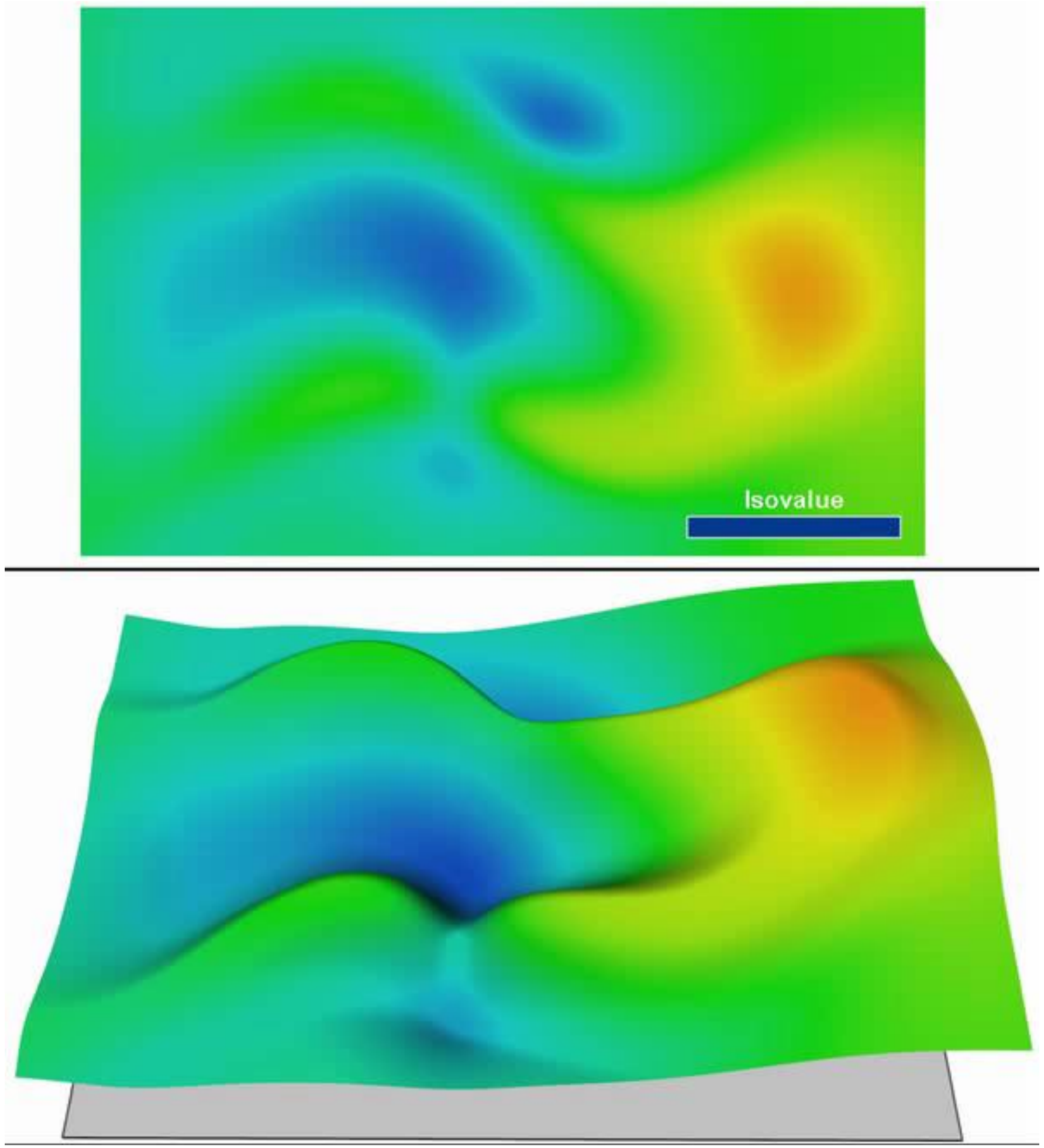

Scalar Field Topology

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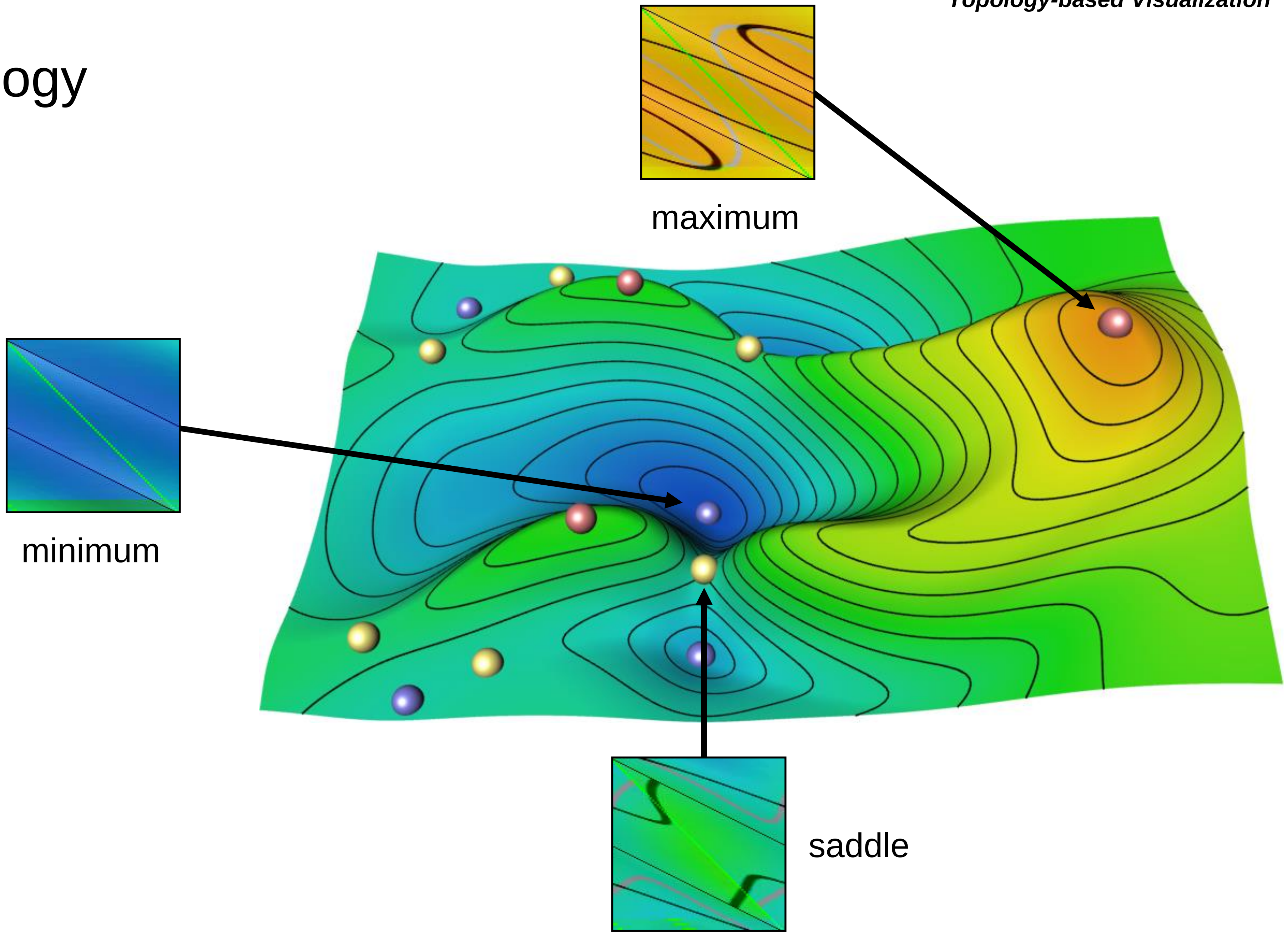
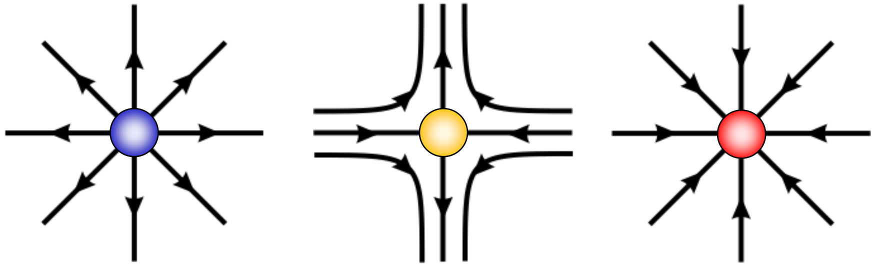


Scalar Field Topology = Vector Field Topology of the Gradient

critical points as defined by the gradient

$$\mathbf{v}(\mathbf{x}_0) = \mathbf{0} \quad \text{with} \quad \mathbf{v}(\mathbf{x}_0 \pm \boldsymbol{\epsilon}) \neq \mathbf{0}$$

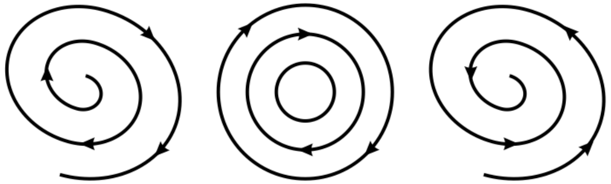


Repelling node
$R_1, R_2 > 0$
$I_1 = I_2 = 0$

Saddle point
$R_1 < 0, R_2 > 0$
$I_1 = I_2 = 0$

Attracting node
$R_1, R_2 < 0$
$I_1 = I_2 = 0$

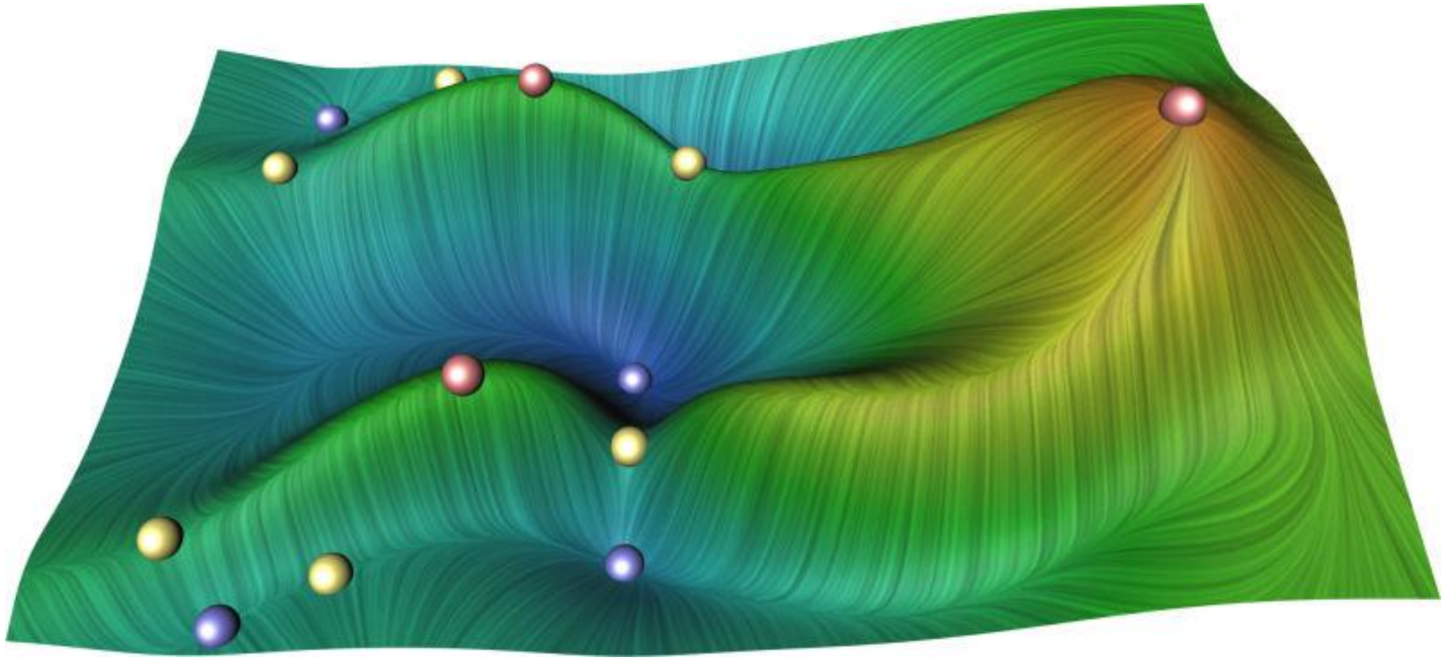
general vector fields



Repelling focus
$R_1 = R_2 > 0$
$I_1 = -I_2 \neq 0$

Center
$R_1 = R_2 = 0$
$I_1 = -I_2 \neq 0$

Attracting focus
$R_1 = R_2 < 0$
$I_1 = -I_2 \neq 0$



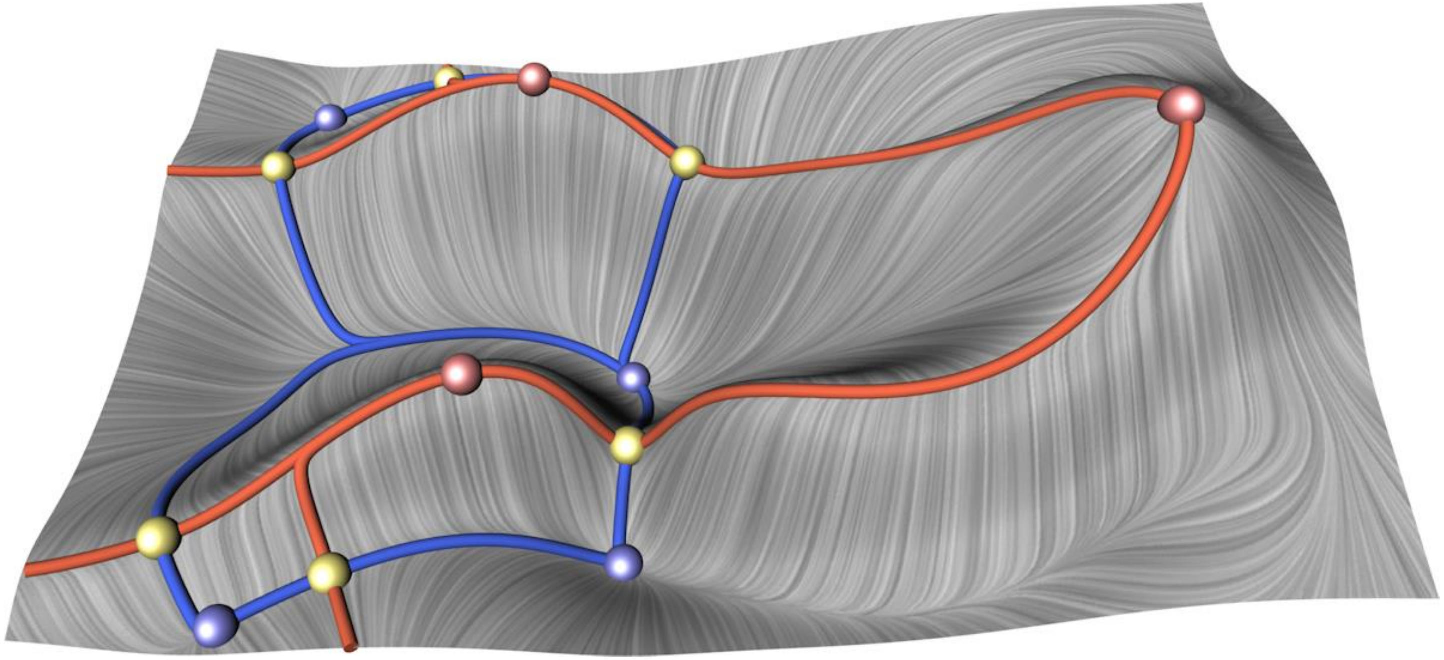
Scalar Field Topology = Vector Field Topology of the Gradient

separatrices

starting from saddles

tangent curves of the gradient

follow the steepest ascend



Scalar Field Topology = Vector Field Topology of the Gradient

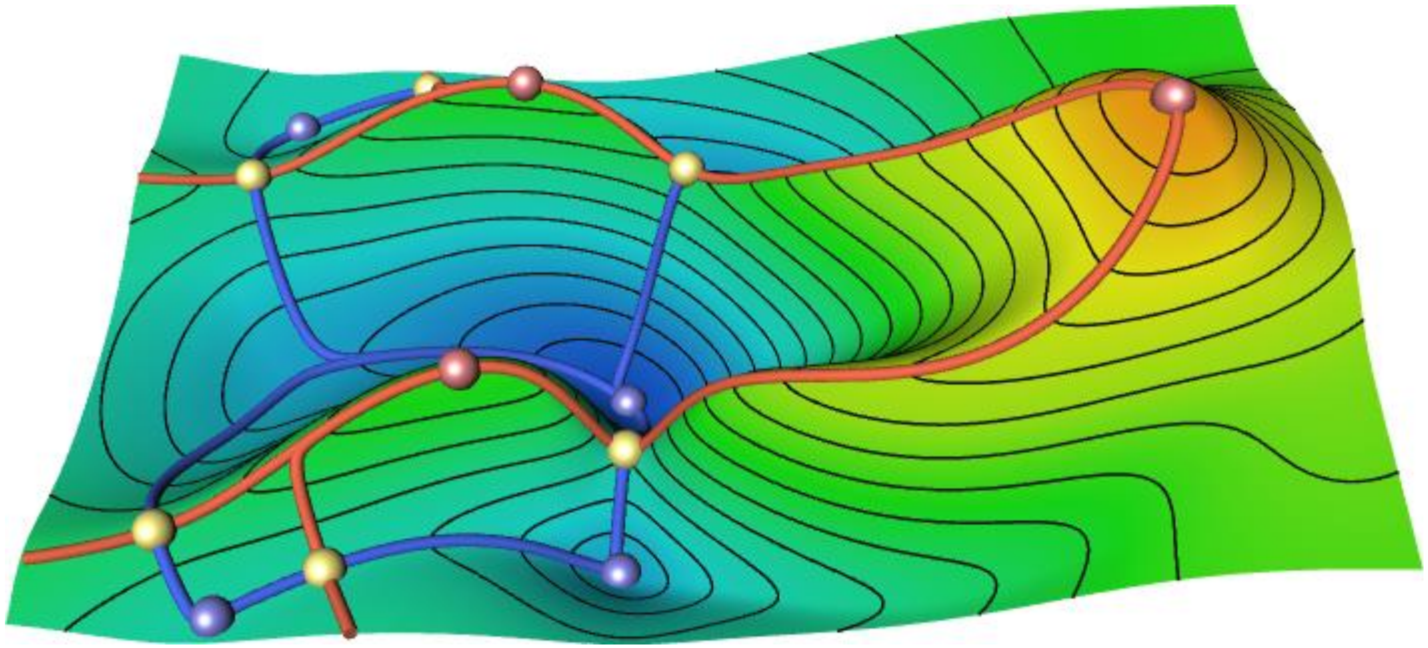
separatrices

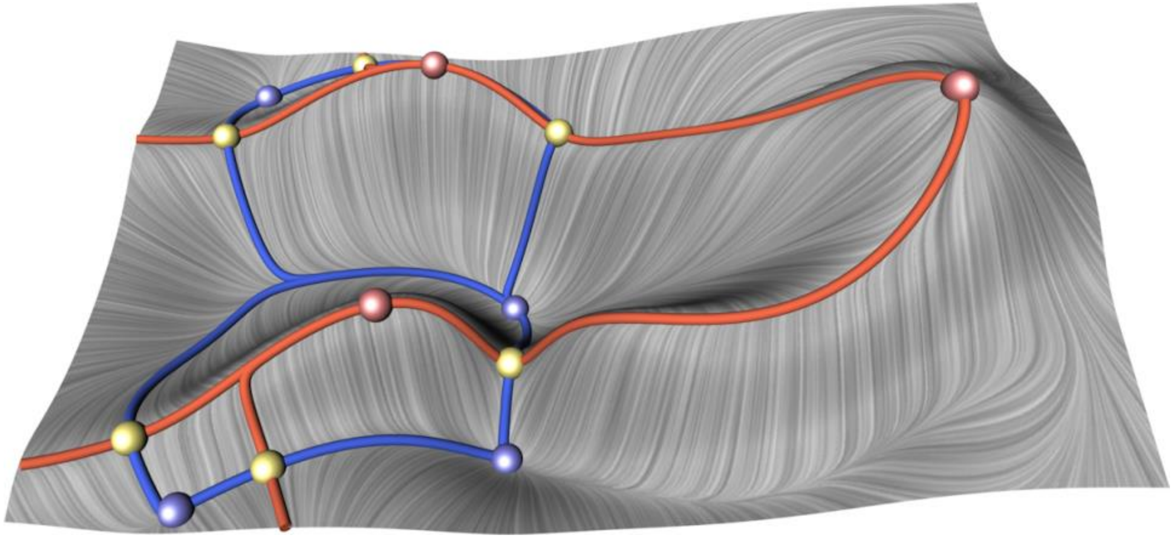
starting from saddles

tangent curves of the gradient

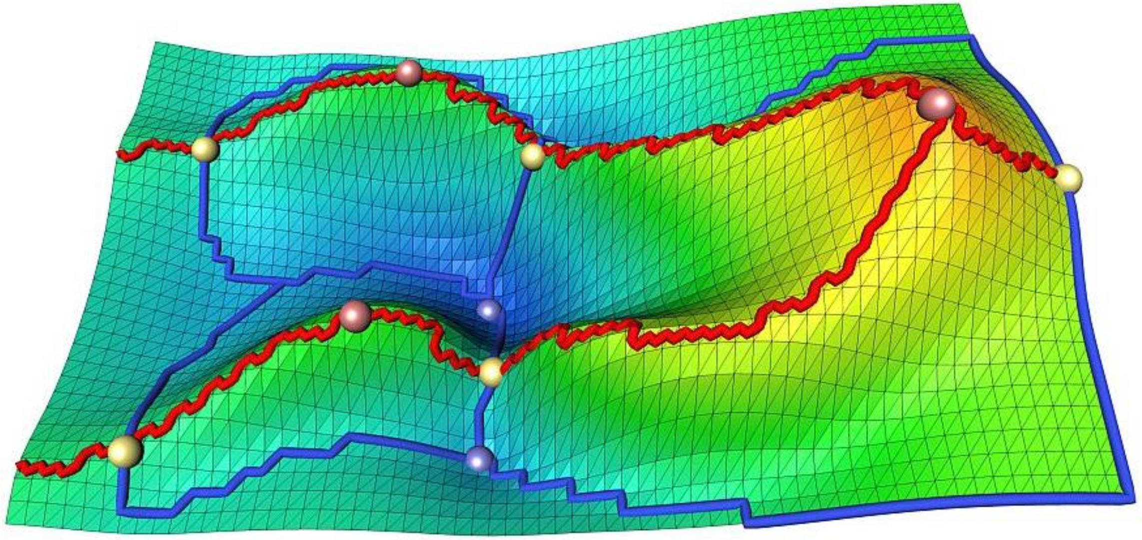
follow the steepest ascend

perpendicular to contours





Continuous view



Discrete view

Discrete Morse theory (R. Forman)

completely combinatorial

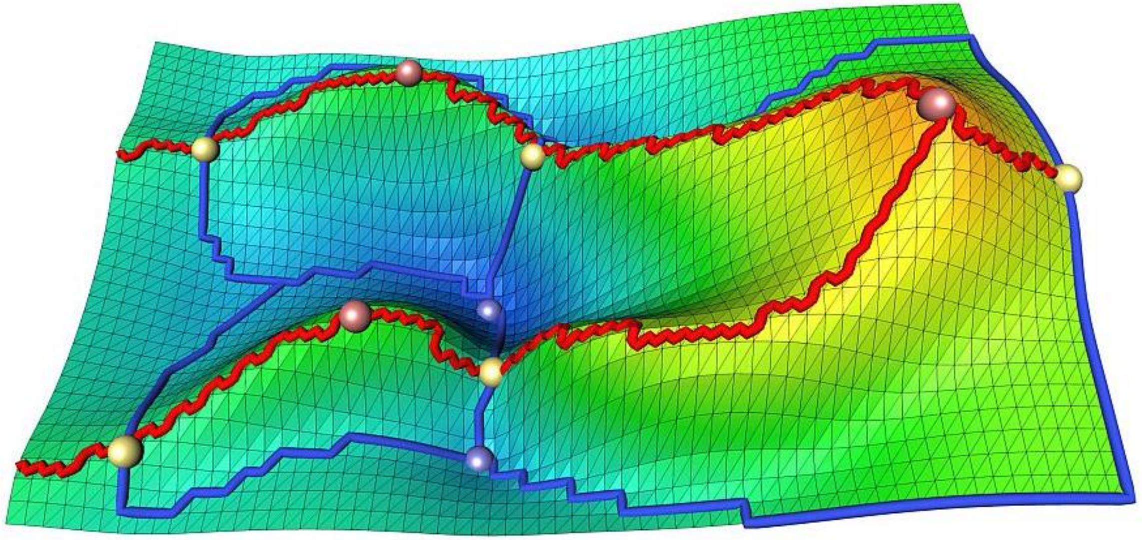
no numerics

no derivatives

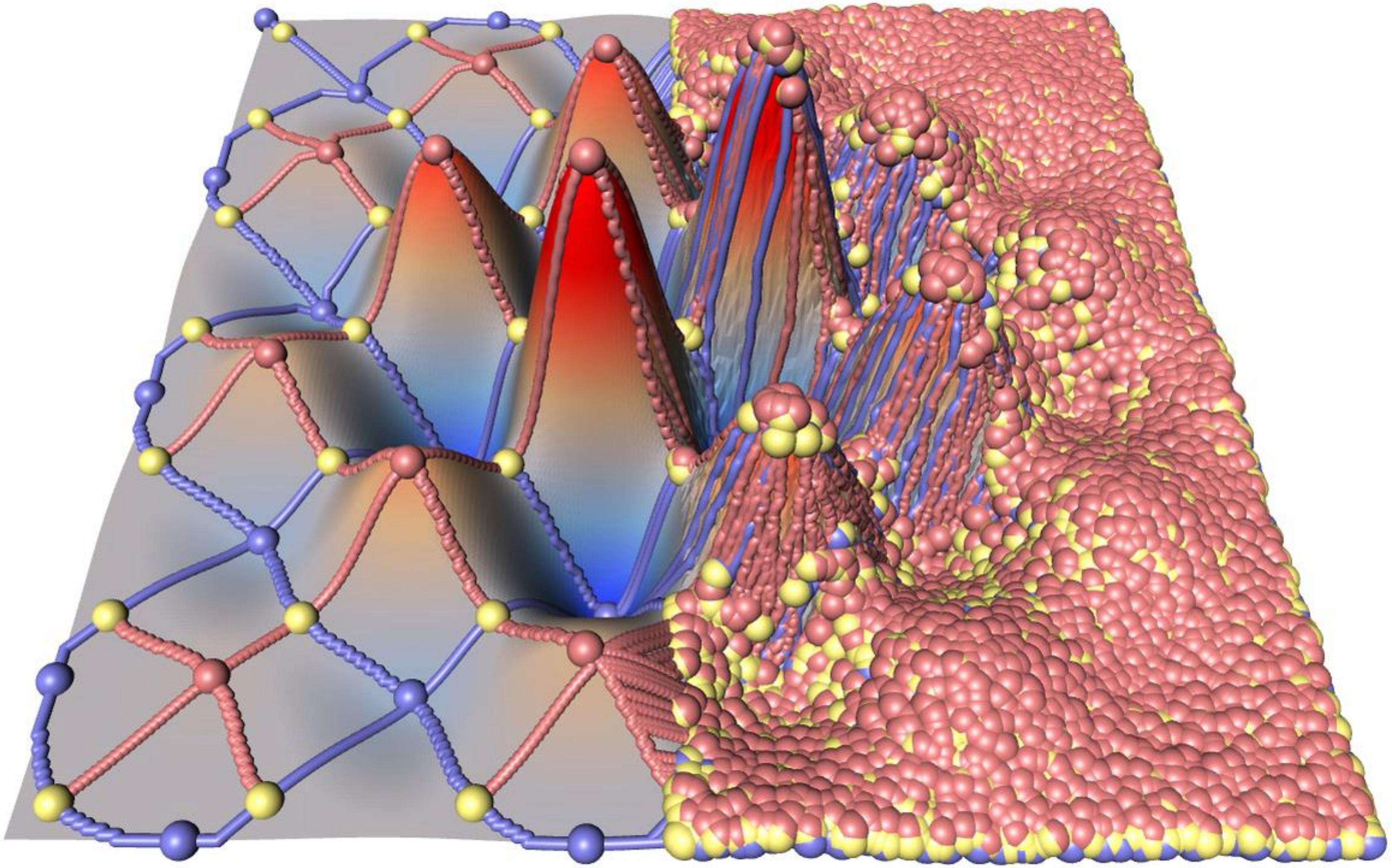
no interpolation

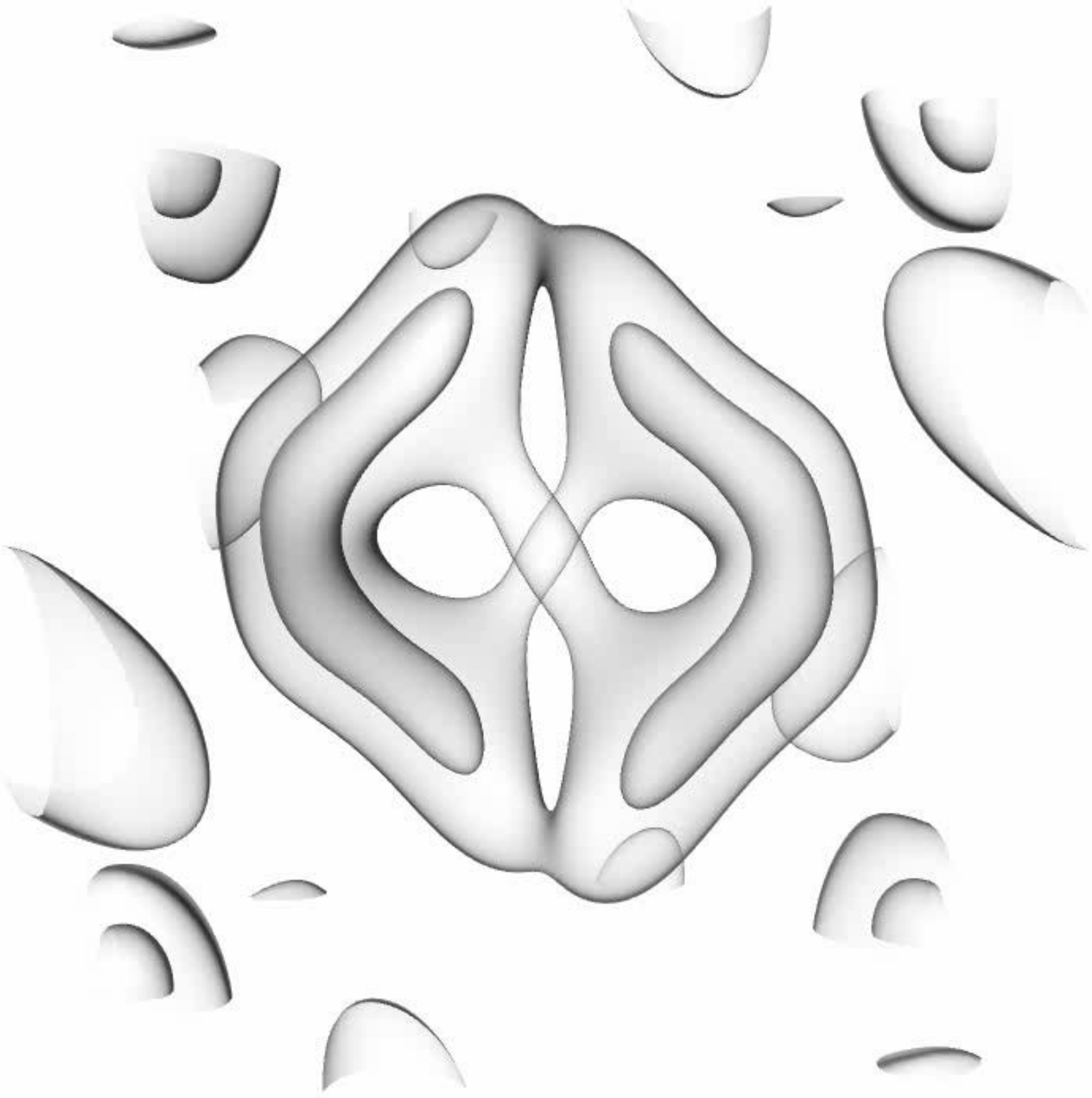
no integration of separatrices

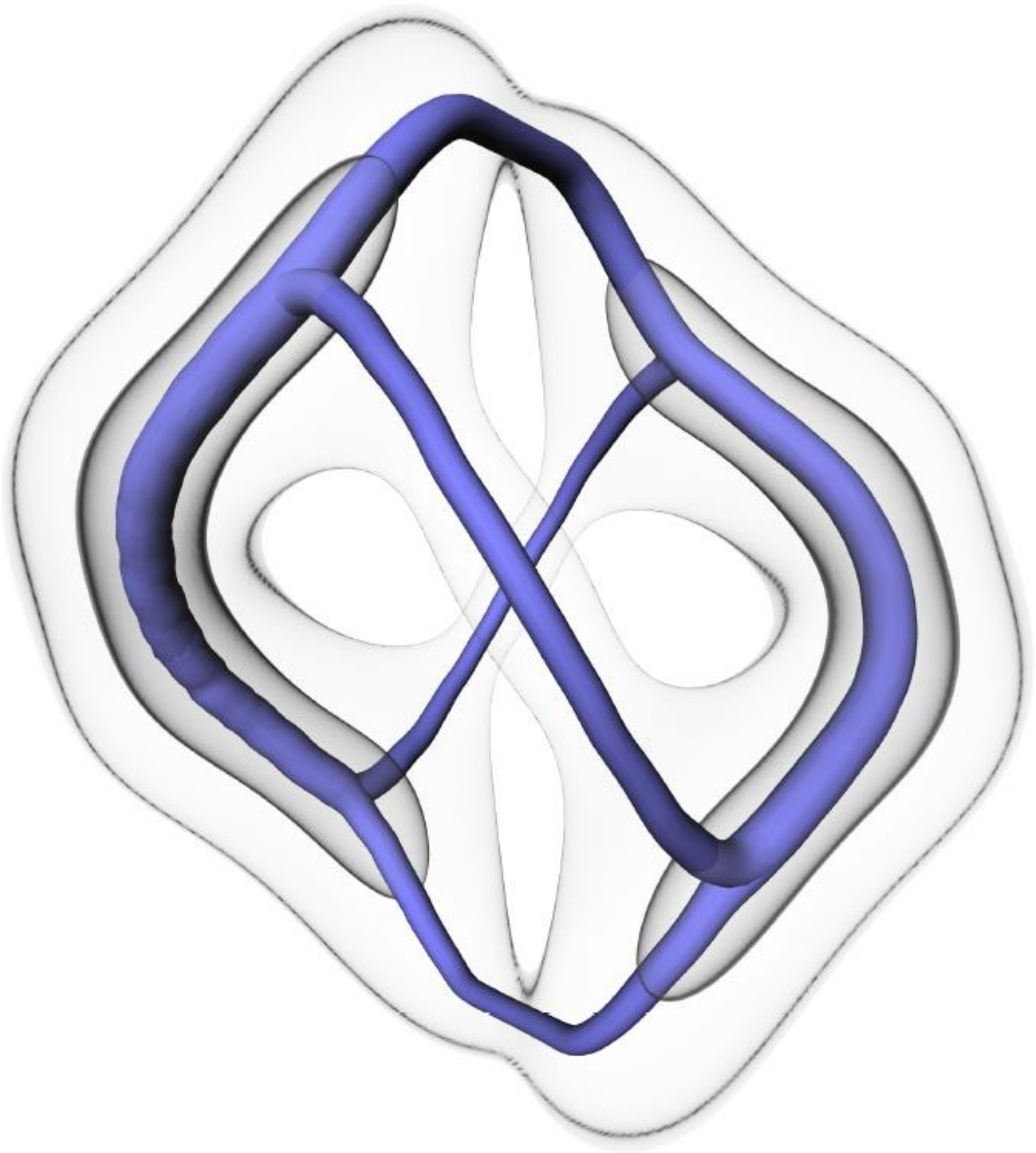
no parameters

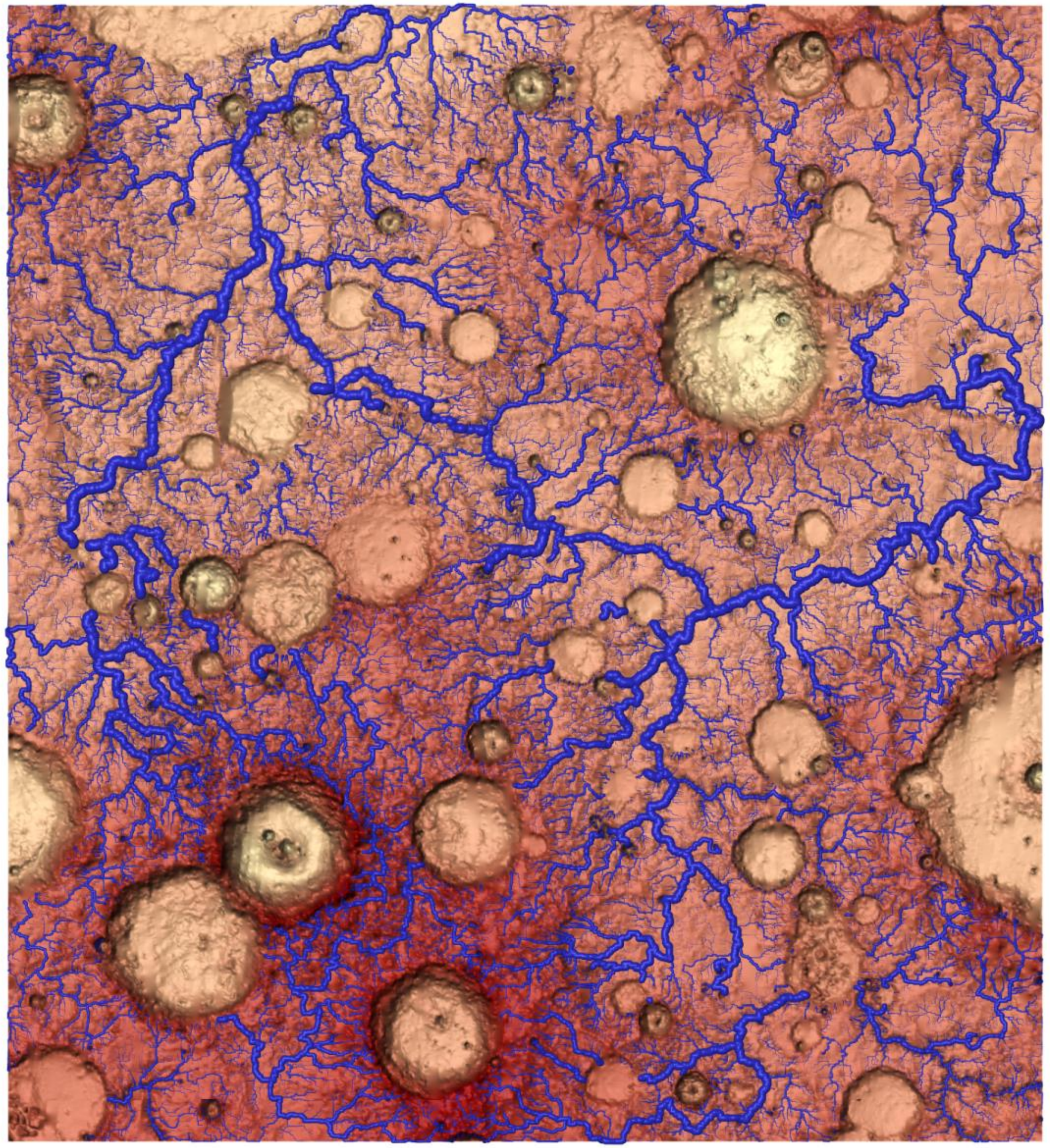


Discrete view









Valleys in a subregion of the Martian surface:

- Noisy data yields complex network
- Separatrix persistence allows differentiation between prominent and noise-induced valleys

Summary

- Topology-based methods focus on the structure of data
- Vector Field Topology 2D / 3D
  - sectors of different flow behavior
  - critical points
  - separatrices
  - interactions with boundary
  - 3D: saddle connectors
- General classification of critical points
- Poincaré index
- Scalar Field Topology
  - vector field topology of the gradient
  - discrete / combinatorial methods for extraction
  - noise removal