Visualization, DD2257

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Derived Quantities

scalar field

$$s : \mathbb{E}^n \rightarrow \mathbb{R}$$

$$s(\mathbf{x})$$
with $\mathbf{x} \in \mathbb{E}^n$

vector field

$$\mathbf{v} : \mathbb{E}^n \rightarrow \mathbb{R}^m$$

$$\mathbf{v}(\mathbf{x}) = \begin{pmatrix} c_1(\mathbf{x}) \\ \vdots \\ c_m(\mathbf{x}) \end{pmatrix}$$
with $\mathbf{x} \in \mathbb{E}^n$

tensor field

$$\mathbf{T} : \mathbb{E}^n \rightarrow \mathbb{R}^{m \times b}$$

$$\mathbf{T}(\mathbf{x}) = \begin{pmatrix} c_{11}(\mathbf{x}) & \dots & c_{1b}(\mathbf{x}) \\ \vdots & & \vdots \\ c_{m1}(\mathbf{x}) & \dots & c_{mb}(\mathbf{x}) \end{pmatrix}$$
with $\mathbf{x} \in \mathbb{E}^n$

scalar field

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with $\mathbf{x} \in \mathbb{E}^n$

The first derivative of a scalar field is a vector field called **gradient**.
It consists of the partial derivatives of the scalar function $s(\mathbf{x})$ for each dimension of the observation space.

$$s(x, y) \dashrightarrow \nabla s(x, y) = \begin{pmatrix} \frac{\partial s}{\partial x} \\ \frac{\partial s}{\partial y} \end{pmatrix} = \begin{pmatrix} s_x \\ s_y \end{pmatrix}$$

2D scalar field

gradient

scalar field

$$s : \mathbb{E}^n \rightarrow \mathbb{R}$$

$$s(\mathbf{x})$$
with $\mathbf{x} \in \mathbb{E}^n$

vector field

$$\mathbf{v} : \mathbb{E}^n \rightarrow \mathbb{R}^m$$

$$\begin{pmatrix} c_1(\mathbf{x}) \end{pmatrix}$$

tensor field

$$\mathbf{T} : \mathbb{E}^n \rightarrow \mathbb{R}^{m \times b}$$

$$\mathbf{T}(\mathbf{x}) = \begin{pmatrix} c_{11}(\mathbf{x}) & \dots & c_{1b}(\mathbf{x}) \\ \vdots & & \vdots \\ c_{m1}(\mathbf{x}) & \dots & c_{mb}(\mathbf{x}) \end{pmatrix}$$
with $\mathbf{x} \in \mathbb{E}^n$

The second derivative of a scalar field is a tensor field called **Hessian**. It consists of the partial derivatives of $s(\mathbf{x})$ derived twice for each dimension of the observation space.

$$s(x, y)$$

2D scalar field

$$\nabla s(x, y) = \begin{pmatrix} \frac{\partial s}{\partial x} \\ \frac{\partial s}{\partial y} \end{pmatrix} = \begin{pmatrix} s_x \\ s_y \end{pmatrix}$$

gradient

$$\nabla^2 s(x, y) = \begin{pmatrix} s_{xx} & s_{xy} \\ s_{yx} & s_{yy} \end{pmatrix}$$

Hessian

scalar field

$$s : \mathbb{E}^n \rightarrow \mathbb{R}$$

$$s(\mathbf{x})$$
with $\mathbf{x} \in \mathbb{E}^n$

vector field

$$\mathbf{v} : \mathbb{E}^n \rightarrow \mathbb{R}^m$$

$$\Big(c_1(\mathbf{x})$$

The first derivative of a vector field is a tensor field called **Jacobian**. It consists of the partial derivatives of $\mathbf{v}(\mathbf{x})$ for each dimension of the observation space.

$$\mathbf{v}(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$$

2D vector field

tensor field

$$\mathbf{T} : \mathbb{E}^n \rightarrow \mathbb{R}^{m \times b}$$

$$\mathbf{T}(\mathbf{x}) = \begin{pmatrix} c_{11}(\mathbf{x}) & \dots & c_{1b}(\mathbf{x}) \\ \vdots & & \vdots \\ c_{m1}(\mathbf{x}) & \dots & c_{mb}(\mathbf{x}) \end{pmatrix}$$
with $\mathbf{x} \in \mathbb{E}^n$

$$\nabla \mathbf{v}(x, y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

Jacobian

$$s(x, y)$$

2D scalar field

$$\mathbf{v}(x, y) = \begin{pmatrix} u \\ v \end{pmatrix}$$

2D vector field

Gradient of a 2D scalar field

$$\mathbf{J}(x, y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

Jacobian of a 2D vector field

Hessian of a 2D scalar field

$$s(x, y, z)$$

3D scalar field

$$\mathbf{v}(x, y, z) = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

3D vector field

Gradient of a 3D scalar field

$$\mathbf{J}(x, y, z) = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$$

Jacobian of a 3D vector field

Hessian of a 3D scalar field

- **Divergence of v:**
- scalar field
- observe transport of a small ball around a point
  - expanding volume ➔ positive divergence
  - contracting volume ➔ negative divergence
  - constant volume ➔ zero divergence

$$\operatorname{div}\ \mathbf{v} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta w}{\delta z} = u_x + v_y + w_z$$

$\operatorname{div}\ \mathbf{v} \equiv 0 \Leftrightarrow \mathbf{v}$ is incompressible

- **Laplacian of a scalar field:**
- Scalar field
- Divergence of the gradient of the scalar field

$$Lf = \mathrm{div}\,\mathbf{grad}\ f = \mathrm{div}\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

$$= \frac{\delta^2 f}{\delta x^2} + \frac{\delta^2 f}{\delta y^2} + \frac{\delta^2 f}{\delta z^2} = f_{xx} + f_{yy} + f_{zz}$$

- **Interpretation of Laplacian:**
- Measure of the difference between the average value of *f* in the immediate neighborhood of the point and the precise value of the field at the point.

- **Properties of the Laplacian of a scalar field:**
- L invariant under rotation and translation of the underlying coordinate system
- $L\,f \equiv \mathbf{0} \Leftrightarrow f$ is harmonic function

- **Curl of v:**
- vector field
- also called rotation (rot) or vorticity
- indication of how the field swirls at a point

$$\mathbf{curl}\ \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ u & v & w \end{vmatrix} = \begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix}$$

- **Curl of v:**

- paddle wheel model:
  - insert paddle wheel in a flow
  - orient such that its rate of rotation is maximal
  - ➔ **curl v** is parallel to main rotation axis
  - ➔ **|curl v|** is corresponds to rate of rotation

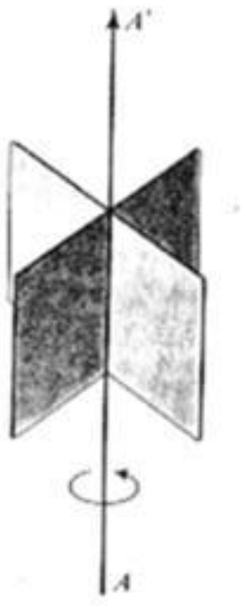


Figure 53

- golf ball model
  - consider golf ball in v
  - is transported and rotates
  - ➔ **curl v** is parallel to main rotation axis
  - ➔ **|curl v|** is corresponds to rate of rotation

- **Properties of curl:**

- **curl v** $\equiv$ **0** $\Leftrightarrow$ **v** is irrotational or curl-free

- **v** = grad *f* $\Leftrightarrow$ **v** is conservative

- Conservative is subclass of curl-free,
  since **curl** grad *f* $\equiv$ **0** for any scalar field *f*

- **The Nabla operator:**
- also called “Del”-operator
- abbreviation: ∇
- symbolically written as:

$$\nabla = \mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z} = \begin{pmatrix} \dfrac{\delta}{\delta x} \\ \dfrac{\delta}{\delta y} \\ \dfrac{\delta}{\delta z} \end{pmatrix}$$

- **The Nabla operator:**

- Allows us to write the other operators as:

$$\mathbf{grad}\, f = \nabla f$$

$$\mathrm{div}\, \mathbf{v} = \nabla \cdot \mathbf{v}$$

$$\mathbf{curl}\, \mathbf{v} = \nabla \times \mathbf{v}$$

$$L\, f = \mathrm{div}\,(\mathbf{grad}\, f) = \nabla \cdot (\nabla f) = \nabla^2 f$$

$$\mathbf{J}_{\mathbf{v}} = \nabla \mathbf{v}$$

- **Scalar and vector identities:**

$$\nabla(f+g)=\nabla f+\nabla g$$

$$\nabla(cf)=c\nabla f$$ for a constant c

$$\nabla(fg)=f\nabla g+g\nabla f$$

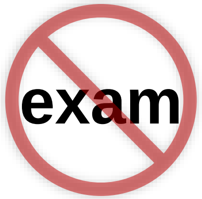
$$\nabla(f/g)=(g\nabla f-f\nabla g)/g^2$$ at points $\mathbf{x}$ where $g(\mathbf{x})\neq 0$

$$\operatorname{div}(\mathbf{v}+\mathbf{w})=\operatorname{div}\mathbf{v}+\operatorname{div}\mathbf{w}$$

$$\mathbf{curl}(\mathbf{v}+\mathbf{w})=\mathbf{curl}\ \mathbf{v}+\mathbf{curl}\ \mathbf{w}$$

$$\operatorname{div}(f\,\mathbf{v})=f\operatorname{div}\mathbf{v}+\mathbf{v}\cdot\nabla f$$

$$\operatorname{div}(\mathbf{v}\times\mathbf{w})=\mathbf{w}\cdot\mathbf{curl}\ \mathbf{v}-\mathbf{v}\cdot\mathbf{curl}\ \mathbf{w}$$



- **Scalar and vector identities (cont'd):**

$$\operatorname{div}\ \mathbf{curl}\ \mathbf{v} = 0$$

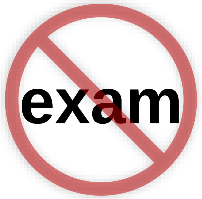
$$\mathbf{curl}\left(f\ \mathbf{v}\right) = f\ \mathbf{curl}\ \mathbf{v} + \nabla f \times \mathbf{v}$$

$$\mathbf{curl}\ \nabla f = \mathbf{0}$$

$$\nabla^2\left(f\ g\right) = f\nabla^2 g + g\nabla^2 f + 2\left(\nabla f \cdot \nabla g\right)$$

$$\operatorname{div}\left(\nabla f \times \nabla g\right) = 0$$

$$\operatorname{div}\left(f\ \nabla g - g\ \nabla f\right) = f\ \nabla^2 g - g\ \nabla^2 f$$



- **Decomposition of Jacobian Matrix:**

- $\mathbf{J_v}$ can be decomposed into symmetric and antisymmetric part:

$$\mathbf{J_v} = \mathbf{S} + \mathbf{\Omega}$$ with

$$\mathbf{S} = \frac{1}{2}\left(\mathbf{J_v} + \mathbf{J_v}^{\mathrm{T}}\right)$$ symmetric part (shear contribution)

$$\mathbf{\Omega} = \frac{1}{2}\left(\mathbf{J_v} - \mathbf{J_v}^{\mathrm{T}}\right) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$ antisymmetric part (rotational contribution)

with $$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \mathbf{curl}\ \mathbf{v} = \begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix}$$

- **Vortex-Strain duality:**
- Ω dominates S: high vortical activity
- S dominates Ω: high strain

$$\mathbf{J_v} = \mathbf{S} + \mathbf{\Omega}$$

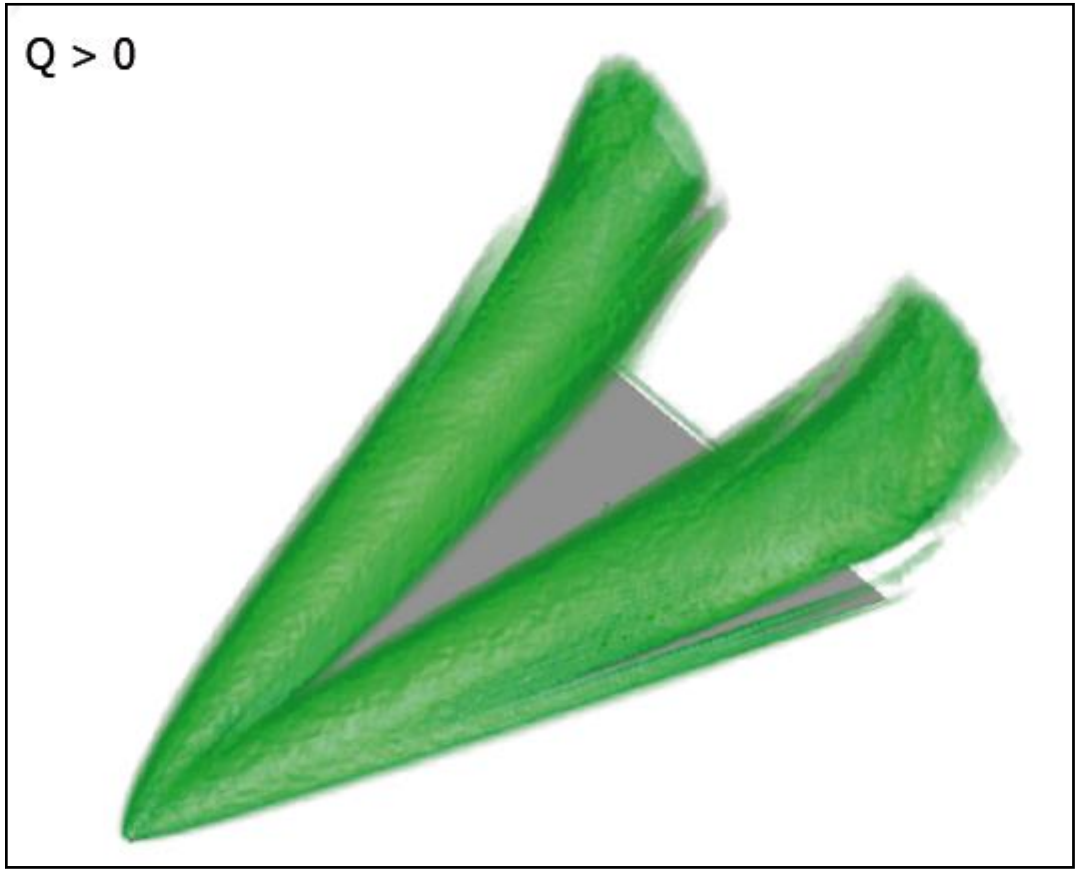
$$\mathbf{S} = \frac{1}{2}\left(\mathbf{J_v} + \mathbf{J_v}^{\mathrm{T}}\right)$$

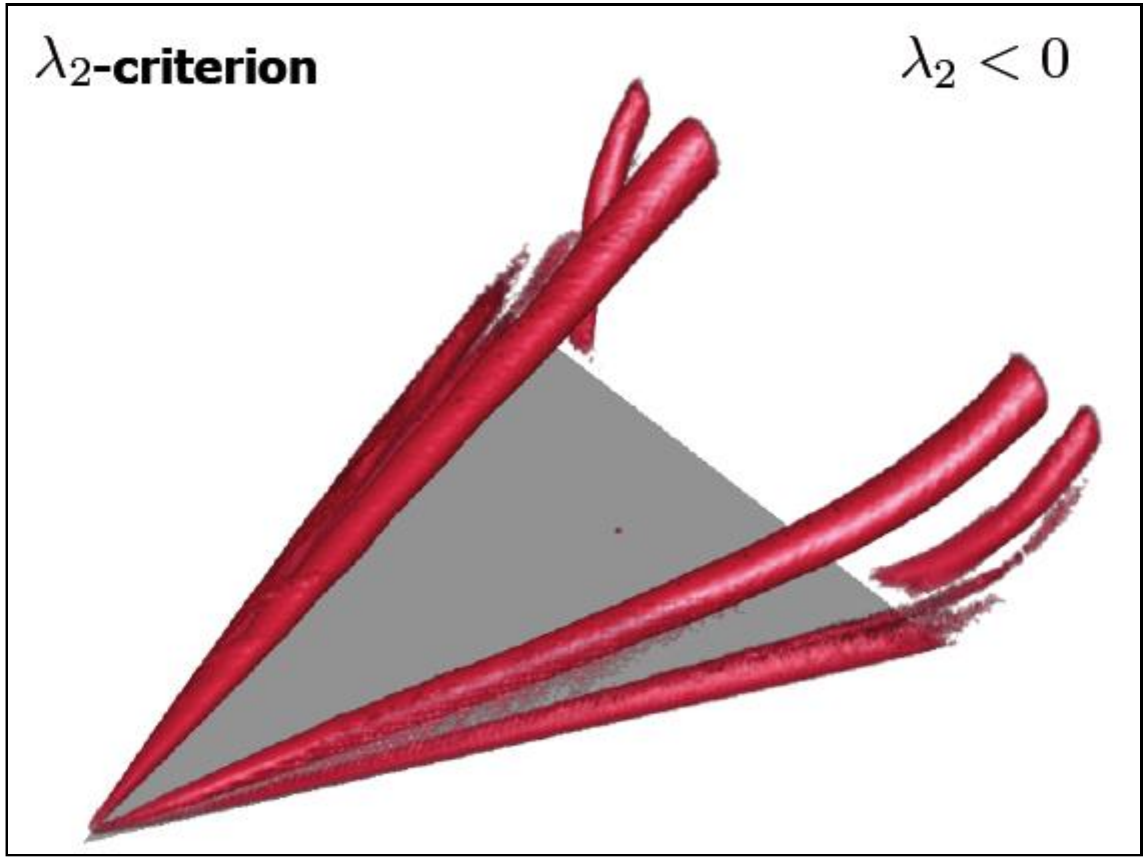
$$\mathbf{\Omega} = \frac{1}{2}\left(\mathbf{J_v} - \mathbf{J_v}^{\mathrm{T}}\right) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

- **Q-criterion, or, Okubo-Weiss parameter:**
  - Q > 0: vortex region, since vorticity magnitude dominates the rate of strain
  - Q < 0: region of high stretching, since rate of strain dominates vorticity magnitude
  - Captures vortex-strain duality

$$Q = \frac{1}{2}(\|\Omega\|^2 - \|\mathbf{S}\|^2) = \|\omega\|^2 - \frac{1}{2}\|\mathbf{S}\|^2$$

- **$\lambda_2$criterion:**
  - Second largest eigenvalue of the symmetric tensor $S^2 + \Omega^2$
  - Vortices can be found where $\lambda_2 < 0$
  - $\lambda_2 > 0$ lacks physical interpretation
  - Does not capture stretching and folding of fluid particles, i.e., does not describe the vortex-strain duality


Q > 0


λ_2-criterion
$\lambda_2 < 0$

Summary

- Derived quantities for scalar and vector fields
  - many more
- Based on derivatives
- Divergence
- Laplacian
- Curl
- Vortex-Strain duality
  - vortex regions in flow fields