VEKTORANALYS
HT 2021

Övningar: gradienten, grundläggande begrepp och parametrisering av kurvor och ytor

Kursvecka 1

Kapitel 1-5 (*Vektoranalys*, 1:e uppl, Frassinetti/Scheffel)

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PROBLEM 1

Calculate the gradient of the following scalar field:

$$\phi(x,y) = e^{-\left(x^2+y^2\right)}$$

(a) What is the direction of the maximum increase in point P: (-1,1)?
(b) What is the maximum increase in point P: (-1,1)?

SOLUTION to problem 1

(a) The direction of the maximum increase is the direction of the gradient *(theorem 4.1)*

$$grad\,\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right) = \left(-2xe^{-\left(x^2+y^2\right)}, -2ye^{-\left(x^2+y^2\right)}\right)$$

In P we have: $grad\,\phi = -2e^{-\left((-1)^2+1^2\right)}\left(-1,1\right) = -2e^{-2}\left(-1,1\right)$

The direction is: $\hat{d} = \frac{(1,-1)}{\sqrt{2}}$

(b) The maximum increase in P is the absolute value of the gradient in P *(theorem 4.1)*

$$\left|grad\,\phi\right| = \left|-2e^{-2}\left(-1,1\right)\right| = 2e^{-2}\sqrt{2}$$

PROBLEM 2

Betrakta skalärfältet: $f(x,y)=xy+x^2$

(a) Beräkna gradienten av $f(x,y)$.

(b) Beräkna riktningsderivatan df/ds i punkten P: $(1,2)$ om riktningen är: $\bar{d} = (1,1)$

Solution to problem 2

(a) Använd definitionen av gradienten (s. 75 i Vektoranalys, Frassinetti/Scheffel, ekv. 4.3)

$$grad(f) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (y + 2x, x)$$

(b) I punkten *P* är gradienten: $grad(f)_P = (2 + 2 \cdot 1,1) = (4,1)$

Definitionen av riktningsderivatan är (s. 75, ekv. 4.5): $\frac{df}{ds} = grad(f) \cdot \hat{\mathbf{d}}$

Observera att riktningen måste vara normaliserad: $\hat{d} = \frac{\bar{d}}{|\bar{d}|} = \frac{(1,1)}{\sqrt{1^2+1^2}} = \frac{(1,1)}{\sqrt{2}}$

Insättning i punkten *P* ger: $\left.\frac{df}{ds}\right|_P = grad(f)_P \cdot \hat{d} = (4,1) \cdot \frac{1}{\sqrt{2}}(1,1) = \frac{1}{\sqrt{2}}(4+1) = \frac{5}{\sqrt{2}}$

PROBLEM 3

(A) Show a simple parametric description $\overline{r} = \overline{r}(u)$ for the curve:

$$\begin{cases} 4x - y^2 = 0 & \text{(1)} \\ x^2 + y^2 - z = 0 & \text{(2)} \end{cases}$$

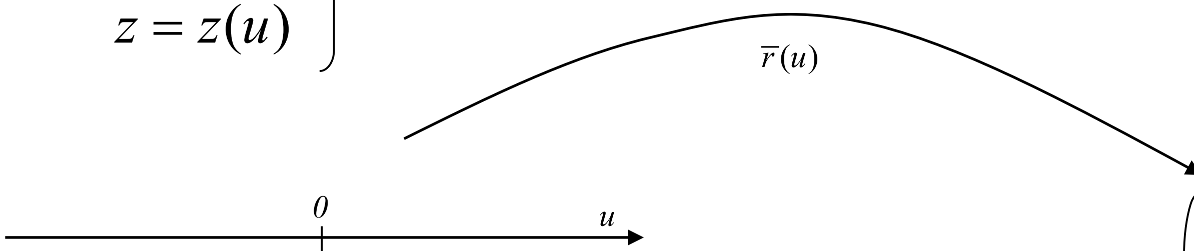
From the point (0,0,0) to the point (1,2,5)

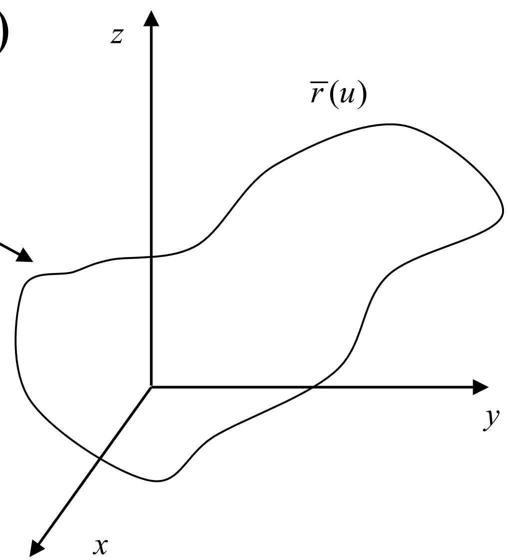
(B) Calculate the vector tangent to the curve in the point $P: \left(\frac{1}{4}, 1, \frac{17}{16} \right)$

SOLUTION to problem 3 (part A)

A “parameterization” means that we have to introduce a new variable (u for example). The “old” variables $x, y,$ and z will be dependent on u.

$$\left.\begin{aligned} x &= x(u) \\ y &= y(u) \\ z &= z(u) \end{aligned}\right\} \Longrightarrow \bar{r} = \bar{r}(x(u), y(u), z(u)) = \bar{r}(u)$$

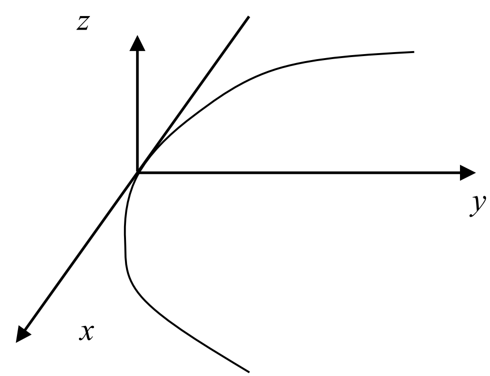




Example:

A parabola located in the xy-plane

$$\left.\begin{aligned} x &= u \\ y &= u^2 \\ z &= 0 \end{aligned}\right\} \Longrightarrow \bar{r}(u) = (u, u^2, 0)$$



SOLUTION to problem 3 (part A)

curve:

$$\begin{cases} 4x - y^2 = 0 & (1) \\ x^2 + y^2 - z = 0 & (2) \end{cases}$$

from the point (0,0,0) to the point (1,2,5)

For example, we can choose: $u=y$

From equation (1) $\Rightarrow$ $u^2 = 4x \quad \Rightarrow \quad x = \dfrac{u^2}{4}$

From equation (2) $\Rightarrow$ $z = x^2 + y^2 = \dfrac{u^4}{16} + u^2$

So we obtain: $\bar{r}(u) = \left(\dfrac{u^2}{4}, u, \dfrac{u^4}{16} + u^2 \right)$

From the point (0,0,0) to the point (1,2,5)

$(0,0,0) \Rightarrow u=0$
$(1,2,5) \Rightarrow u=2$

The curve is:

$$\boxed{\begin{array}{l} \bar{r}(u) = \left(\dfrac{u^2}{4}, u, \dfrac{u^4}{16} + u^2 \right) \\ u: 0 \to 2 \end{array}}$$

SOLUTION to problem 3 (part B)

The tangent in a point is the value of the derivative (calculated in the parameter u) in that point.

$$\bar{t} = \frac{d\bar{r}}{du} = \left(\frac{2u}{4}, 1, \frac{4u^3}{16} + 2u \right) = \left(\frac{u}{2}, 1, \frac{u^3}{4} + 2u \right)$$

The tangent has to be calculated in the point $\left(\frac{1}{4}, 1, \frac{17}{16} \right)$

since $u=y$, we have $u=1$

Therefore:

$$\bar{t} = \left(\frac{1}{2}, 1, \frac{1}{4} + 2 \right) = \left(\frac{1}{2}, 1, \frac{9}{4} \right)$$

PROBLEM 4

Consider the following surface:

$$x^2 - 2y^2 - 2z = 0 \qquad (1)$$

Calculate:

(A) The equations of the normal line to the surface in the point $P\ \ (2,1,1)$
(B) The equation of the tangent plane to the surface in the point P

SOLUTION to problem 4 (part A)

A normal line is a line that intersects the surface in the point. The direction of the line is perpendicular to the surface.

How to calculate the direction perpendicular to a surface?

Theorem 4.2 (in Frassinetti/Scheffel): The gradient of a scalar field $\phi(x,y,z)$ in the point P is orthogonal to the level surface $\phi=c$ in P.

Consider the level surface $\phi = x^2 - 2y^2 - 2z = 0$

The normal line is parallel to the gradient

$$grad\,\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = (2x, -4y, -2)$$

In $P\ (2,1,1)$ $\quad grad\,\phi = (4, -4, -2)$

The direction of the normal line is: $\bar{n}_P = (4, -4, -2)$

The equation of the normal line is: $\bar{r}(u) = (2,1,1) + (4,-4,-2)u = (4u+2, -4u+1, -2u+1)$

SOLUTION to problem 4 (part B)

A tangent plane is a plane that is parallel to the surface in the point.

From "*basic*" geometry, given a vector $\overline{v} = (A, B, C)$

Then the plane $Ax + By + Cz + D = 0$ is perpendicular to $\overline{v}$

Therefore, using $\overline{v} = \overline{n}_P = (4, -4, -2)$

we have that the plane $4x - 4y - 2z + D = 0$ is perpendicular to $\overline{n}_P$

D is chosen in order that the plane passes through the point $P\ (2,1,1)$:

$$4 \cdot 2 - 4 \cdot 1 - 2 \cdot 1 + D = 0 \qquad \Rightarrow D=-2$$

$2x - 2y - z - 1 = 0$ passes through P and is tangent to the surface

PROBLEM 5

Write a parameterization of the following surface

$$\begin{cases} y = x^2 + z^2 \\ y < 2 \end{cases}$$

SOLUTION to problem 5

$$\begin{cases} x = \rho\cos\theta \\ z = \rho\sin\theta \\ y = x^2 + z^2 = \rho^2 \end{cases} \Rightarrow \quad \overline{r}(\rho,\theta) = \left(\rho\cos\theta, \rho^2, \rho\sin\theta\right) \quad \text{and} \quad \begin{cases} \theta : 0 \to 2\pi \\ \rho : 0 \to \sqrt{2} \end{cases}$$

PROBLEM 6

Write a parameterization of the following curve

$$\begin{cases} \dfrac{x^2}{4} + \dfrac{y^2}{16} + \dfrac{z^2}{9} = 1 \\ z = 0 \\ y > 0 \end{cases}$$

SOLUTION to problem 6

$$\begin{cases} x = 2\cos\varphi \\ y = 4\sin\varphi \\ z = 0 \end{cases} \Rightarrow \quad \overline{r}(\varphi) = \left(2\cos\varphi, 4\sin\varphi, 0\right) \quad \text{and} \quad \varphi : 0 \to \pi$$

PROBLEM 7

Write a parameterization of the following surface

$$z = xy$$

SOLUTION to problem 7

$$\begin{cases} x = u \\ y = v \\ z = uv \end{cases} \quad \Rightarrow \qquad \bar{r}(u,v) = (u, v, uv)$$

PROBLEM 8

Write a parameterization of the curve defined by:

$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ x - y = 0 \end{cases}$$

from point $P_0 \quad \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$ till point $P_1 \quad (0,0,1)$

(the coordinates of the points are given in a cartesian coordinate system)

SOLUTION to problem 8

$$\begin{cases} x = \frac{\sqrt{2}}{2}\sin\theta \\ y = \frac{\sqrt{2}}{2}\sin\theta \\ z = \cos\theta \end{cases} \qquad \bar{r}(\theta) = \left(\frac{\sqrt{2}}{2}\sin\theta, \frac{\sqrt{2}}{2}\sin\theta, \cos\theta\right) \qquad \left.\begin{array}{l} \text{in } \mathrm{P}_0 \quad z = 0 \Rightarrow \theta = \frac{\pi}{2} \\ \text{in } \mathrm{P}_1 \quad z = 1 \Rightarrow \theta = 0 \end{array}\right\} \Rightarrow \qquad \theta : \frac{\pi}{2} \to 0$$

PROBLEM 9

Kraften på en ledare L med den elektriska strömmen I, i det magnetiska fältet är:

$$\overline{F} = I\int_L \left(d\overline{l} \times \overline{B}\right)$$

Beräkna kraften på en cirkulär strömslinga L (radie R och centrum i origo). L ligger i xy-planet (z=0) och omsluter z-axeln en gång. Magnetiska fältet $\overline{B}$ definieras i cylinderkoordinater:

$$\overline{B} = B_0\rho\left(\cos\varphi\,\hat{e}_z + \sin\varphi\,\hat{e}_\varphi\right)$$

Använd följande steg:

(a) uttryck $d\overline{l}$ i ett cylindriskt koordinatsystem

(b) beräkna $d\overline{l} \times \overline{B}$ i ett cylindriskt koordinatsystem

(c) integrera och beräkna $\overline{F}$ (Du kan här använda ett kartesiskt koord-system)

SOLUTION to problem 9

$$d\bar{l} = \rho d\varphi \hat{e}_\varphi$$

$$d\bar{l} \times \bar{B} = \rho d\varphi \hat{e}_\varphi \times B_0 \rho(\cos\varphi \hat{e}_z + \sin\varphi \hat{e}_\varphi) = B_0 \rho^2 \cos\varphi d\varphi \hat{e}_\rho$$

$$\bar{F} = I \int_0^{2\pi} B_0 \rho^2 \cos\varphi d\varphi \hat{e}_\rho = IB_0 R^2 \int_0^{2\pi} \cos\varphi \hat{e}_\rho d\varphi =$$

$$= IB_0 R^2 \int_0^{2\pi} \cos\varphi (\cos\varphi \hat{e}_x + \sin\varphi \hat{e}_y) d\varphi =$$

$$IB_0 R^2 \int_0^{2\pi} \cos^2\varphi \hat{e}_x d\varphi + IB_0 R^2 \underbrace{\int_0^{2\pi} \cos\varphi \sin\varphi \hat{e}_y d\varphi}_{=0} = IB_0 R^2 \hat{e}_x \int_0^{2\pi} \cos^2\varphi d\varphi =$$

$$= IB_0 R^2 \hat{e}_x \left[\frac{\varphi}{2} + \frac{\sin 2\varphi}{4}\right]_0^{2\pi} = IB_0 R^2 \pi \hat{e}_x$$

PROBLEM 10

The scalar fields f, g, h are given by:

$$f(\overline{r}) = 2xy - y^2z^2 + 2xz$$
$$g(\overline{r}) = x^3y + y^3z - xz^3$$
$$h(\overline{r}) = x^3y^2z + x^2yz^3 + xy^3z^2$$

(a) Calculate the direction $\hat{n}$ for which the directional derivative of f and g in the point P (-1,0,1) is zero simultaneously

(b) Calculate the directional derivative of h in P along the direction $\hat{n}$

SOLUTION

(a) The directional derivative of the scalar field ϕ in the direction $\hat{n}$ is:

$$\frac{d\phi}{ds} = grad\,\phi \cdot \hat{n}$$

Therefore, we need to find the direction $\hat{n}$ for which: $\begin{cases} grad(f)\cdot\hat{n} = 0 \\ grad(g)\cdot\hat{n} = 0 \end{cases}$

Let's calculate the gradient of f and g:

$$grad\ f = \left(2y + 2z, 2x - 2yz^2, -2y^2z + 2x\right)$$
$$grad\ g = \left(3x^2y - z^3, x^3 + 3y^2z, y^3 - 3xz^2\right)$$

In the point P (-1,0,1) the gradients are:

$$(grad\ f)_P = (2,-2,-2)$$
$$(grad\ g)_P = (-1,-1,3)$$

If $\hat{n} = (a,b,c)$ we obtain:

$$(grad\ f)_P \cdot \hat{n} = (2,-2,-2)\cdot(a,b,c) = 0$$
$$(grad\ g)_P \cdot \hat{n} = (-1,-1,3)\cdot(a,b,c) = 0$$

$$\left.\begin{array}{l} a-b-c=0 \\ a+b-3c=0 \end{array}\right\} \Rightarrow \begin{cases} a=2c \\ b=c \end{cases}$$

Therefore: $\hat{n} = (2c,c,c)$

Normalizing: $\boxed{\hat{n} = \dfrac{(2,1,1)}{\sqrt{6}}}$

(b) The directional derivative of the h in the direction $\hat{n}$ in the point P is:

$$\left(\frac{dh}{ds}\right)_P = (gradh)_P \cdot \hat{n} = (0,1,0)\cdot\frac{(2,1,1)}{\sqrt{6}} = \frac{1}{\sqrt{6}}$$