A Proof of that Linearizability is a Compositional Condition

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Compositionality

- A correctness condition CC is compositional if:
  - History H satisfies CC iff every register subhistory H|x satisfies CC
- Linearizability is compositional:
  - $LIN(H) \Leftrightarrow \forall x: LIN(H|x)$
- We will prove that this is the case

Definition of Linearizability

- Remember the definition of linearizability:
  - LIN(H) iff there exists a sequential history S satisfying the following requirements:
    - (1) S is legal
    - (2) S and H are equivalent
    - (3) If $o_1 <_H o_2$ then $o_1 <_S o_2$
  - $o_1 <_H o_2$ denotes that $res(o_1) \rightarrow_H inv(o_2)$
    - $e_1 \rightarrow_H e_2$ denotes that e_1 precedes e_2 in H

The Proof

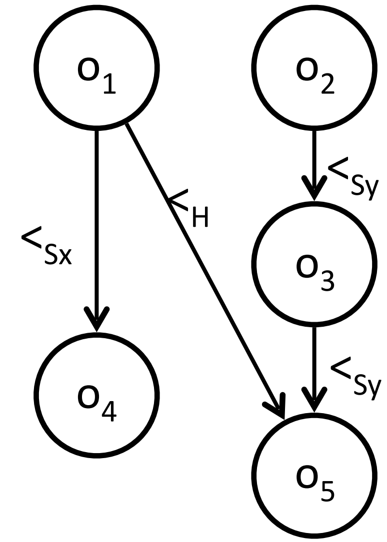
- We must show that:
  - LIN(H) $\Rightarrow \forall x$: LIN(H|x), and
  - LIN(H) $\Leftarrow \forall x$: LIN(H|x)
- The $\Rightarrow$ direction follows trivially
  - Exercise: Complete the proof
    - Assume LIN(H), implying that there exists a sequential history S satisfying requirements (1)-(3)
    - Show that subhistory S|x satisfies the requirements of a sequential history for LIN(H|x)

The $\Leftarrow$ Direction

- We must show that LIN(H) $\Leftarrow$ $\forall x$: LIN(H|x)
- Assume that the right-hand side holds:
  - For each x, there exists a sequential history Sx that satisfies the requirements of LIN(H|x)
    - (1) Sx is legal
    - (2) Sx and H|x are equivalent
    - (3) If $o_1 <_{H|x} o_2$ then $o_1 <_{Sx} o_2$
- We will construct a sequential history S that satisfies the requirements of LIN(H)

Constructing Operation Graph

- Create a graph, whose vertices are operations in H, and edges are added as follows:
  - Add an edge from o to o’ if o $<_{Sx}$ o’
  - Add an edge from o to o’ if o $<_H$ o’
  - We refer to an edge as a $<_{Sx}$ edge or a $<_H$ edge (there can be zero, one, or two edges from o to o’)



Constructing Sequential History S

- If the constructed graph is acyclic, then a topological sort can be performed
  - Creates a total order on operations, compatible with the partial ordering in the graph
  - History S is created directly from this total ordering
- Sequential history S created in this way satisfies the requirements for LIN(H) by construction:
  - S is legal since the total ordering in $<_{Sx}$ is legal,
  - S and H are equivalent,
  - $o_1 <_H o_2$ implies $o_1 <_S o_2$

Acyclic

- We need to show that any graph constructed as described is acyclic
  - So that the graph can be topologically sorted
  - Proof by contradiction: assume a minimal cycle exists of a certain length, and reach contradiction

Cycle of Length n=2

- Cycles with two operations:
  - $o_1 <_H o_2 <_H o_1$
    - Not possible, as $<_H$ is a partial order
  - $o_1 <_{Sx} o_2 <_{Sx} o_1$
    - Not possible, as $<_{Sx}$ is a total order
  - $o_1 <_H o_2 <_{Sx} o_1$
    - As o_1 and o_2 are both ops on x, then $o_1 <_H o_2$ implies that $o_1 <_{Sx} o_2$, which is contradicted in previous case

Cycle of Length n=3

- Cycles with three operations:
  - $o_1 <_H o_2 <_H o_3 <_H o_1$
    - Not minimal as $<_H$ is partial order
    - Similar contradiction if $<_{Sx}$ instead of $<_H$
  - $o_1 <_H o_2 <_H o_3 <_{Sx} o_1$
    - Not minimal as $<_H$ is partial order
  - In fact any cycle of length three must have two consecutive edges of same type ($<_H$ or $<_{Sx}$), and therefore cannot be minimal

Cycle of Length n≥4

- Consider a cycle of arbitrary length n≥4
- At some point in the cycle there is a section:
  - $o_1 <_H o_2 <_{Sx} o_3 <_H o_4$
  - We will show that this cycle is not minimal, as there must exist an edge $o_1 <_H o_4$
- Focus on edge $o_2 <_{Sx} o_3$, there are two cases:
  - Either $o_2 <_H o_3$, and hence $o_1 <_H o_4$ by transitivity of $<_H$
  - Or, not($o_3 <_H o_2$), this case is handled on next slide

Cycle of Length n≥4, case 2

- Second case, continued from previous slide:
  - not($o_3 <_H o_2$) implies that not(res(o_3) $\rightarrow_H$ inv(o_2))
  - As $\rightarrow_H$ is a total order, we have inv(o_2) $\rightarrow_H$ res(o_3)
  - Together with $o_1 <_H o_2$, and $o_3 <_H o_4$, we have:
    - res(o_1) $\rightarrow_H$ inv(o_2) $\rightarrow_H$ res(o_3) $\rightarrow_H$ inv(o_4)
    - Implying that res(o_1) $\rightarrow_H$ inv(o_4) $\Rightarrow$ $o_1 <_H o_4$
  - Hence, the cycle containing $o_1 <_H o_2 <_{Sx} o_3 <_H o_4$ is not minimal

Summary

- Cycles of all lengths have been contradicted and the graph is therefore acyclic
- It can be topologically sorted into a sequential history S that meets requirements of LIN(H)

- We have proven that:
  - Linearizability is compositional
    - $LIN(H) \Leftrightarrow \forall x: LIN(H|x)$