SF 1684 Algebra and Geometry
Lecture 8

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Topics for Today

1. Determinants
2. Calculating Determinants: Cofactor Expansion and Row Reduction Formula
3. Determinants and Invertibility

Another Function on Matrices: the Determinant

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One of the most important function on matrices is called the determinant.

Definition

The **determinent** of a square matrix, denoted $\det(A)$ or $|A|$, is the *sum of all signed elementary products* of A.

Elementary Products

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An elementary product of a square matrix is a product of it's entries with one entry coming from each row and no two in the same column.

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Examples:

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$$

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$$

$\underline{a_{11}a_{22}}$ $\quad$ $\underline{a_{12}a_{21}}$

$\underline{a_{11}a_{22}a_{33}}$ $\quad$ $\underline{a_{11}a_{23}a_{32}}$

$\underline{a_{12}a_{21}a_{33}}$ $\quad$ $\underline{a_{12}a_{23}a_{31}}$

$\underline{a_{13}a_{22}a_{32}}$ $\quad$ $\underline{a_{13}a_{22}a_{31}}$

Parity of Permutations

We note that we can rearrange all of our elementary products to be of the form

$$a_{1,j_1} a_{2,j_2} \cdots a_{n,j_n}$$

where the $\{j_1, j_2, \ldots, j_n\}$ is a permutation of $\{1, 2, \ldots, n\}$.

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$\{2,1,4,3\} \to (1,2,4,3) \to (1,2,3,4)$

$\{2, 1, 4, 3\}$ is an even permutation

$\{5,3,4,2,1\} \xrightarrow{1} \{1,3,4,2,5\} \xrightarrow{2} \{1,2,4,3,5\} \xrightarrow{3} \{1,2,3,4,5\}$

$\{5, 3, 4, 2, 1\}$ is an odd permutation

Sign of an Elementary Product

Definition

Given an elementary product

$$a_{1,j_1} a_{2,j_2} \cdots a_{n,j_n}$$

we define the **sign** of the product to be "$+$" if $\{j_1, \ldots, j_n\}$ is even and "$-$" if $\{j_1, \ldots, j_n\}$ is odd.

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The sign of $a_{1,2}a_{2,1}a_{3,4}a_{4,3}$ is "$+$" since $\{2, 1, 4, 3\}$ is an even permutation.

The sign of $a_{1,5}a_{2,3}a_{3,4}a_{4,2}a_{5,1}$ is "$-$" since $\{5, 3, 4, 2, 1\}$ is an odd permutation.

Determinant of 2×2 and 3×3 Matrices

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$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$$

$$+a_{11}a_{22} \quad - \quad a_{12}a_{21}$$

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \underline{a_{11}a_{22} - a_{12}a_{21}}$$

$$\det\left(\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}\right) = \begin{array}{l} +a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} \\ +a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} \\ +a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \end{array}$$

Method for Computing 2×2 Determinants

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$ad - bc = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$ad - bc = ad - cb$$

Compute the determinant of $A = \begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix}$.

$$\det(A)$$

$$= 3 \times 1 - 5 \times 2$$

$$= 3 - 10 = -7$$

Method for Computing 3 × 3 Determinants

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \implies \begin{matrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,1} & a_{3,2} \end{matrix}$$

$$\det(\quad) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

WARNING: This generalizes <u>only</u> to the 3x3!
Can not do a similar construction for higher <u>dimensions</u>!

Exercise

Compute the determinant of

$$A = \begin{pmatrix} 3 & 0 & 2 \\ 1 & -1 & 3 \\ 2 & 2 & 0 \end{pmatrix}$$

$$\begin{matrix} 3 & 0 & 2 & 3 & 0 \\ 1 & -1 & 3 & 1 & -1 \\ 2 & 2 & 0 & 2 & 2 \end{matrix}$$

$$\det(A) =$$

$$3 \times -1 \times 0 + 0 \times 3 \times 2 + 2 \times 1 \times 2 - 0 \times 1 \times 0 - 3 \times 3 \times 2 - 2 \times -1 \times 2$$

$$0 + 0 + 4 - 0 - 18 + 4 = -10$$

Computing Determinants: Cofactors and Minors

Definition

For a square matrix A, we define the (i,j)-**th minor**, denote $M_{i,j}$ to be the determinant of the matrix obtained by removing the i-th row and j-th column. The (i,j)-**th cofactor**, denoted $C_{i,j}$ is then $(-1)^{i+j}M_{i,j}$

$$M_{3,1} = \det\begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} = 0\times 3 - 2\times -1 = 2$$

$$C_{31} = 2$$

$$A = \begin{pmatrix} 3 & 0 & 2 \\ 1 & -1 & 3 \\ 2 & 2 & 0 \end{pmatrix}$$

$$M_{1,1} = \det\begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix} = (-1)\times 0 - 3\times 2 = -6 \longrightarrow C_{1,1} = (-1)^{1+1} M_{1,1} = 1 \times -6 = -6$$

$$M_{2,1} = \det\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = 0\times 0 - 2\times 2 = -4 \longrightarrow C_{21} = (-1)^{2+1} M_{21} = (-1)\times(-4) = 4$$

$$M_{3,2} = \det\begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix} = 3\times 3 - 2\times 1 = 9-2 = 7$$

$$C_{3,2} = -7$$

Computing Determinant: Cofactor Expansion

Theorem

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$$\det(A) = a_{i,1}C_{i,1} + a_{i,2}C_{i,2} + \cdots + a_{i,n}C_{i,n}$$

Expanding the determinant along the ith row

$$A = \begin{bmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

$$a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$

$$\parallel$$

$$\det(A)$$

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$$= (-1)^{i+1}a_{i,1}M_{i,1} + (-1)^{i+2}a_{i,2}M_{i,2} + \cdots + (-1)^{i+n}a_{i,n}M_{i,n}$$

Moreover, for any j

$$\det(A) = a_{1,j}C_{1,j} + a_{2,j}C_{2,j} + \cdots + a_{n,j}C_{n,j}$$

Expanding along the j th column

$$\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

$$\underbrace{a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}}_{\det(A)}$$

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Generic Example: 3×3

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Expand along the 1^{st} row, $\det(A) =$

$$+ a_{11} \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

Expand along the 2^{nd} column, $\det(A) =$

$$- a_{12} \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{22} \det\begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} - a_{32} \det\begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix}$$

Specific Example: 4×4

Calculate the determinant of

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix} \qquad A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix} \qquad \begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

$$1 \det\begin{bmatrix} 5 & 6 & 7 \\ 0 & 8 & 9 \\ 0 & 0 & 10 \end{bmatrix} - \cancel{0 \det\begin{bmatrix} 2 & 3 & 4 \\ 0 & 8 & 9 \\ 0 & 0 & 10 \end{bmatrix}} + \cancel{0 \det\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 0 & 0 & 10 \end{bmatrix}}$$

$$\cancel{+0 \det\begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix}} - \cancel{0 \det\begin{pmatrix} 5 & 7 \\ 0 & 9 \end{pmatrix}} + 10 \cdot \det\begin{pmatrix} 5 & 6 \\ 0 & 8 \end{pmatrix} - \cancel{0 \det\begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 0 & 8 & 9 \end{pmatrix}}$$

$$\det(A) = 10 \cdot \det\begin{pmatrix} 5 & 6 \\ 0 & 8 \end{pmatrix} = 10(5 \times 8 - 0 \cdot 6) = 400$$

$$400 = 1 \times 5 \times 8 \times 10$$

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Note that diagonal matrices are also upper triangular and so

$$\det(D) = \det\left(\begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}\right) = d_1 d_2 \cdots d_n$$

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Further, the identity matrix I_n, is the diagonal matrix with $d_1 = d_2 = \cdots = d_n = 1$ and so we may conclude $\det(I_n) = 1$.

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3. *If the row operation is adding one row to another then* $\det(B) = \det(A)$.

Sketch of Proof

(1)

$$\begin{matrix} + & - & + & - & + & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & + & \cdots \end{matrix}$$

expand along first row get
$+ \; - \; + \; - \; +$ pattern
swap first and second row and
expand along the second row
$- \; + \; - \; + \; - \; +$ patte

(2) $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \vdots & & \end{pmatrix}$ $\quad B = \begin{pmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ & \vdots & & \end{pmatrix}$

expand B along first row $= c\,a_{11} \det(\;) \cdots + c\,a_{1n} \det(\;)$

$= c\,(a_{11} \det(\;) + \cdots + a_{1n} \det(\;))$

$= c\,(\det A)$

(3) i) Use part 1 to show that if two rows are the same
the det is 0.

ii) if $A = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$ $\quad B = \begin{pmatrix} r_1 + r_2 \\ \vdots \\ r_n \end{pmatrix} \Rightarrow \det B = \det \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} + \det \begin{pmatrix} r_2 \\ \vdots \\ r_n \end{pmatrix}$

Exercise

Row reduce A to REF and then calculate the determinate

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \\ 2 & -1 & -2 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 - R_1 \end{matrix}$$

$\det(A)$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -4 \\ 0 & -5 & -8 & -8 \\ 0 & -1 & -2 & -3 \end{bmatrix} \quad \begin{matrix} R_4 \leftrightarrow R_2 \\ R_2 \Rightarrow R_4 \end{matrix}$$

$-\det(A)$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -5 & -8 & -8 \\ 0 & -2 & -2 & -4 \end{bmatrix} \quad \begin{matrix} R_3 - 5R_1 \\ R_4 - 2R_1 \end{matrix}$$

$-\det(A)$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 2 & 2 \end{bmatrix} \quad \tfrac{1}{2} R_3$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$= -\frac{1}{2}\det(A)$

More Work Space

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 2 & 2 \end{pmatrix} \xrightarrow{R_4 - 2R_3} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & -5 \end{pmatrix} = B$$

$$= -\tfrac{1}{2}\det(A)$$

$$-\frac{1}{2}\det A = \det(B) = 1 \times -1 \times 1 \times -5$$

$$= 5$$

$$\det A = -10$$

Properties of Determinants

Theorem

Let A be an $n \times n$ matrix

1. $\det(A^T) = \det(A)$

$B = A^T \Rightarrow \det(B) = \det(A)$

Expanding the determinant of A along a column is the same as expanding the determinant of A^T along a row

Properties of Determinants

Theorem

Let A be an $n \times n$ matrix

1. $\det(A^T) = \det(A)$
2. *If A has a row or column of* 0*'s then* $\det(A) = 0$

if A has a row of zeros then

expanding along this row gives

$\det A = 0 \cdot \det(\ ) - 0 \det(\ ) + \cdots + 0 \cdot \det(\ ) = 0$

likewise if a column is all 0's

Properties of Determinants

Theorem

Let A be an $n \times n$ matrix

1. $\det(A^T) = \det(A)$
2. *If A has a row or column of 0's then* $\det(A) = 0$
3. *If A has two proportional rows, then* $\det(A) = 0$

If A has two proportional rows then
let B be the matrix obtained from subtraction
the two rows. So B has a row of zeros
and by previous theorem $\det(A) = \det(B) = 0$

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3. *If A has two proportional rows, then* $\det(A) = 0$
4. *If A has two proportional columns, then* $\det(A) = 0$

If A has two proportional columns then A^T has two proportional rows. and

So $\det(A^T) = 0$ and so $\det(A) = \det(A^T) = 0$

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3. *If A has two proportional rows, then* $\det(A) = 0$
4. *If A has two proportional columns, then* $\det(A) = 0$
5. $\det(cA) = c^n \det(A)$.

$\cancel{\det(cA) \neq c \cdot \det(A)}$

cA can be thought of multiplying each row by c. And by previous thm each time we multiply a row, we obtain an extra c.

Big Theorem

Theorem

An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

Proof: A is invertible if and only if A has RREF of I_n

which means that there is a sequence of row operations that reduce A to I_n

each row operation will either multiply by a non zero constant (with -1 being an option) or not change the determinant.

non zero

A is invertible $\Longleftrightarrow$ $\det(A) = C \det(I_n) = C \neq 0$

Major Theorem

Theorem

Let A be an $n \times n$ matrix. The the following are equivalent

1. $A\vec{x} = \vec{b}$ *has a unique solution for every* $\vec{b}$
2. $A\vec{x} = 0$ *has a unique solution*
3. $rk(A) = n$
4. *The RREF of A is I_n*
5. *A is invertible*

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Theorem

Let A be an $n \times n$ matrix. The the following are equivalent

1. $A\vec{x} = \vec{b}$ *has a unique solution for every* $\vec{b}$
2. $A\vec{x} = 0$ *has a unique solution*
3. $rk(A) = n$
4. *The RREF of A is I_n*
5. *A is invertible*
6. *The columns of A are linearly independent*

Major Theorem

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7. $\det(A) \neq 0$

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5. *A is invertible*
6. *The columns of A are linearly independent*
7. $\det(A) \neq 0$
8. *The row vectors of A are linearly independent*

A is invertible
$\Leftrightarrow$ det$(A) \neq 0$
$\Leftrightarrow$ det$(A^T) \neq 0$
$\Leftrightarrow$ columns of A^T are lin. indep
$\Leftrightarrow$ rows of A are lin ind.

Properties of Determinants 2

Theorem

1. $\det(AB) = \det(A)\det(B)$
2. $\det(A^{-1}) = \frac{1}{\det(A)}$, *provided A^{-1} exists.*

1) If A is not invertible then AB is also not invertible and $\det(AB) = 0 = \det A \det B$

If A is invertible, then write it as a product of elementary matrices and the result follows by row elementary row operations effect B.

2) $AA^{-1} = I_n \Rightarrow 1 = \det(I_n) = \det(AA^{-1}) = \det(A)\det(A^{-1})$

$\det(A) \cdot \det(A^{-1}) = 1 \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$.

More Work Space