

# SF 1684 Algebra and Geometry

## Lecture 8

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# Topics for Today

- ① Determinants
- ② Calculating Determinants: Cofactor Expansion and Row Reduction Formula
- ③ Determinants and Invertibility

# Another Function on Matrices: the Determinant

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## Definition

The **determinant** of a square matrix, denoted  $\det(A)$  or  $|A|$ , is the *sum of all signed elementary products* of  $A$ .

# Elementary Products

## Definition

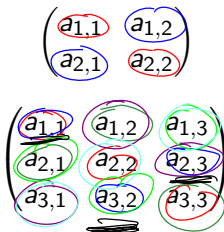
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Examples:



$$\underline{a_{1,1} a_{2,2}}$$

$$\underline{a_{1,2} a_{2,1}}$$

$$\underline{a_{1,1} a_{2,2} a_{3,3}}$$

$$\underline{a_{1,1} a_{2,3} a_{3,1}}$$

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$$\underline{a_{1,3} a_{2,2} a_{3,1}}$$

# Parity of Permutations

We note that we can rearrange all of our elementary products to be of the form

$$a_{1,j_1} a_{2,j_2} \cdots a_{n,j_n}$$

where the  $\{j_1, j_2, \dots, j_n\}$  is a permutation of  $\{1, 2, \dots, n\}$ .



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$$\{2, 1, 4, 3\} \xrightarrow{1} (1, 2, 4, 3) \xrightarrow{2} (1, 2, 3, 4)$$

$\{2, 1, 4, 3\}$  is an even permutation

$$\{5, 3, 4, 2, 1\} \xrightarrow{1} (1, 3, 4, 2, 5) \xrightarrow{2} (1, 2, 4, 3, 5)$$

$\{5, 3, 4, 2, 1\}$  is an odd permutation

$$\xrightarrow{2} (1, 2, 4, 3, 5) \xrightarrow{3} (1, 2, 3, 4, 5)$$

# Sign of an Elementary Product

## Definition

Given an elementary product

$$a_{1,j_1} a_{2,j_2} \cdots a_{n,j_n}$$

we define the **sign** of the product to be “+” if  $\{j_1, \dots, j_n\}$  is even and “−” if  $\{j_1, \dots, j_n\}$  is odd.

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The sign of  $a_{1,2}a_{2,1}a_{3,4}a_{4,3}$  is “+” since  $\{2, 1, 4, 3\}$  is an even permutation.

The sign of  $a_{1,5}a_{2,3}a_{3,4}a_{4,2}a_{5,1}$  is “−” since  $\{5, 3, 4, 2, 1\}$  is an odd permutation.

# Determinant of $2 \times 2$ and $3 \times 3$ Matrices

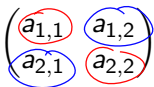
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## Definition

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$$\det \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$$

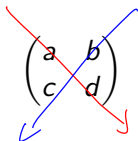
$$+ a_{11} a_{22} - a_{12} a_{21}$$
$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \underline{a_{11} a_{22} - a_{12} a_{21}}$$

$$+ a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32}$$
$$= + a_{12} a_{23} a_{31} - a_{11} a_{21} a_{33}$$
$$+ a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}$$

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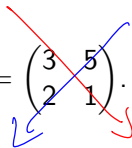
# Method for Computing $2 \times 2$ Determinants



$$\underline{ad - bc = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

$$ad - \underline{bc} = ad - \underline{cb}$$

Compute the determinant of  $A = \begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix}$ .



$$\begin{aligned} \det(A) &= 3 \times 1 - 5 \times 2 \\ &= 3 - 10 = -7 \end{aligned}$$

# Method for Computing $3 \times 3$ Determinants

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$$

$\Rightarrow$

$$\begin{array}{ccccc} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,1} & a_{3,2} \end{array}$$

*(Note: In the original image, the first three columns are crossed out with red lines, and the last two columns are crossed out with blue lines. Blue arrows point from the first three columns to the last two columns, and red arrows point from the last two columns back to the first three columns.)*

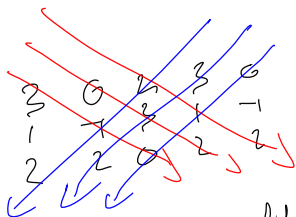
$$\det \begin{pmatrix} \phantom{a_{1,1}} & \phantom{a_{1,2}} & \phantom{a_{1,3}} \end{pmatrix} = a_{1,1} a_{2,2} a_{3,3} + a_{1,2} a_{2,3} a_{3,1} + a_{1,3} a_{2,1} a_{3,2} - a_{1,3} a_{2,2} a_{3,1} - a_{1,1} a_{2,3} a_{3,2} - a_{1,2} a_{2,1} a_{3,3}$$

**WARNING** This generalizes only to the  $3 \times 3$ !  
Can not do a similar construction for high dimensional

# Exercise

Compute the determinant of

$$A = \begin{pmatrix} 3 & 0 & 2 \\ 1 & -1 & 3 \\ 2 & 2 & 0 \end{pmatrix}$$



$$\det(A) =$$

$$3 \times 1 \times 0 + 0 \times 3 \times 2 + 2 \times 1 \times 2 - 0 \times 1 \times 0 - 3 \times 2 \times 2 - 2 \times 1 \times 2$$

$$0 + 0 + 4 - 0 - 12 + 4 = -10$$

# Computing Determinants: Cofactors and Minors

## Definition

For a square matrix  $A$ , we define the  $(i,j)$ -th **minor**, denote  $M_{i,j}$  to be the determinant of the matrix obtained by removing the  $i$ -th row and  $j$ -th column. The  $(i,j)$ -th **cofactor**, denoted  $C_{i,j}$  is then  $(-1)^{i+j} M_{i,j}$

$$M_{2,1} = \det \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} = 0 \times 3 - 2 \times (-1) = 2$$

$$C_{2,1} = 2$$

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 3 \\ 2 & 2 & 0 \end{pmatrix}$$

$$M_{1,1} = \det \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix} = (-1) \times 0 - 3 \times 2 = -6$$

$$\rightarrow C_{1,1} = (-1)^{1+1} M_{1,1} = 1 \times -6 = -6$$

$$M_{2,1} = \det \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = 0 \times 0 - 2 \times 2 = -4$$

$$\rightarrow C_{2,1} = (-1)^{2+1} M_{2,1} = (-1) \times (-4) = 4$$

$$M_{3,2} = \det \begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix} = 3 \times 3 - 2 \times 1 = 9 - 2 = 7$$

$$C_{3,2} = -7$$

# Computing Determinant: Cofactor Expansion

## Theorem

*Let  $A$  be a  $n \times n$  square matrix with entries  $a_{i,j}$ .*

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Let  $A$  be a  $n \times n$  square matrix with entries  $a_{i,j}$ . Then for any  $i$

$$\det(A) = a_{i,1}C_{i,1} + a_{i,2}C_{i,2} + \cdots + a_{i,n}C_{i,n}$$

Expanding the determinant along the  $i$ th row

A diagram of a matrix  $A$  enclosed in large square brackets. A horizontal red line passes through the  $i$ -th row. The entries  $a_{i,1}$ ,  $a_{i,2}$ , and  $a_{i,n}$  are circled in red, blue, and green respectively. Vertical dashed lines separate the columns. The matrix is labeled  $A =$  to the left.

$$a_{i,1}C_{i,1} + a_{i,2}C_{i,2} + \cdots + a_{i,n}C_{i,n}$$

||

$$\det(A)$$

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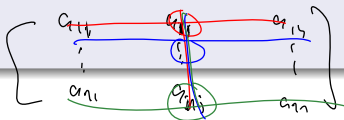
$$\det(A) = a_{i,1} \underline{C_{i,1}} + a_{i,2} \underline{C_{i,2}} + \cdots + a_{i,n} \underline{C_{i,n}}$$

$$= \underline{(-1)^{i+1}} a_{i,1} \underline{M_{i,1}} + \underline{(-1)^{i+2}} a_{i,2} \underline{M_{i,2}} + \cdots + \underline{(-1)^{i+n}} a_{i,n} \underline{M_{i,n}}$$

Moreover, for any  $j$

$$\det(A) = a_{1,j} C_{1,j} + a_{2,j} C_{2,j} + \cdots + a_{n,j} C_{n,j}$$

Expanding along the  $j$ th column



$$a_{1,j} C_{1,j} + a_{2,j} C_{2,j} + \cdots + a_{n,j} C_{n,j} = \det(A)$$

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# Generic Example: $3 \times 3$

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \quad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Expand along the 1<sup>st</sup> row,  $\det(A) =$

$$+ a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

Expand along the 2<sup>nd</sup> column,  $\det(A) =$

$$- a_{12} \det \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} + a_{22} \det \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} - a_{32} \det \begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix}$$

# Specific Example: $4 \times 4$

Calculate the determinant of

$$\begin{array}{cccc} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{array}$$

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

$$\begin{array}{cccc} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{array}$$

$$1 \det \begin{bmatrix} 5 & 6 & 7 \\ 0 & 8 & 9 \\ 0 & 0 & 10 \end{bmatrix}$$

$$- 0 \det \begin{bmatrix} 2 & 3 & 4 \\ 0 & 8 & 9 \\ 0 & 0 & 10 \end{bmatrix}$$

$$+ 0 \det \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 0 & 0 & 10 \end{bmatrix}$$

$$+ 0 \det \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} - 0 \det \begin{bmatrix} 5 & 7 \\ 0 & 9 \end{bmatrix} + 10 \cdot \det \begin{bmatrix} 5 & 6 \\ 0 & 8 \end{bmatrix} - 0 \det \begin{bmatrix} 2 & 3 \\ 5 & 6 \\ 0 & 8 & 9 \end{bmatrix}$$

$$\det(A) = (0 \cdot \det \begin{bmatrix} 5 & 6 \\ 0 & 8 \end{bmatrix}) = 10 (5 \cdot 8 - 0 \cdot 6) = 400$$

$$400 = 1 \times 5 \times 8 \times 10$$

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Note that diagonal matrices are also upper triangular and so

$$\det(D) = \det \left( \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix} \right) = d_1 d_2 \cdots d_n$$

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Further, the identity matrix  $I_n$ , is the diagonal matrix with  $d_1 = d_2 = \cdots = d_n = 1$  and so we may conclude  $\det(I_n) = 1$ .

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- ③ *If the row operation is adding one row to another then*  
 $\det(B) = \det(A)$ .

# Sketch of Proof

①

$$\begin{array}{cccccc} + & - & + & - & + & \dots \\ - & + & - & + & - & \dots \\ + & - & + & - & + & \dots \end{array}$$

expand along first row get  
+ - + - + pattern  
swap first and second row and  
expand along the second row  
- + - + - + pattern

②  $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} \quad B = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}$

expand B along first row  $= a_{11} \det(\dots) + \dots + c_{1n} \det(\dots)$   
 $= c (a_{11} \det(\dots) + \dots + a_{1n} \det(\dots))$   
 $= c (\det A)$

③ i) Use part 1 to show that if two rows are the same  
 the det is 0.  
 ii) if  $A = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$   $B = \begin{pmatrix} r_1 + r_i \\ \vdots \\ r_n \end{pmatrix} \Rightarrow \det B = \det \begin{pmatrix} r_1 \\ \vdots \\ r_i \\ \vdots \\ r_n \end{pmatrix} + \det \begin{pmatrix} r_1 \\ \vdots \\ r_n \\ \vdots \\ r_i \end{pmatrix}$

# Exercise

Row reduce A to REF and then calculate the determinate

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \\ 2 & -1 & -2 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{array}{l} R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 - R_1 \end{array} \quad \frac{-\det(A)}{=}$$

$$\begin{aligned} \det(A) &\downarrow \\ &= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -4 \\ 0 & -5 & -8 & -8 \\ 0 & -1 & -2 & -3 \end{bmatrix} \begin{array}{l} R_1 \leftrightarrow R_3 \\ R_1 \Rightarrow R_3 \end{array} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -5 & -8 & -8 \\ 0 & -2 & -2 & -4 \end{bmatrix} \begin{array}{l} R_3 - 5R_2 \\ R_4 - 2R_2 \end{array} \\ &= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 2 & 2 \end{bmatrix} \begin{array}{l} \frac{1}{2} R_3 \\ \frac{1}{2} R_4 \end{array} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 7/2 \\ 0 & 0 & 2 & 2 \end{bmatrix} \end{aligned}$$

$$-\det(A)'' = \frac{1}{2} \det(A)$$

# More Work Space

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 2 & 2 \end{pmatrix} \xrightarrow{R_4 - 2R_3} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & -5 \end{pmatrix} = B$$

$\frac{1}{2} \det(A)$

$$\frac{1}{2} \det A = \det(B) = 1 \times -1 \times 1 \times -5 = 5$$

$$\det A = -10$$



# Properties of Determinants

## Theorem

Let  $A$  be an  $n \times n$  matrix

①  $\det(A^T) = \det(A)$

$$B = A^T \Rightarrow \det(B) = \det(A)$$

Expanding the determinant <sup>of  $A$</sup>  along a column is the same as expanding the determinant of  $A^T$  along a row

# Properties of Determinants

## Theorem

Let  $A$  be an  $n \times n$  matrix

- 1  $\det(A^T) = \det(A)$
- 2 If  $A$  has a row or column of 0's then  $\det(A) = 0$

If  $A$  has a row of zeros then  
expanding along this row give

$$\det A = 0 \cdot \det(\dots) + 0 \cdot \det(\dots) + \dots + 0 \cdot \det(\dots) = 0$$

likewise if a column is all 0's

# Properties of Determinants

## Theorem

Let  $A$  be an  $n \times n$  matrix

- 1  $\det(A^T) = \det(A)$
- 2 If  $A$  has a row or column of 0's then  $\det(A) = 0$
- 3 If  $A$  has two proportional rows, then  $\det(A) = 0$

If  $A$  has two proportional rows then  
let  $B$  be the matrix obtained from subtraction  
the two rows. So  $B$  has a row of zeros  
and by previous theorem  $\det(A) = \det(B) = 0$

# Properties of Determinants

## Theorem

Let  $A$  be an  $n \times n$  matrix

- 1  $\det(A^T) = \det(A)$
- 2 If  $A$  has a row or column of 0's then  $\det(A) = 0$
- 3 If  $A$  has two proportional rows, then  $\det(A) = 0$
- 4 If  $A$  has two proportional columns, then  $\det(A) = 0$

If  $A$  has two proportional columns then  
 $A^T$  has two proportional rows. and  
so  $\det(A^T) = 0$  and  $\det(A) = \det(A^T) = 0$

# Properties of Determinants

## Theorem

Let  $A$  be an  $n \times n$  matrix

- 1  $\det(A^T) = \det(A)$
- 2 If  $A$  has a row or column of 0's then  $\det(A) = 0$
- 3 If  $A$  has two proportional rows, then  $\det(A) = 0$
- 4 If  $A$  has two proportional columns, then  $\det(A) = 0$
- 5  $\det(cA) = c^n \det(A)$ .

~~$\det(cA) \neq c \cdot \det(A)$~~

$cA$  can be thought of multiplying each row by  $c$ . And by previous theorem each time we multiply a row, we obtain an extra  $c$ .

# Big Theorem

## Theorem

An  $n \times n$  matrix  $A$  is invertible if and only if  $\det(A) \neq 0$ .

Proof:  $A$  is invertible if and only if  
 $A$  has  $\text{RREF}$  of  $I_n$ .

which means that there is a sequence of  
row operations that reduce  $A$  to  $I_n$

Each row operation will either multiply by a  
non zero constant (with  $-1$  being an optional <sup>non zero</sup>) or  
not change the  $\det$   <sub>$\uparrow$</sub>  determinant.

$A$  is invertible  $\Leftrightarrow \det(A) = C \quad \det(I_n) = C \neq 0$

## Theorem

*Let  $A$  be an  $n \times n$  matrix. The the following are equivalent*

- ①  $A\vec{x} = \vec{b}$  has a unique solution for every  $\vec{b}$
- ②  $A\vec{x} = 0$  has a unique solution
- ③  $rk(A) = n$
- ④ The RREF of  $A$  is  $I_n$
- ⑤  $A$  is invertible

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- ⑥ The columns of  $A$  are linearly independent
- ⑦  $\det(A) \neq 0$

# Major Theorem

## Theorem

Let  $A$  be an  $n \times n$  matrix. The the following are equivalent

- ①  $A\vec{x} = \vec{b}$  has a unique solution for every  $\vec{b}$
- ②  $A\vec{x} = 0$  has a unique solution
- ③  $\text{rk}(A) = n$
- ④ The RREF of  $A$  is  $I_n$
- ⑤  $A$  is invertible
- ⑥ The columns of  $A$  are linearly independent
- ⑦  $\det(A) \neq 0$
- ⑧ The row vectors of  $A$  are linearly independent

$A$  is invertible  
 $\Leftrightarrow \det(A) \neq 0$   
 $\Leftrightarrow \det(A^T) \neq 0$   
 $\Leftrightarrow$  columns of  $A^T$   
are lin. inde  
 $\Leftrightarrow$  rows of  $A$   
are lin. ind.

# Properties of Determinants 2

## Theorem

- 1  $\det(AB) = \det(A) \det(B)$
- 2  $\det(A^{-1}) = \frac{1}{\det(A)}$ , provided  $A^{-1}$  exists.

1) If  $A$  is not invertible then  $AB$  is also not invertible and  $\det(AB) = 0 = \det A \det B$

If  $A$  is invertible, then write it as a product of elementary matrices and the result follows by how elementary row operations affect  $\det$ .

$$\begin{aligned} 2) \quad AA^{-1} &= I_n \quad \Rightarrow \quad 1 = \det(I_n) = \det(AA^{-1}) = \det(A) \det(A^{-1}) \\ \det(A) \cdot \det(A^{-1}) &= 1 \quad \Rightarrow \quad \det(A^{-1}) = \frac{1}{\det(A)}. \end{aligned}$$

# More Work Space