Review problems 21.1 – 21.16, Finan.

(21.1) a. Define $g: \mathbb{Z} \to \mathbb{Z}$ by $g(n) = 3n-2$. (i) is g one-to-one? prove or give a counterexample. (ii) is g onto? Proof or counterexample.
b. Define $G: \mathbb{R} \to \mathbb{R}$ by $G(x) = 3x-2$. Is G onto? Proof or counterex.

(i) One-to-one is the property that $x_1 \neq x_2 \Rightarrow g(x_1) \neq g(x_2)$ which is equivalent to $g(x_1) = g(x_2) \Rightarrow x_1 = x_2$ which is true. This can be seen by the computation $3x_1 - 2 = 3x_2 - 2 \Leftrightarrow 3x_1 = 3x_2 \Leftrightarrow x_1 = x_2$. Hence g is one-to-one. (ii) But g is not onto, since $g(n) = 3n-2$ the only integers it assumes are those congruent to $-2 \pmod 3$. For example we never have $g(n) = 0$ for any integer n since that would require $3n-2=0 \Leftrightarrow n = 2/3$ which is not an integer.

b. The function $G(x) = 3x-2$ is onto, for choose an arbitrary $y \in \mathbb{R}$, then $3x-2 = y \Leftrightarrow 3x = y+2 \Leftrightarrow x = \frac{y+2}{3}$ so that G assumes the value y for $x = \frac{y+2}{3}$.

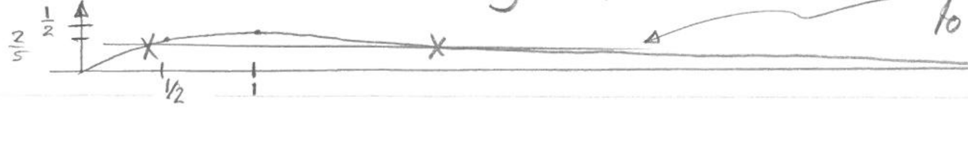
(21.2) Determine whether the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{x+1}{x}$ is one-to-one or not. (Obviously $x \neq 0$.)

Does the implication $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ hold?

$$\frac{x_1+1}{x_1} = \frac{x_2+1}{x_2} \Leftrightarrow \cancel{x_2 x_1} + x_2 = \cancel{x_1 x_2} + x_1 \Leftrightarrow x_1 = x_2,$$ Yes so the function f is one-to-one.

(21.3) Determine whether the function $f(x) = \frac{x}{x^2+1}$ is one-to-one or not. ($f: \mathbb{R} \to \mathbb{R}$)

It cannot be one-to-one since $f(\frac{1}{2}) = \frac{\frac{1}{2}}{(\frac{1}{2})^2+1} = \frac{1}{2(\frac{5}{4})} = \frac{2}{5} < f(1) = \frac{1}{2}$ but $f(x) \to 0$ as $x \to \infty$ so the graph of f has the following appearance:



it is then possible to find $x_1 \neq x_2$ with $f(x_1) = f(x_2)$

f is not one-to-one.

21.4 Let $f: \mathbb{R} \to \mathbb{Z}$ be the floor function: $f(x) = \lfloor x \rfloor$.

a) Is f one-to-one? No since $f(0) = f(0.5) = 0$ and $0 \neq 0.5$.

b) Is f onto? Yes, since for every $y \in \mathbb{Z}$ we have an $x \in \mathbb{R}$ with $f(x) = y$, namely $x = y$.

21.5 Let $\Sigma = \{0,1\}$ and let $l: \Sigma^* \to \mathbb{N}$ denote the length function.

a) Is l one-to-one? Prove or give a counter example.

b) I l onto? Prove or give a counterexample.

a) l is not one-to-one since $l(0) = 1 = l(1)$ and the strings "0" and "1" are distinct.

b) l is onto since for each $y \in \mathbb{N} = \{0,1,2,\ldots\}$ we can take a string s of y zeroes, we then get $l(s) = y$.

21.6 If $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are one-to-one, is $f+g$ also one-to-one?

No since $f(x) = x$ and $g(x) = -x$ are both one-to-one but $f+g = 0$ which is constant and hence not one-to-one.

21.7 Define $F: P\{a,b,c\} \to \mathbb{N}$ to be the number of elements of a subset of $P\{a,b,c\}$. (Probably we mean $\{a,b,c\}$.)

a) Is F one-to-one?

No, since $F(\{a\}) = 1 = F(\{b\})$ and $\{a\} \neq \{b\}$.

b) Is F onto?

No; since $F(X) \leq 3$ for all $X \in P(\{a,b,c\})$ — the maximum number of elements is 3. (Hence $F(X) \neq 4$ for all $X \in P(\{a,b,c\})$.)

21.8 If $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are onto functions, is $f+g$ also onto?

No, and the same functions as in 21.6 exemplifies this.

(21.9) Let $\Sigma = \{a, b\}$, and let $l: \Sigma^* \to \mathbb{N}$ be the length function. Let $f: \mathbb{N} \to \{0,1,2\}$ be the hash function $f(n) = n \bmod 3$. Find $(f \circ l)(abaa)$, $(f \circ l)(baaab)$, and $(f \circ l)(aaa)$.

$(f \circ l)(abaa) = f(l(abaa)) = f(4) = 4 \bmod 3 = \underline{\underline{1}}$.

$(f \circ l)(baaab) = f(l(baaab)) = f(5) = 5 \bmod 3 = \underline{\underline{2}}$.

$(f \circ l)(aaa) = f(l(aaa)) = f(3) = 3 \bmod 3 = \underline{\underline{0}}$.

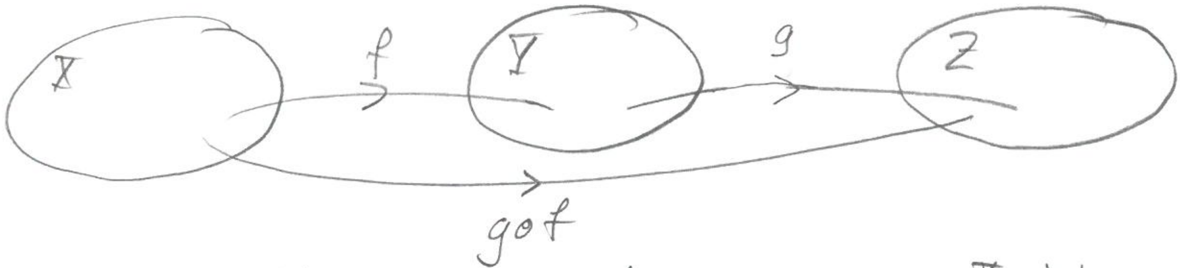
(21.10) Show that the function $F^{-1}: \mathbb{R} \to \mathbb{R}$ given by $F^{-1}(y) = \frac{y-2}{3}$ is the inverse of the function $F(x) = 3x+2$.

We show this by showing that $F \circ F^{-1}$ ~~{~~ and $F^{-1} \circ F$ ~~}~~ is the identity function on $\mathbb{R}$. ~~(The statements are equivalent)~~

Show that $F^{-1}(F(x)) = x$ for all $x \in \mathbb{R}$: & $F(F^{-1}(y)) = y$

$$\begin{cases} F^{-1}(F(x)) = F^{-1}(3x+2) = \frac{(3x+2)-2}{3} = \frac{3x}{3} = \underline{\underline{x}}. \\ F(F^{-1}(y)) = 3 \cdot F^{-1}(y) + 2 = 3 \cdot \frac{y-2}{3} + 2 = \underline{\underline{y}} \end{cases}$$

(21.11) If $f: X \to Y$ and $g: Y \to Z$ are functions and $g \circ f: X \to Z$ is one-to-one, must both f & g be one-to-one? Prove or give a counterexample.

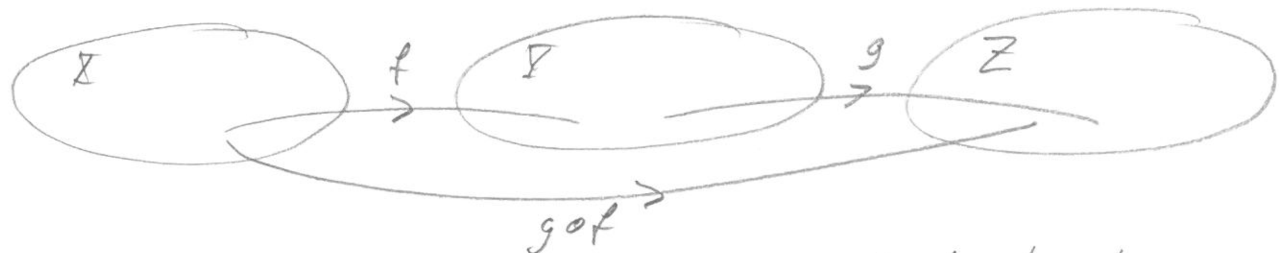


The answer is No because f might map X into a subset of Y, $f(X)$ and for $g \circ f$ to be one-to-one it suffices that the restriction of g to $f(X)$ is one-to-one.

Example: $X = Z = \mathbb{R}$, $Y = \mathbb{R}^*$. $f(x) = e^x$ and $g(y) = \ln|y|$

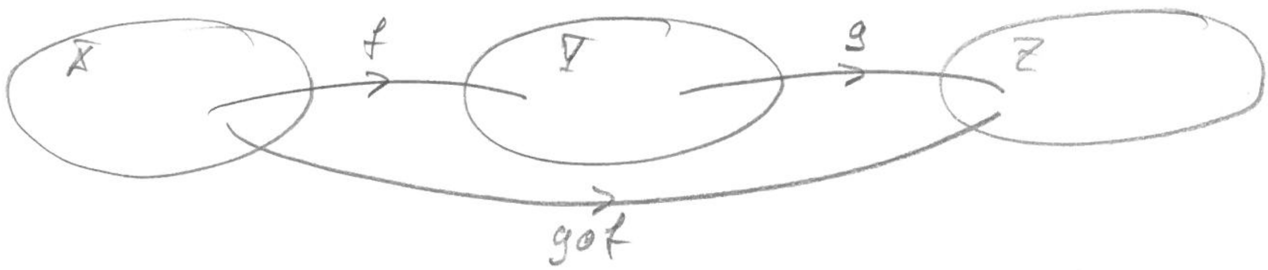
$g: \mathbb{R}^* \to \mathbb{R}$ is not one-to-one, because $g(-e) = \ln|-e| = 1 = \ln|e| = g(e)$. But $g \circ f = \ln|e^x| = \ln e^x = x$ which is certainly a one-to-one function.

(21.12) If $f: X \to Y$ and $g: Y \to Z$ are functions and $g \circ f: X \to Z$ is onto, must both f and g be onto? Prove or give a counterexample.



g must certainly be onto, but must f also be onto? No set $X = \mathbb{R}$, $Y = \mathbb{R} \setminus \{\pi/2 + n\pi \,;\, n = 0, \pm 1, \pm 2, \ldots\}$ and $Z = \mathbb{R}$ and put $g(y) = \tan(y)$ and $f(x) = \arctan(x)$. Then $g \circ f(x) = \tan(\arctan(x)) = x$ which is onto but $f: X \to Y$ only assumes values in $(-\frac{\pi}{2}, \frac{\pi}{2})$ not in $(\frac{\pi}{2}, \frac{3\pi}{2})$ and so on.

(21.13) If $f: X \to Y$ and $g: Y \to Z$ are functions and $g \circ f: X \to Z$ is one-to-one must f be one-to-one?

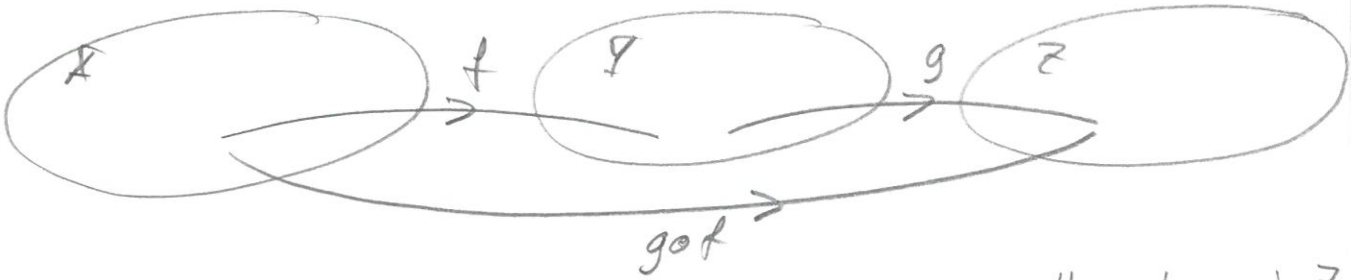


Do we have $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$? (Given that $g \circ f$ is one-to-one?) Yes we do, since

$$f(x_1) = f(x_2) \Rightarrow g(f(x_1)) = g(f(x_2)) \Leftrightarrow$$

$$(g \circ f)(x_1) = (g \circ f)(x_2),$$ and since $g \circ f$ is one-to-one, $x_1 = x_2$ from which the one-to-one-ness of f also follows.

(21.14) If $f: X \to Y$ and $g: Y \to Z$ are functions and $g\circ f: X \to Z$ is onto, must also g be onto?



Clearly <u>yes</u> because $g\circ f$ must assume all values in Z, if g leaves out one value, this value cannot be assumed by $g\circ f$.

(21.15) Let $f: W \to X$, $g: X \to Y$, $h: Y \to Z$ be functions. Must $h\circ(g\circ f) = (h\circ g)\circ f$? Prove or give a counterexample.

Both functions $h\circ(g\circ f)$ and $(h\circ g)\circ f$ have W as domain and Z as co-domain. If, for each $w \in W$ we have $(h\circ(g\circ f))(x) = ((h\circ g)\circ f)(x)$ the functions are the same.

By definition

$(h\circ(g\circ f))(x) = h((g\circ f)(x)) = h(g(f(x)))$ and

$((h\circ g)\circ f)(x) = (h\circ g)(f(x)) = h(g(f(x)))$ and these

are obviously the same. Hence $h\circ(g\circ f) = (h\circ g)\circ f$.

(21.16) Let $f: X \to Y$ and $g: Y \to Z$ be two bijective functions. Show that $(g\circ f)^{-1}$ exists and $(g\circ f) = f^{-1}\circ g^{-1}$.

We need to show that $((g\circ f)\circ(f^{-1}\circ g^{-1}))(z) = z$ for all $z \in Z$. By the previous problem, $((g\circ f)\circ(f^{-1}\circ g^{-1}))(z) =$

$= (g\circ\underbrace{(f\circ f^{-1})}_{=\iota}\circ g^{-1})(z) = (g\circ \iota \circ g^{-1})(z) = (g\circ g^{-1})(z) = \iota(z)$

$= z$, where ι is the identity function. Hence $(f^{-1}\circ g^{-1})$ is the inverse of $(g\circ f)$.

and $((f^{-1}\circ g^{-1})\circ(g\circ f))(x) = x$ ... similar!