Fundamentals of Counting and Probability Theory

The major goal of this chapter is to establish several techniques for counting large finite sets without actually listing their elements. Also, the fundamentals of probability theory are discussed.

31 Elements of Counting

For a set X, $|X|$ denotes the number of elements of X. It is easy to see that for any two sets A and B we have the following result known as the **Inclusion - Exclusion Principle**

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Indeed, $|A|$ gives the number of elements in A including those that are common to A and B. The same holds for $|B|$. Hence, $|A| + |B|$ includes twice the number of common elements. Hence, to get an accurate count of the elements of $A \cup B$, it is necessary to subtract $|A \cap B|$ from $|A| + |B|$.
Note that if A and B are disjoint then $|A \cap B| = 0$ and consequently $|A \cup B| = |A| + |B|$.

Example 31.1 *(The Addition Rule)*
Show by induction on n, that if $\{A_1, A_2, \cdots, A_n\}$ is a collection of pairwise disjoint sets then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$

Solution.
Basis of induction: For $n = 2$ the result holds by the Inclusion-Exclusion

Principle.
Induction hypothesis: Suppose that for any collection $\{A_1, A_2, \cdots, A_n\}$ of pairwise disjoint sets we have

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$

Induction step: Let $\{A_1, A_2, \cdots, A_n, A_{n+1}\}$ be a collection of pairwise disjoint sets. Since $(A_1 \cup A_2 \cup \cdots \cup A_n) \cap A_{n+1} = (A_1 \cap A_{n+1}) \cup \cdots \cup (A_n \cap A_{n+1}) = \emptyset$, by the Inclusion-Exclusion Principle and the induction hypothesis we have

$$\begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}| &= |A_1 \cup A_2 \cup \cdots \cup A_n| + |A_{n+1}| \\ &= |A_1| + |A_2| + \cdots + |A_n| + |A_{n+1}| \quad \blacksquare \end{aligned}$$

Å så finns det något som saknas här...

Example 31.2
A total of 35 programmers interviewed for a job; 25 knew FORTRAN, 28 knew PASCAL, and 2 knew neither languages. How many knew both languages?

Solution.
Let A be the group of programmers that knew FORTRAN, B those who knew PASCAL. Then $A \cap B$ is the group of programmers who knew both languages. By the Inclusion-Exclusion Principle we have

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

That is,

$$33 = 25 + 28 - |A \cap B|.$$

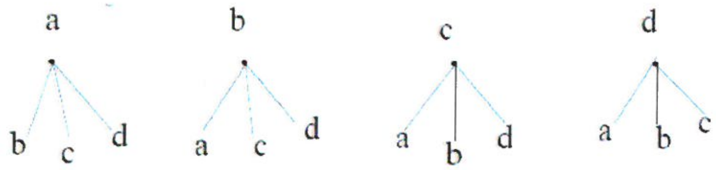
Solving for $|A \cap B|$ we find $|A \cap B| = 20$. $\blacksquare$
Another important rule of counting is the **multiplication rule**. It states that if a decision consists of k steps, where the first step can be made in n_1 different ways, the second step in n_2 ways, $\cdots$, the kth step in n_k ways, then the decision itself can be made in $n_1 n_2 \cdots n_k$ ways.

Example 31.3
a. How many possible outcomes are there if 2 distinguishable dice are rolled?
b. Suppose that a state's license plates consist of 3 letters followed by four digits. How many different plates can be manufactured? (no repetitions)

Solution.
a. By the multiplication rule there are $6 \times 6 = 36$ possible outcomes.
b. By the multiplication rule there are $26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7$ possible license plates. ■

Example 31.4
Let $\Sigma = \{a, b, c, d\}$ be an alphabet with 4 letters. Let Σ^2 be the set of all words of length 2 with letters from Σ. Find the number of all words of length 2 where the letters are not repeated. First use the product rule. List the words by means of a **tree diagram**.

Solution.
By the multiplication rule there are $4 \times 3 = 12$ different words. Constructing a tree diagram



we find that the words are

$$\{ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc\}$$ ■

An **r-permutation** of n objects, in symbol $P(n, r)$, is an ordered selection of r objects from a given n objects.

Example 31.5
a. Use the product rule to show that $P(n, r) = \frac{n!}{(n-r)!}$.
b. Find all possible 2-permutations of the set $\{1, 2, 3\}$.

Solution.
a. We can treat a permutation as a decision with r steps. The first step can be made in n different ways, the second in $n - 1$ different ways, ..., the rth in $n - r + 1$ different ways. Thus, by the multiplication rule there are $n(n-1) \cdots (n - r + 1)$ r-permutations of n objects. That is, $P(n, r) = n(n-1) \cdots (n - r + 1) = \frac{n!}{(n-r)!}$.
b. $P(3, 2) = \frac{3!}{(3-2)!} = 6$. ■

Example 31.6
How many license plates are there that start with three letters followed by 4 digits (no repetitions)?

Solution.
$P(26,3)\cdot P(10,4) = 78,624,000.$ ■

An **r-combination** of n objects, in symbol $C(n,r)$, is an unordered selection of r of the n objects. Thus, $C(n,r)$ is the number of ways of choosing r objects from n given objects without taking order in account. But the number of different ways that r objects can be ordered is $r!$. Since there are $C(n,r)$ groups of r objects from a given n objects, the number of ordered selection of r objects from n given objects is $r!C(n,r) = P(n,r)$. Thus

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}.$$

Example 31.7
In how many different ways can a hand of 5 cards be selected from a deck of 52 cards?(no repetition)

Solution.
$C(52,5) = 2,598,960.$ ■

Example 31.8
Prove the following identities:
a. $C(n,0) = C(n,n) = 1$ and $C(n,1) = C(n,n-1) = n$.
b. Symmetry property: $C(n,r) = C(n,n-r), r \leq n$.
c. Pascal's identity: $C(n+1,k) = C(n,k-1) + C(n,k), n \geq k$.

Solution.
a. Follows immediately from the definition of of $C(n,r)$.
b. Indeed, we have

$$\begin{aligned} C(n,n-r) &= \tfrac{n!}{(n-r)!(n-n+r)!} \\ &= \tfrac{n!}{r!(n-r)!} \\ &= C(n,r) \end{aligned}$$

c.

$$\begin{aligned}
C(n,k-1)+C(n,k) &= \tfrac{n!}{(k-1)!(n-k+1)!}+\tfrac{n!}{k!(n-k)!}\\
&= \tfrac{n!k}{k!(n-k+1)!}+\tfrac{n!(n-k+1)}{k!(n-k+1)!}\\
&= \tfrac{n!}{k!(n-k+1)!}(k+n-k+1)\\
&= \tfrac{(n+1)!}{k!(n+1-k)!}=C(n+1,k) \quad \blacksquare
\end{aligned}$$

Pascal's identity allows one to construct the following triangle known as Pascal's triangle (for $n = 4$) as follows

$$\begin{array}{lllllllll}
1 & & & & & & & & \\
1 & \rightarrow & 1 & & & & & & \\
1 & \rightarrow & 2 & \rightarrow & 1 & & & & \\
1 & \rightarrow & 3 & \rightarrow & 3 & \rightarrow 1 & & & \\
1 & \rightarrow & 4 & \rightarrow & 6 & \rightarrow 4 & \rightarrow & 1 &
\end{array}$$

The following theorem provides an expansion of $(x+y)^n$ where n is a non-negative integer.

Theorem 31.1 *(Binomial Theorem)*
Let x and y be variables, and let n be a positive integer. Then

$$(x+y)^n = \sum_{k=0}^{n} C(n,k)x^{n-k}y^k$$

where $C(n,k)$ is called the **binomial coefficient**.

Proof.
The proof is by induction.

Basis of induction: For $n = 1$ we have

$$(x+y)^1 = \sum_{k=0}^{1} C(1,k)x^{1-k}y^k = x+y.$$

Induction hypothesis: Suppose that the theorem is true for n.
Induction step: Let us show that it is still true for $n+1$. That is

$$(x+y)^{n+1} = \sum_{k=0}^{n+1} C(n+1,k)x^{n-k+1}y^k.$$

Indeed, we have

$$\begin{aligned}
(x+y)^{n+1} &= (x+y)(x+y)^n = x(x+y)^n + y(x+y)^n\\
&= x\sum_{k=0}^{n} C(n,k)x^{n-k}y^k + y\sum_{k=0}^{n} C(n,k)x^{n-k}y^k\\
&= \sum_{k=0}^{n} C(n,k)x^{n-k+1}y^k + \sum_{k=0}^{n} C(n,k)x^{n-k}y^{k+1}\\
&= C(n,0)x^{n+1} + C(n,1)x^n y + C(n,2)x^{n-1}y^2\\
&+ \cdots + C(n,n)xy^n + C(n,0)x^n y\\
&+ C(n,1)x^{n-1}y^2 + \cdots + C(n,n-1)xy^n\\
&+ C(n,n)y^{n+1}\\
&= C(n+1,0)x^{n+1} + C(n+1,1)x^n y + C(n+1,2)x^{n-1}y^2\\
&+ \cdots + C(n+1,n)xy^n + C(n+1,n+1)y^{n+1}\\
&= \sum_{k=0}^{n+1} C(n+1,k)x^{n-k+1}y^k.
\end{aligned}$$

Example 31.9

Expand $(x+y)^6$ using the binomial theorem.

Solution.

By the Binomial Theorem and Pascal's triangle we have

$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6 \blacksquare$$

Example 31.10

a. Show that $\sum_{k=0}^{n} C(n,k) = 2^n$.

b. Show that $\sum_{k=0}^{n} (-1)^k C(n,k) = 0$.

Solution.

a. Letting $x = y = 1$ in the binomial theorem we find

$$2^n = (1+1)^n = \sum_{k=0}^{n} C(n,k).$$

b. This follows from the binomial theorem by letting $x = 1$ and $y = -1$ $\blacksquare$

Review Problems

Problem 31.1
a. How many ways can we get a sum of 4 or a sum of 8 when two distinguishable dice are rolled?
b. How many ways can we get a sum of 8 when two undistinguishable dice are rolled?

Problem 31.2
a. How many 4-digit numbers can be formed using the digits, $1, 2, \cdots, 9$ (with repetitions)? How many can be formed if no digit can be repeated?
b. How many different license plates are there that involve $1, 2,$ or 3 letters followed by 4 digits (with repetitions)?

Problem 31.3
a. In how many ways can 4 cards be drawn, with replacement, from a deck of 52 cards?
b. In how many ways can 4 cards be drawn, without replacement, from a deck of 52 cards?

Problem 31.4
In how many ways can 7 women and 3 men be arranged in a row if the three men must always stand next to each other.

Problem 31.5
A menu in a Chinese restaurant allows you to order exactly two of eight main dishes as part of the dinner special. How many different combinations of main dishes could you order?

Problem 31.6
Find the coefficient of a^5b^7 in the binomial expansion of $(a - 2b)^{12}$.

Problem 31.7
Use the binomial theorem to prove that

$$3^n = \sum_{k=0}^{n} 2^k C(n, k).$$