

Elements of Graph Theory

In this chapter we present the basic concepts related to graphs and trees such as the degree of a vertex, connectedness, Euler and Hamiltonian circuits, isomorphisms of graphs, rooted and spanning trees.

34 Graphs, Paths, and Circuits

An **undirected graph** G consists of a set V_G of **vertices** and a set E_G of **edges** such that each edge $e \in E_G$ is associated with an unordered pair of vertices, called its **endpoints**.

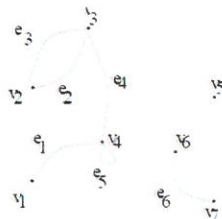
A **directed graph** or **digraph** G consists of a set V_G of vertices and a set E_G of edges such that each edge $e \in E_G$ is associated with an ordered pair of vertices.

We denote a graph by $G = (V_G, E_G)$.

Two vertices are said to be **adjacent** if there is an edge connecting the two vertices. Two edges associated to the same vertices are called **parallel**. An edge incident to a single vertex is called a **loop**. A vertex that is not incident on any edge is called an **isolated** vertex. A graph with neither loops nor parallel edges is called a **simple** graph.

Example 34.1

Consider the following graph G



- Find E_G and V_G .
- List the isolated vertices.
- List the loops.
- List the parallel edges.
- List the vertices adjacent to v_3 .
- Find all edges incident on v_4 .

Solution.

- $E_G = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $V_G = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$.
- There is only one isolated vertex, v_5 .
- There is only one loop, e_5 .
- $\{e_2, e_3\}$.
- $\{v_2, v_4\}$.
- $\{e_1, e_4, e_6\}$. ■

Example 34.2

Which one of the following graphs is simple.

a.



b.

**Solution.**

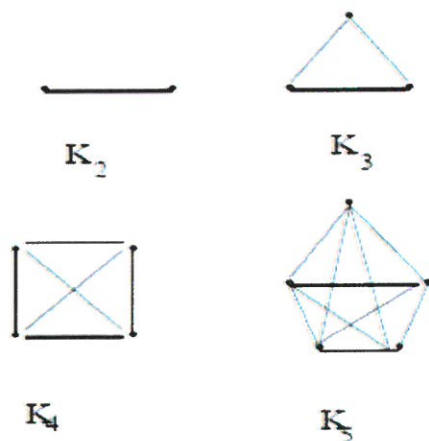
- G is not simple since it has a loop and parallel edges.
- G is simple. ■

A **complete graph** on n vertices, denoted by K_n , is the simple graph that contains exactly one edge between each pair of distinct vertices.

Example 34.3

Draw K_2 , K_3 , K_4 , and K_5 .

Solution.

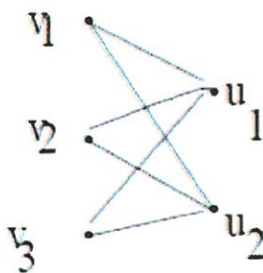


■

A graph in which the vertices can be partitioned into two disjoint sets V_1 and V_2 with every edge incident on one vertex in V_1 and one vertex of V_2 is called **bipartite graph**.

Example 34.4

a. Show that the graph G is bipartite.



b. Show that K_3 is not bipartite.

Solution.

a. Clear from the definition and the graph.

b. Any two sets of vertices of K_3 will have one set with at least two vertices. Thus, according to the definition of bipartite graph, K_3 is not bipartite. ■

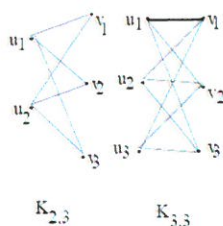
A **complete bipartite graph** $K_{m,n}$, is the graph that has its vertex set partitioned into two disjoint subsets of m and n vertices, respectively. More-

over, there is an edge between two vertices if and only if one vertex is in the first set and the other vertex is in the second set.

Example 34.5

Draw $K_{2,3}$, $K_{3,3}$.

Solution.

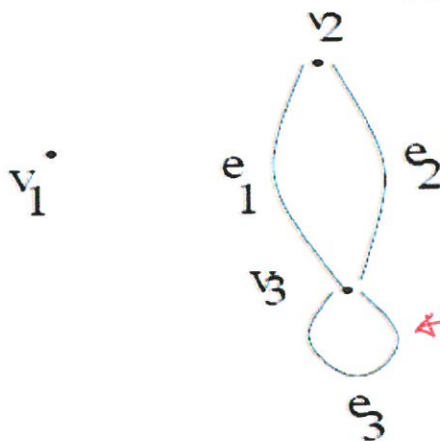


■

The **degree** of a vertex v in an undirected graph, in symbol $\deg(v)$, is the number of edges incident on it. By definition, a loop at a vertex contributes twice to the degree of that vertex. The **total degree** of G is the sum of the degrees of all the vertices of G .

Example 34.6

What are the degrees of the vertices in the following graph



Solution.

$\deg(v_1) = 0, \deg(v_2) = 2, \deg(v_3) = 4$. ■

← detta är en loop
 så i vår mening
 är inte ~~detta en~~
 den här en riktig
 graf utan en
pseudograf.

Theorem 34.1

For any graph $G = (V_G, E_G)$ we have

$$2|E_G| = \sum_{v \in V(G)} \deg(v).$$

Proof.

Suppose that $V_G = \{v_1, v_2, \dots, v_n\}$ and $|E_G| = m$. Let $e \in E_G$. If e is a loop then it contributes 2 to the total degree of G . If e is not a loop then let v_i and v_j denote the endpoints of e . Then e contributes 1 to $\deg(v_i)$ and contributes 1 to the $\deg(v_j)$. Therefore, e contributes 2 to the total degree of G . Since e was chosen arbitrarily, this shows that each edge of G contributes 2 to the total degree of G . Thus,

$$2|E_G| = \sum_{v \in V(G)} \deg(v) \blacksquare$$

The following is easily deduced from the previous theorem.

Theorem 34.2

In any graph there are an even number of vertices of odd degree.

Proof.

Let $G = (V_G, E_G)$ be a graph. By the previous theorem, the sum of all the degrees of the vertices is $T = 2|E_G|$, an even number. Let E be the sum of the numbers $\deg(v)$, each which is even and O the sum of numbers $\deg(v)$ each which is odd. Then $T = E + O$. That is, $O = T - E$. Since both T and E are even, O is also even. This implies that there must be an even number of the odd degrees. Hence, there must be an even number of vertices with odd degree. ■

Example 34.7

Find a formula for the number of edges in K_n .

Solution.

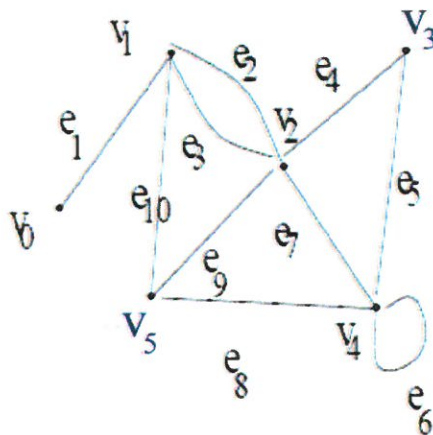
Since G is complete, each vertex is adjacent to the remaining vertices. Thus, the degree of each of the n vertices is $n - 1$, and we have the sum of the degrees of all of the vertices being $n(n - 1)$. By Theorem 34.1, $n(n - 1) = 2|E_G|$.

This completes a proof of the theorem ■

In an undirected graph G a sequence P of the form $v_0e_1v_1e_2\cdots v_{n-1}e_nv_n$ with no edge repeated is called a **path of length n** or a path connecting v_0 to v_n . If P is a path such that $v_0 = v_n$ then it is called a **circuit** or a **cycle**. A path or circuit is **simple** if it does not contain the same vertex more than once. A graph that does not contain any circuit is called **acyclic**.

Example 34.8

In the graph below, determine whether the following sequences are paths, simple paths, circuits, or simple circuits.



- $v_0e_1v_1e_{10}v_5e_9v_2e_2v_1$.
- $v_3e_5v_4e_8v_5e_{10}v_1e_3v_2$.
- $v_1e_2v_2e_3v_1$.
- $v_5e_9v_2e_4v_3e_5v_4e_6v_4e_8v_5$.

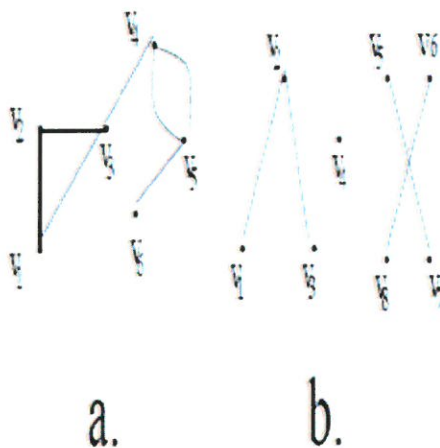
Solution.

- a path (no repeated edge), not a simple path (repeated vertex v_1), not a circuit
- a simple path
- a simple circuit
- a circuit, not a simple circuit (vertex v_4 is repeated) ■

An undirected graph is called **connected** if there is a path between every pair of distinct vertices of the graph. A graph that is not connected is said to be **disconnected**.

Example 34.9

Determine which graph is connected and which one is disconnected.

**Solution.**

a. Connected.

b. Disconnected since there is no path connecting the vertices v_1 and v_4 . ■

A simple path that contains all edges of a graph G is called an **Euler path**. If this path is also a circuit, it is called an **Euler circuit**.

Theorem 34.3

If a graph G has an Euler circuit then every vertex of the graph has even degree.

Proof.

Let G be a graph with an Euler circuit. Start at some vertex on the circuit and follow the circuit from vertex to vertex, erasing each edge as you go along it. When you go through a vertex you erase one edge going in and one edge going out, or else you erase a loop. Either way, the erasure reduces the degree of the vertex by 2. Eventually every edge gets erased and all the vertices have degree 0. So all vertices must have had even degree to begin with. ■

It follows from the above theorem that if a graph has a vertex with odd degree then the graph can not have an Euler circuit.

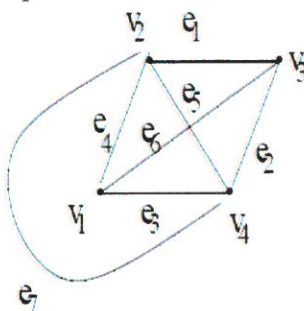
The following provides a converse to the above theorem.

Theorem 34.4 (*Euler Theorem*)

If all the vertices of a connected graph have even degree, then the graph has an Euler circuit.

Example 34.10

Show that the following graph has no Euler circuit.

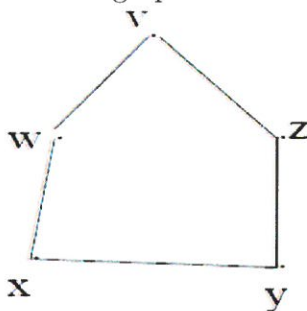
**Solution.**

Vertices v_1 and v_3 both have degree 3, which is odd. Hence, by the remark following the previous theorem, this graph does not have an Euler circuit. ■

A path is called a **Hamiltonian path** if it visits every vertex of the graph exactly once. A circuit that visits every vertex exactly once except for the last vertex which duplicates the first one is called a **Hamiltonian circuit**.

Example 34.11

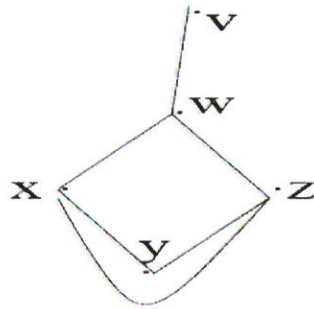
Find a Hamiltonian circuit in the graph

**Solution.**

$vwxyzv$ ■

Example 34.12

Show that the following graph has a Hamiltonian path but no Hamiltonian circuit.

**Solution.**

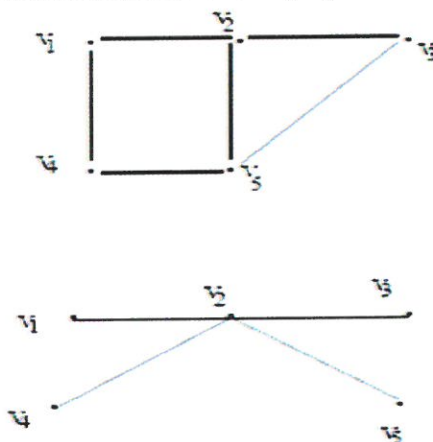
$vwxyz$ is a Hamiltonian path. There is no Hamiltonian circuit since no cycle goes through v . ■

Review Problems

Problem 34.1

The **union** of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The **intersection** of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$.

Find the union and the intersection of the graphs



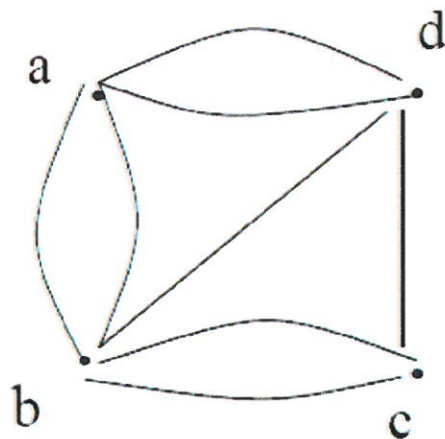
Problem 34.2

Graphs can be represented using matrices. The **adjacency** matrix of a graph G with n vertices is an $n \times n$ matrix A_G such that each entry a_{ij} is the number of edges connecting v_i and v_j . Thus, $a_{ij} = 0$ if there is no edge from v_i to v_j .

a. Draw a graph with the adjacency matrix

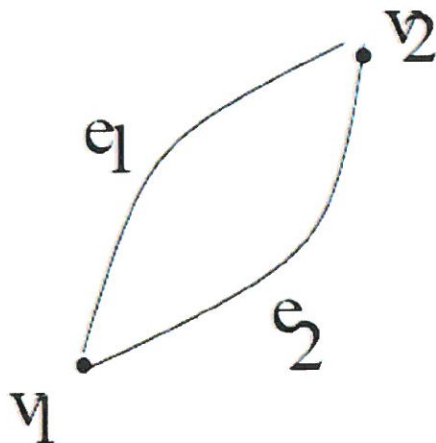
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

b. Use an adjacency matrix to represent the graph

**Problem 34.3**

A graph $H = (V_H, E_H)$ is a **subgraph** of $G = (V_G, E_G)$ if and only if $V_H \subseteq V_G$ and $E_H \subseteq E_G$.

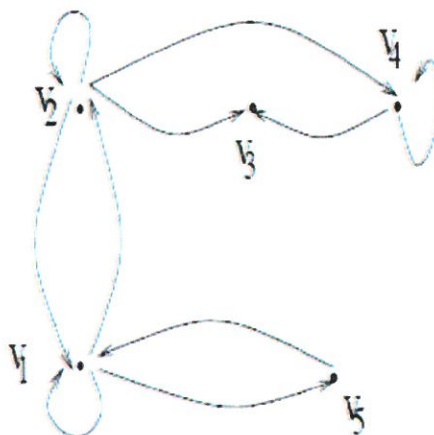
Find all nonempty subgraphs of the graph



When (u, v) is an edge in a directed graph G then u is called the **initial vertex** and v is called the **terminal vertex**. In a directed graph, the **in-degree** of a vertex v , denoted by $\deg^-(v)$, is the number of edges with v as their terminal vertex. Similarly, the **out-degree** of v , denoted by $\deg^+(v)$, is the number of edges with v as an initial vertex. Note that $\deg(v) = \deg^+(v) + \deg^-(v)$.

Problem 34.4

Find the in-degree and out-degree of each of the vertices in the graph G with directed edges.

**Problem 34.5**

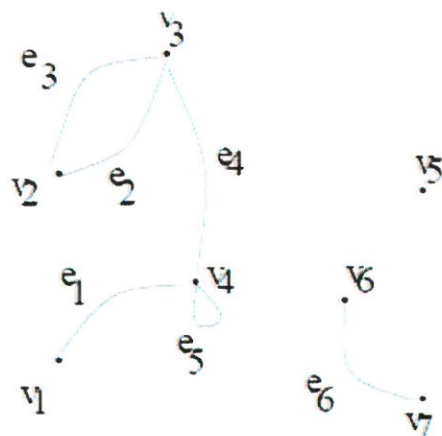
Show that for a digraph $G = (V_G, E_G)$ we have

$$|E_G| = \sum_{v \in V(G)} \deg^-(v) = \sum_{v \in V(G)} \deg^+(v).$$

Another useful matrix representation of a graph is known as the **incidence matrix**. It is constructed as follows. We label the rows with the vertices and the columns with the edges. The entry for row v and column e is 1 if e is incident on v and 0 otherwise. If e is a loop at v we assign the value 2. It is easy to see that the sum of entries of each column is 2 and that the sum of entries of a row gives the degree of the vertex corresponding to that row.

Problem 34.6

Find the incidence matrix corresponding to the graph

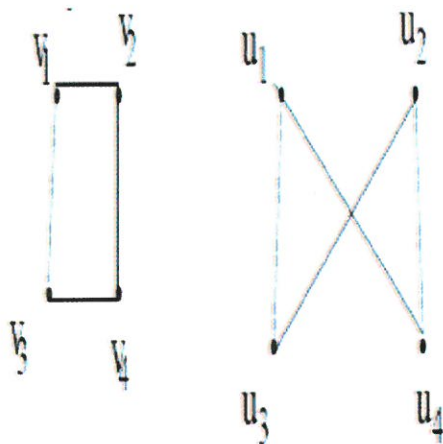
**Problem 34.7**

If each vertex of an undirected graph has degree k then the graph is called a **regular** graph of degree k .

How many edges are there in a graph with 10 vertices each of degree 6?

Problem 34.8

Two simple graphs G_1 and G_2 are **isomorphic**, in symbol, $G_1 \simeq G_2$, if there is one-to-one onto function, $f : V(G_1) \rightarrow V(G_2)$ and $(u, v) \in E_{G_1}$ if and only if $(f(u), f(v)) \in E_{G_2}$. Show that the following graphs are isomorphic.



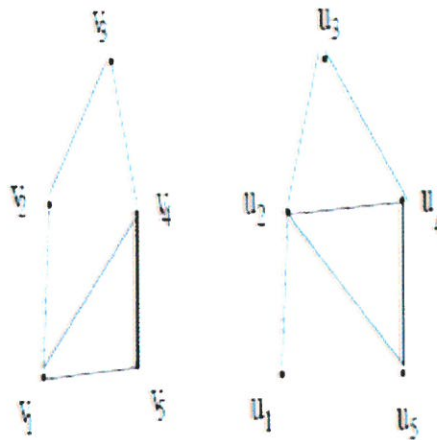
Warning: The number of vertices, the number of edges, and the degrees of the vertices are all invariants under isomorphism. If any of these quantities

differ in two graphs, these graphs cannot be isomorphic. However, when these invariants are the same, it does not necessarily mean that the two graphs are isomorphic.

The isomorphism between two graphs $G_1 = (V_{G_1}, E_{G_1})$ and $G_2 = (V_{G_2}, E_{G_2})$ with parallel edges or loops requires two bijections $f : V_{G_1} \rightarrow V_{G_2}$ and $g : E_{G_1} \rightarrow E_{G_2}$ such that if $e \in E_{G_1}$ is an edge with endpoints (u, v) then $g(e) \in E_{G_2}$ is an edge with endpoints $(f(u), f(v))$.

Problem 34.9

Show that the following graphs are not isomorphic.



Problem 34.10

Show that the following graph has no Hamiltonian path.

