11 Method of Proof by Induction

With the emphasis on structured programming has come the development of an area called **program verification,** which means your program is correct as you are writing it.
One technique essential to program verification is **mathematical induction**, a method of proof that has been useful in every area of mathematics as well.
Consider an arbitrary loop in Pascal starting with the statement

$$FOR\ I\ := 1\ TO\ N\ DO$$

If you want to verify that the loop does something regardless of the particular integral value of N, you need mathematical induction.
Also, sums of the form

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

are very useful in analysis of algorithms and a proof of this formula is mathematical induction.
Next we examine this method. We want to prove that a predicate $P(n)$ is true for any nonnegative integer $n \geq n_0$. The steps of mathematical induction are as follows:

(i) (Basis of induction) Show that $P(n_0)$ is true.
(ii) (Induction hypothesis) Assume $P(n)$ is true.
(iii) (Induction step) Show that $P(n+1)$ is true.

Example 11.1
Use the technique of mathematical induction to show that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}, \quad n \geq 1.$$

Solution.
Let $P(n) : 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. Then
(i) (Basis of induction) $P(1) : 1 = \frac{1(1+1)}{2}$. That is, $P(1)$is true.
(ii) (Induction hypothesis) Assume $P(n)$ is true. That is, $P(n) : 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Referens till induktions-axiomet saknas!

INTE BRA!

(iii) (Induction step) We must show that $P(n+1) : 1+2+3+\cdots+n+1 = \frac{(n+1)(n+2)}{2}$. Indeed,

$$1+2+\cdots+n+(n+1) = (1+2+\cdots+n)+n+1 = \frac{n(n+1)}{2}+(n+1) = \frac{(n+1)(n+2)}{2} \blacksquare$$

Example 11.2 *(Geometric progression)*

a. Use induction to show $P(n) : \sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}$, $n \geq 0$ where $r \neq 1$.

b. Show that $1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} \leq 2$, for all $n \geq 1$.

Solution.

a. We use the method of proof by mathematical induction.

(i) (Basis of induction) $a = a\frac{1-r^{0+1}}{1-r} = \sum_{k=0}^{0} ar^k$. That is, $P(0)$is true.

(ii) (Induction hypothesis) Assume $P(n)$ is true. That is, $\sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}$.

(iii) (Induction step) We must show that $P(n+1)$ is true. That is, $\sum_{k=0}^{n+1} ar^k = \frac{a(1-r^{n+2})}{1-r}$. Indeed,

$$\begin{aligned}\sum_{k=0}^{n+1} ar^k =& \sum_{k=0}^{n} ar^k + ar^{n+1}\\ =& a\frac{1-r^{n+1}}{1-r} + ar^{n+1}\frac{1-r}{1-r}\\ =& a\frac{1-r^{n+1}+r^{n+1}-r^{n+2}}{1-r}\\ =& a\frac{1-r^{n+2}}{1-r}.\end{aligned}$$

b. By a. we have

$$\begin{aligned}1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}} =& \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}\\ =& 2(1-(\frac{1}{2})^n)\\ =& 2-\frac{1}{2^{n-1}}\\ \leq& 2 \blacksquare\end{aligned}$$

Example 11.3 *(Arithmetic progression)*
Use induction to show that $P(n) : \sum_{k=1}^{n}(a+(k-1)r) = \frac{n}{2}[2a+(n-1)r],\ n \geq 1$.

Solution.
We use the method of proof by mathematical induction.

(i) (Basis of induction) $a = \frac{1}{2}[2a + (1-1)r] = \sum_{k=1}^{1}(a + (k-1)r)$. That is, $P(1)$is true.
(ii) (Induction hypothesis) Assume $P(n)$ is true. That is, $\sum_{k=1}^{n}(a+(k-1)r) = \frac{n}{2}[2a + (n-1)r]$.
(iii) (Induction step) We must show that $P(n+1)$ is true. That is, $\sum_{k=1}^{n+1}(a + (k-1)r) = \frac{(n+1)}{2}[2a + nr]$. Indeed,

$$\begin{aligned}\sum_{k=1}^{n+1}(a + (k-1)r) =& \sum_{k=1}^{n}(a + (k-1)r) + a + (n+1-1)r \\ =& \frac{n}{2}[2a + (n-1)r] + a + nr \\ =& \frac{2an + n^2r - nr + 2a + 2nr}{2} \\ =& \frac{2a(n+1) + n(n+1)r}{2} \\ =& \frac{n+1}{2}[2a + nr] \blacksquare\end{aligned}$$

We next exhibit a theorem whose proof uses mathematical induction.

Theorem 11.1
For all integers $n \geq 1$, $2^{2n} - 1$ is divisible by 3.

Proof.
Let $P(n) : 2^{2n} - 1$ is divisible by 3. Then
(i) (Basis of induction) $P(1)$ is true since 3 is divisible by 3.
(ii) (Induction hypothesis) Assume $P(n)$ is true. That is, $2^{2n} - 1$ is divisible by 3.
(iii) (Induction step) We must show that $2^{2n+2} - 1$ is divisible by 3. Indeed,

$$\begin{aligned}2^{2n+2}-1 =& 2^{2n}(4)-1\\ =& 2^{2n}(3+1)-1\\ =& 2^{2n}\cdot 3+(2^{2n}-1)\\ =& 2^{2n}\cdot 3+P(n)\end{aligned}$$

Since $3|(2^{2n}-1)$ and $3|(2^{2n}\cdot 3)$ we have $3|(2^{2n}\cdot 3+2^{2n}-1)$. This ends a proof of the theorem ■ DETTA BEVIS SAKNAR REFERENS TILL INDUKTIONSAXIOMET!

Example 11.4

a. Use induction to prove that $n < 2^n$ for all non-negative integers n.

b. Use induction to prove that $2^n < n!$ for all non-negative integers $n \geq 4$.

Solution.

a. Let $P(n) : n < 2^n$ We want to show that $P(n)$ is valid for all $n \geq 0$. By the method of mathematical induction we have

(i) (Basis of induction) $2^0 - 0 = 1 > 0$. That is, $0 < 2^0$. Thus, $P(0)$is true.

(ii) (Induction hypothesis) Assume $P(n)$ is true. That is, $n < 2^n$.

(iii) (Induction step) We must show that $P(n+1)$ is also true. That is, $n+1 < 2^{n+1}$. Indeed,

$$\begin{aligned}2^{n+1}-(n+1) =& 2^n cdot 2 - n - 1\\ =& 2^n(1+1)-n-1\\ =& (2^n-n)+2^n-1\\ >& 2^n-1\\ >& 0\end{aligned}$$

where we used the fact that $2^n - n > 0$.

b. Let $P(n) : 2^n < n!$. We want to show that $P(n)$ is valid for all $n \geq 4$. By the method of mathematical induction we have

(i) (Basis of induction) $4! - 2^4 = 8 > 0$. That is, $P(4)$is true.

(ii) (Induction hypothesis) Assume $P(n)$ is true. That is, $2^n < n!,\ n \geq 4$.

(iii) (Induction step) We must show that $P(n+1)$ is true. That is, $2^{n+1} < (n+1)!$. Indeed,

$$\begin{aligned}(n+1)! - 2^{n+1} =&(n+1)n! - 2^n(1+1)\\ =&n! - 2^n + nn! - 2^n\\ >&nn! - 2^n\\ >&n! - 2^n\\ >&0\end{aligned}$$

where we have used the fact that if $n \geq 1$ then $nn! \geq n!$ ■ Och, ja referensen till induktions-axiomet saknas!

Example 11.5 *(Bernoulli's inequality)*
Let $h > -1$. Use induction to show that

$$(1+nh) \leq (1+h)^n, \quad n \geq 0.$$

Solution.
Let $P(n) : (1+nh) \leq (1+h)^n$. We want to show that $P(n)$ is valid for all nonnegative integers. (i) (Basis of induction) $(1+h)^0 - (1+0h) = 0$. That is, $P(0)$is true.
(ii) (Induction hypothesis) Assume $P(n)$ is true. That is, $(1+nh) \leq (1+h)^n$.
(iii) (Induction step) We must show that $P(n+1)$ is true. That is, $(1+(n+1)h) \leq (1+h)^{n+1}$. Indeed,

$$\begin{aligned}(1+h)^{n+1} - (1+(n+1)h) =&(1+h)(1+h)^n - nh - 1 - h\\ \geq&(1+h)(1+nh) - nh - 1 - h\\ =&nh^2\\ \geq&0 \blacksquare\end{aligned}$$

Gissa vad som saknas!

Example 11.6
Define the following sequence of numbers: $a_1 = 2$ and for $n \geq 2$, $a_n = 5a_{n-1}$. Find a formula for a_n and then prove its validity by mathematical induction.

Solution.
Listing the first few terms we find, $a_1 = 2, a_2 = 10, a_3 = 50, a_4 = 250$. Thus, $a_n = 2.5^{n-1}$. We will show that $P(n) : a_n = 2 \cdot 5^{n-1}$ is valid for all $n \geq 1$ by the method of mathematical induction.
(i) (Basis of induction) $a_1 = 2 = 2.5^{1-1}$. That is, $P(1)$ is true.

(ii) (Induction hypothesis) Assume $P(n)$ is true. That is, $a_n = 2.5^{n-1}$
(iii) (Induction step) We must show that $a_{n+1} = 2.5^n$. Indeed,

$$\begin{aligned} a_{n+1} &= 5a_n \\ &= 5(2.5^{n-1}) \\ &= 2.5^n \blacksquare \end{aligned}$$

Och ja... induktions-axiomet!

Review Problems

Problem 11.1
Use the method of induction to show that

$$2 + 4 + 6 + \cdots + 2n = n^2 + n$$

for all integers $n \geq 1$.

Problem 11.2
Use mathematical induction to prove that

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

for all integers $n \geq 0$.

Problem 11.3
Use mathematical induction to show that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all integers $n \geq 1$.

Problem 11.4
Use mathematical induction to show that

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

for all integers $n \geq 1$.

Problem 11.5
Use mathematical induction to show that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

for all integers $n \geq 1$.

Problem 11.6
Use the formula

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

to find the value of the sum

$$3 + 4 + \cdots + 1,000.$$

Problem 11.7
Find the value of the geometric sum

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}.$$

Problem 11.8
Let $S(n) = \sum_{k=1}^{n} \frac{k}{(k+1)!}$. Evaluate $S(1), S(2), S(3), S(4)$, and $S(5)$. Make a conjecture about a formula for this sum for general n, and prove your conjecture by mathematical induction.

Problem 11.9
For each positive integer n let $P(n)$ be the proposition $4^n - 1$ is divisible by 3.
a. Write $P(1)$. Is $P(1)$ true?
b. Write $P(k)$.
c. Write $P(k+1)$.
d. In a proof by mathematical induction that this divisibility property holds for all integers $n \geq 1$, what must be shown in the induction step?

Problem 11.10
For each positive integer n let $P(n)$ be the proposition $2^{3n} - 1$ is divisible by 7. Prove this property by mathematical induction.

Problem 11.11
Show that $2^n < (n+2)!$ for all integers $n \geq 0$.

Problem 11.12
a. Use mathematical induction to show that $n^3 > 2n + 1$ for all integers $n \geq 2$.
b. Use mathematical induction to show that $n! > n^2$ for all integers $n \geq 4$.

Problem 11.13
A sequence $a_1, a_2, \cdots$ is defined recursively by $a_1 = 3$ and $a_n = 7a_{n-1}$ for $n \geq 2$. Show that $a_n = 3 \cdot 7^{n-1}$ for all integers $n \geq 1$.

Now, let A be a matrix of size $m \times n$ and entries a_{ij}; B is a matrix of size $n \times p$ and entries b_{ij}. Then the **product** matrix is a matrix of size $m \times p$ and entries

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

that is c_{ij} is obtained by multiplying componentwise the entries of the ith row of A by the entries of the jth column of B. It is very important to keep in mind that the number of columns of the first matrix must be equal to the number of rows of the second matrix; otherwise the product is undefined.

Problem 14.6
Consider the matrices

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{pmatrix}, B = \begin{pmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{pmatrix}.$$

Compute, if possible, AB and BA.

Problem 14.7
Prove by induction on $n \geq 1$ that

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 2^n & n2^{n-1} \\ 0 & 2^n \end{pmatrix}.$$

22 Recursion

A **recurrence relation** for a sequence $a_0, a_1, \cdots$ is a relation that defines a_n in terms of $a_0, a_1, \cdots, a_{n-1}$. The formula relating a_n to earlier values in the sequence is called the **generating rule**. The assignment of a value to one of the a's is called an **initial condition.**

Example 22.1
The Fibonacci sequence

$$1, 1, 2, 3, 5, \cdots$$

is a sequence in which every number after the first two is the sum of the preceding two numbers. Find the generating rule and the initial conditions.

Solution.
The initial conditions are $a_0 = a_1 = 1$ and the generating rule is $a_n = a_{n-1} + a_{n-2}, n \geq 2$. ■

Example 22.2
Let $n \geq 0$ and find the number s_n of words from the alphabet $\Sigma = \{0, 1\}$ of length n not containing the pattern 11 as a subword.

Solution.
Clearly, $s_0 = 1$(empty word) and $s_1 = 2$. We will find a recurrence relation for $s_n, n \geq 2$. Any word of length n with letters from Σ begins with either 0 or 1. If the word begins with 0, then the remaining $n - 1$ letters can be any sequence of 0's or 1's except that 11 cannot happen. If the word begins with 1 then the next letter must be 0 since 11 can not happen; the remaining $n - 2$ letters can be any sequence of 0's and 1's with the exception that 11 is not allowed. Thus the above two categories form a partition of the set of all words of length n with letters from Σ and that do not contain 11. This implies the recurrence relation

$$s_n = s_{n-1} + s_{n-2}, \quad n \geq 2 ■$$

A **solution** to a recurrence relation is an explicit formula for a_n in terms of n.
The most basic method for finding the solution of a sequence defined recursively is by using **iteration.** The iteration method consists of starting with the initial values of the sequence and then calculate successive terms of the

sequence until a pattern is observed. At that point one guesses an explicit formula for the sequence and then uses mathematical induction to prove its validity.

Example 22.3
Find a solution for the recurrence relation

$$\begin{aligned} a_0 =& 1 \\ a_n =& a_{n-1} + 2, \quad n \geq 1 \end{aligned}$$

Solution.
Listing the first five terms of the sequence one finds

$$\begin{aligned} a_0 =& 1 \\ a_1 =& 1 + 2 \\ a_2 =& 1 + 4 \\ a_3 =& 1 + 6 \\ a_4 =& 1 + 8 \end{aligned}$$

Hence, a guess is $a_n = 2n + 1, n \geq 0$. It remains to show that this formula is valid by using mathematical induction.

Basis of induction: For $n = 0, a_0 = 1 = 2(0) + 1$.
Induction hypothesis: Suppose that $a_n = 2n + 1$.
Induction step: We must show that $a_{n+1} = 2(n + 1) + 1$. By the definition of a_{n+1} we have $a_{n+1} = a_n + 2 = 2n + 1 + 2 = 2(n + 1) + 1$. ■

Example 22.4
Consider the arithmetic sequence

$$a_n = a_{n-1} + d, \quad n \geq 1$$

where a_0 is the initial value. Find an explicit formula for a_n.

Solution. Listing the first four terms of the sequence after a_0 we find

$$\begin{aligned} a_1 =& a_0 + d \\ a_2 =& a_0 + 2d \\ a_3 =& a_0 + 3d \\ a_4 =& a_0 + 4d \end{aligned}$$

Hence, a guess is $a_n = a_0 + nd$. Next, we prove the validity of this formula by induction.

Basis of induction: For $n = 0, a_0 = a_0 + (0)d$.
Induction hypothesis: Suppose that $a_n = a_0 + nd$.
Induction step: We must show that $a_{n+1} = a_0 + (n+1)d$. By the definition of a_{n+1} we have $a_{n+1} = a_n + d = a_0 + nd + d = a_0 + (n+1)d$. ■ Här saknas något. Vad?

Example 22.5
Consider the geometric sequence

$$a_n = ra_{n-1}, \quad n \geq 1$$

where a_0 is the initial value. Find an explicit formula for a_n.

Solution.
Listing the first four terms of the sequence after a_0 we find

$$\begin{aligned} a_1 &= ra_0 \\ a_2 &= r^2 a_0 \\ a_3 &= r^3 a_0 \\ a_4 &= r^4 a_0 \end{aligned}$$

Hence, a guess is $a_n = r^n a_0$. Next, we prove the validity of this formula by induction.

Basis of induction: For $n = 0, a_0 = r^0 a_0$.
Induction hypothesis: Suppose that $a_n = r^n a_0$.
Induction step: We must show that $a_{n+1} = r^{n+1} a_0$. By the definition of a_{n+1} we have $a_{n+1} = ra_n = r(r^n a_0) = r^{n+1} a_0$. ■ Ind.ax!

Example 22.6
Find a solution to the recurrence relation

$$\begin{aligned} a_0 &= 0 \\ a_n &= a_{n-1} + (n-1), \quad n \geq 1 \end{aligned}$$

Solution.
Writing the first five terms of the sequence we find

$$\begin{aligned}a_0 =&0\\ a_1 =&0\\ a_2 =&0+1\\ a_3 =&0+1+2\\ a_4 =&0+1+2+3\end{aligned}$$

We guess that

$$a_n = 0+1+2+\cdots+(n-1) = \frac{n(n-1)}{2}.$$

We next show that the formula is valid by using induction on $n \geq 0$.

Basis of induction: $a_0 = 0 = \frac{0(0-1)}{2}$.
Induction hypothesis: Suppose that $a_n = \frac{n(n-1)}{2}$.
Induction step: We must show that $a_{n+1} = \frac{n(n+1)}{2}$. Indeed,

$$\begin{aligned}a_{n+1} =&a_n + n\\ =&\frac{n(n-1)}{2} + n\\ =&\frac{n(n+1)}{2} \blacksquare\end{aligned}$$

!

Example 22.7
Consider the recurrence relation

$$\begin{aligned}a_0 =&1\\ a_n =&2a_{n-1} + n,\ n \geq 1\end{aligned}$$

Is it true that $a_n = 2^n + n$ is a solution to the given recurrence relation?

Solution.
This is false since $a_2 = 2a_1 + 2 = 2(2a_0 + 1) + 2 = 8 \neq 2^2 + 2$ $\blacksquare$

Example 22.8
Define a sequence, $a_1, a_2, \cdots$, recursively as follows:

$$\begin{aligned}a_1 =&1\\ a_n =&2 \cdot a_{\lfloor \frac{n}{2} \rfloor},\quad n \geq 2\end{aligned}$$

a. Use iteration to guess an explicit formula for this sequence.
b. Use induction to prove the validity of the formula found in a.

Solution.
Computing the first few terms of the sequence we find

$$\begin{aligned} a_1 =& 1 \\ a_2 =& 2 \\ a_3 =& 2 \\ a_4 =& 4 \\ a_5 =& 4 \\ a_6 =& 4 \\ a_7 =& 4 \\ a_8 =& \cdots = a_{15} = 8 \end{aligned}$$

Hence, for $2^i \leq n < 2^{i+1}, a_n = 2^i$. Moreover, $i \leq \log_2 n < i+1$ so that $i = \lfloor \log_2 n \rfloor$ and a formula for a_n is

$$a_n = 2^{\lfloor \log_2 n \rfloor}, \ n \geq 1.$$

b. We prove the above formula by mathematical induction.

Basis of induction: For $n = 1$, $a_1 = 1 = 2^{\lfloor \log_2 1 \rfloor}$.
Induction hypothesis: Suppose that $a_n = 2^{\lfloor \log_2 n \rfloor}$.
Induction step: We must show that $a_{n+1} = 2^{\lfloor \log_2 (n+1) \rfloor}$. Indeed, for n odd (i.e. $n+1$ even) we have

$$\begin{aligned} a_{n+1} =& 2 \cdot a_{\lfloor \frac{n+1}{2} \rfloor} \\ =& 2 \cdot a_{\frac{n+1}{2}} \\ =& 2 \cdot 2^{\lfloor \log_2 \frac{n+1}{2} \rfloor} \\ =& 2^{\lfloor \log_2 (n+1) - 1 \rfloor + 1} \\ =& 2^{\lfloor \log_2 (n+1) \rfloor - 1 + 1} \\ =& 2^{\lfloor \log_2 (n+1) \rfloor} \end{aligned}$$

A similar argument holds when n is even. ■

When iteration does not apply, other methods are available for finding explicit formulas for special classes of recursively defined sequences. The method explained below works for sequences of the form

$$a_n = Aa_{n-1} + Ba_{n-2} \tag{22.1}$$

where n is greater than or equal to some fixed nonnegative integer k and A and B are real numbers with $B \neq 0$. Such an equation is called a **second-order linear homogeneous recurrence relation with constant coefficients**.

Example 22.9
Does the Fibonacci sequence satisfy a second-order linear homogeneous relation with constant coefficients?

Solution.
Recall that the Fibonacci sequence is defined recursively by $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$ and $a_0 = a_1 = 1$. Thus, a_n satisfies a second-order linear homogeneous relation with $A = B = 1$ ■

The following theorem gives a technique for finding solutions to (22.1).

Theorem 22.1
Equation (22.1) is satisfied by the sequence $1, t, t^2, \cdots, t^n, \cdots$ where $t \neq 0$ if and only if t is a solution to the **characteristic equation**

$$t^2 - At - B = 0 \tag{22.2}$$

Proof.
($\Longrightarrow$): Suppose that t is a nonzero real number such that the sequence $1, t, t^2, \cdots$ satisfies (22.1). We will show that t satisfies the equation $t^2 - At - B = 0$. Indeed, for $n \geq k$ we have

$$t^n = At^{n-1} + Bt^{n-2}.$$

Since $t \neq 0$ we can divide through by t^{n-2} and obtain $t^2 - At - B = 0$.
($\Longleftarrow$) : Suppose that t is a nonzero real number such that $t^2 - At - B = 0$. Multiply both sides of this equation by t^{n-2} to obtain

$$t^n = At^{n-1} + Bt^{n-2}.$$

This says that the sequence $1, t, t^2, \cdots$ satisfies (22.1) ■

Example 22.10
Consider the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}, \quad n \geq 2.$$

Find two sequences that satisfy the given generating rule and have the form $1, t, t^2, \cdots$.

Solution.
According to the previous theorem t must satisfy the characteristic equation

$$t^2 - t - 2 = 0.$$

Solving for t we find $t = 2$ or $t = -1$. So the two solutions to the given recurrence sequence are $1, 2, 2^2, \cdots, 2^n, \cdots$ and $1, -1, \cdots, (-1)^n, \cdots$ ■

Are there other solutions than the ones provided by Theorem 22.1? The answer is yes according to the following theorem.

Theorem 22.2
If s_n and t_n are solutions to (22.1) then for any real numbers C and D the sequence

$$a_n = Cs_n + Dt_n, \quad n \geq 0$$

is also a solution.

Proof.
Since s_n and t_n are solutions to (22.1), for $n \geq 2$ we have

$$\begin{aligned} s_n =& As_{n-1} + Bs_{n-2} \\ t_n =& At_{n-1} + Bt_{n-2} \end{aligned}$$

Therefore,

$$\begin{aligned} Aa_{n-1} + Ba_{n-2} =& A(Cs_{n-1} + Dt_{n-1}) + B(Cs_{n-2} + Dt_{n-2}) \\ =& C(As_{n-1} + Bs_{n-2}) + D(At_{n-1} + Bt_{n-2}) \\ =& Cs_n + Dt_n = a_n \end{aligned}$$

so that a_n satisfies (22.1) ■

Example 22.11
Find a solution to the recurrence relation

$$\begin{aligned} a_0 =& 1, a_1 = 8 \\ a_n =& a_{n-1} + 2a_{n-2}, \quad n \geq 2. \end{aligned}$$

Solution.
By the previous theorem and Example 22.10, $a_n = C2^n + D(-1)^n, \ n \geq 2$ is a solution to the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}.$$

If a_n satisfies the system then we must have

$$\begin{array}{rcl} a_0 & = & C2^0 + D(-1)^0 \\ a_1 & = & C2^1 + D(-1)^1 \end{array}$$

This yields the system

$$\left\{ \begin{array}{rcl} C + D & = & 1 \\ 2C - D & = & 8 \end{array} \right.$$

Solving this system to find $C = 3$ and $D = -2$. Hence, $a_n = 3 \cdot 2^n - 2(-1)^n$.
■

Example 22.12
Find an explicit formula for the Fibonacci sequence

$$\begin{aligned} a_0 =& a_1 = 1 \\ a_n =& a_{n-1} + a_{n-2} \end{aligned}$$

Solution.
The roots of the characteristic equation

$$t^2 - t - 1 = 0$$

are $t = \frac{1-\sqrt{5}}{2}$ and $t = \frac{1+\sqrt{5}}{2}$. Thus,

$$a_n = C(\frac{1+\sqrt{5}}{2})^n + D(\frac{1-\sqrt{5}}{2})^n$$

is a solution to

$$a_n = a_{n-1} + a_{n-2}.$$

Using the values of a_0 and a_1 we obtain the system

$$\begin{cases} C + D & = 1 \\ C(\frac{1+\sqrt{5}}{2}) + D(\frac{1-\sqrt{5}}{2}) & = 1. \end{cases}$$

Solving this system to obtain

$$C = \frac{1+\sqrt{5}}{2\sqrt{5}} \quad \textit{and} \quad D = -\frac{1-\sqrt{5}}{2\sqrt{5}}.$$

Hence,

$$a_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{n+1} - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{n+1} \blacksquare$$

Next, we discuss the case when the characteristic equation has a single root.

Theorem 22.3
Let A and B be real numbers and suppose that the characteristic equation

$$t^2 - At - B = 0$$

has a single root r. Then the sequences $\{1, r, r^2, \cdots\}$ and $\{0, r, 2r^2, 3r^3, \cdots, nr^n, \cdots\}$ both satisfy the recurrence relation

$$a_n = Aa_{n-1} + Ba_{n-2}.$$

Proof.
Since r is a root to the characteristic equation, the sequence $\{1, r, r^2, \cdots\}$ is a solution to the recurrence relation

$$a_n = Aa_{n-1} + Ba_{n-2}.$$

Now, since r is the only solution to the characteristic equation we have

$$(t - r)^2 = t^2 - At - B.$$

This implies that $A = 2r$ and $B = -r^2$. Let $s_n = nr^n, \quad n \geq 0$. Then

$$\begin{aligned} As_{n-1} + Bs_{n-2} =& A(n-1)r^{n-1} + B(n-2)r^{n-2} \\ =& 2r(n-1)r^{n-1} - r^2(n-2)r^{n-2} \\ =& 2(n-1)r^n - (n-2)r^n \\ =& nr^n = s_n \end{aligned}$$

So s_n is a solution to $a_n = Aa_{n-1} + Ba_{n-2}$. $\blacksquare$

Example 22.13
Find an explicit formula for

$$\begin{aligned} a_0 =& 1, a_1 = 3 \\ a_n =& 4a_{n-1} - 4a_{n-2}, \quad n \geq 2 \end{aligned}$$

Solution.
Solving the characteristic equation

$$t^2 - 4t + 4 = 0$$

we find the single root $r = 2$. Thus,

$$a_n = C2^n + Dn2^n$$

is a solution to the equation $a_n = 4a_{n-1} - 4a_{n-2}$. Since $a_0 = 1$ and $a_1 = 3$, we obtain the following system of equations:

$$\begin{aligned} C =& 1 \\ 2C + 2D =& 3 \end{aligned}$$

Solving this system to obtain $C = 1$ and $D = \frac{1}{2}$. Hence, $a_n = 2^n + \frac{n}{2}2^n$. ■

Example 22.14
Let $A_1, A_2, \cdots, A_n$ be subsets of a set S.
a. Give a recursion definition for $\cup_{i=1}^n A_i$.
b. Give a recursion definition for $\cap_{i=1}^n A_i$.

Solution.
a. $\cup_{i=1}^1 A_i = A_1$ and $\cup_{i=1}^n A_i = (\cup_{i=1}^{n-1} A_i) \cup A_n, \quad n \geq 2$.
b. $\cap_{i=1}^1 A_i = A_1$ and $\cap_{i=1}^n A_i = (\cap_{i=1}^{n-1} A_i) \cap A_n, \quad n \geq 2$.■

Example 22.15
Use mathematical induction to prove the following generalized De Morgan's law.

$$(\cup_{i=1}^n A_i)^c = \cap_{i=1}^n A_i^c$$

Solution.
Basis of induction: $(\cup_{i=1}^1 A_i)^c = A_1^c = \cap_{i=1}^1 A_i^c$.

Induction hypothesis: Suppose that $(\cup_{i=1}^{n} A_i)^c = \cap_{i=1}^{n} A_i^c$.
Induction step: We must show that $(\cup_{i=1}^{n+1} A_i)^c = \cap_{i=1}^{n+1} A_i^c$. Indeed,

$$\begin{aligned}(\cup_{i=1}^{n+1} A_i)^c =&((\cup_{i=1}^{n} A_i) \cup A_{n+1})^c\\ =&(\cup_{i=1}^{n} A_i)^c) \cap A_{n+1}^c\\ =&(\cap_{i=1}^{n} A_i^c) \cap A_{n+1}^c\\ =& \cap_{i=1}^{n+1} A_i^c \blacksquare\end{aligned}$$

Example 22.16
Let $a_1, a_2, \cdots, a_n$ be numbers.
a. Give a recursion definition for $\sum_{i=1}^{n} a_i$.
b. Give a recursion definition for $\Pi_{i=1}^{n} a_i$.

Solution.
a. $\sum_{i=1}^{1} a_i = a_1$ and $\sum_{i=1}^{n} a_i = (\sum_{i=1}^{n-1} a_i) + a_n, \quad n \geq 2.$
b. $\Pi_{i=1}^{1} a_i = a_1$ and $\Pi_{i=1}^{n} a_i = (\Pi_{i=1}^{n-1} a_i) \cdot a_n, \quad n \geq 2.\blacksquare$

Example 22.17
A function is said to be defined **recursively** or to be a **recursive function** if its rule of definition refers to itself. Define the factorial function recursively.

Solution.
We have

$$\begin{aligned}f(0) =&1\\ f(n) =&nf(n-1), \quad n \geq 1 \blacksquare\end{aligned}$$

Example 22.18
Let $G : \mathbb{N} \to \mathbf{Z}$ be the relation given by

$$G(n) = \begin{cases} 1, & \text{if } n = 1\\ 1 + G(\frac{n}{2}), & \text{if n is even}\\ G(3n-1), & \text{if } n > 1 \text{ is odd}\end{cases}$$

Show that G is not a function.

Solution.
Assume that G is a function so that $G(5)$ exists. Listing the first five values

of G we find

$$\begin{aligned}
G(1) =& 1 \\
G(2) =& 2 \\
G(3) =& G(8) = 1 + G(4) = 2 + G(2) = 4 \\
G(4) =& 1 + G(2) = 3 \\
G(5) =& G(14) = 1 + G(7) \\
=& 1 + G(20) \\
=& 2 + G(10) \\
=& 3 + G(5)
\end{aligned}$$

But the last equality implies that $0 = 3$ which is impossible. Hence, G does not define a function. ■

Review Problems

Problem 22.1
Find the first four terms of the following recursively defined sequence:

$$\begin{aligned} v_1 =& 1, v_2 = 2 \\ v_n =& v_{n-1} + v_{n-2} + 1, \quad n \geq 3. \end{aligned}$$

Problem 22.2
Prove each of the following for the Fibonacci sequence:
a. $F_k^2 - F_{k-1}^2 = F_kF_{k+1} - F_{k+1}F_{k-1}, \quad k \geq 1.$
b. $F_{k+1}^2 - F_k^2 - F_{k-1}^2 = 2F_kF_{k-1}, \quad k \geq 1.$
c. $F_{k+1}^2 - F_k^2 = F_{k-1}F_{k+2}, \quad k \leq 1.$
d. $F_{n+2}F_n - F_{n+1}^2 = (-1)^n$ for all $n \geq 0.$

Problem 22.3
Find $\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$ where $F_0, F_1, F_2, \cdots$ is the Fibonacci sequence. (Assume that the limit exists.)

Problem 22.4
Define $x_0, x_1, x_2, \cdots$ as follows:

$$x_n = \sqrt{2 + x_{n-1}}, \quad x_0 = 0.$$

Find $\lim_{n\to\infty} x_n$.

Problem 22.5
a. Make a list of all bit strings of lengths zero, one, two, three, and four that do not contain the pattern 111.
b. For each $n \geq 0$ let $d_n =$ the number of bit strings of length n that do not contain the bit pattern 111. Find d_0, d_1, d_2, d_3, and d_4.
c. Find a recurrence relation for $d_0, d_1, d_2, \cdots$
d. Use the results of (b) of (c) to find the number of bit strings of length five that do not contain the pattern 111.

Problem 22.6
Find a formula for each of the following sums:
a. $1 + 2 + \cdots + (n-1), \quad n \geq 2.$
b. $3 + 2 + 4 + 6 + 8 + \cdots + 2n, \quad n \geq 1.$
c. $3 \cdot 1 + 3 \cdot 2 + 3 \cdot 3 + \cdots 3 \cdot n, \quad n \geq 1.$

Problem 22.7
Find a formula for each of the following sums:
a. $1 + 2 + 2^2 + \cdots + 2^{n-1}, \quad n \geq 1.$
b. $3^{n-1} + 3^{n-2} + \cdots + 3^2 + 3 + 1, \quad n \geq 1.$
c. $2^n + 3 \cdot 2^{n-2} + 3 \cdot 2^{n-3} + \cdots + 3 \cdot 2^2 + 3 \cdot 2 + 3, \quad n \geq 1.$
d. $2^n - 2^{n-1} + 2^{n-2} - 2^{n-3} + \cdots + (-1)^{n-1} \cdot 2 + (-1)^n, \quad n \geq 1.$

Problem 22.8
Use iteration to guess a formula for the following recursively defined sequence and then use mathematical induction to prove the validity of your formula: $c_1 = 1, c_n = 3c_{n-1} + 1$, for all $n \geq 2$.

Problem 22.9
Use iteration to guess a formula for the following recursively defined sequence and then use mathematical induction to prove the validity of your formula: $w_0 = 1, w_n = 2^n - w_{n-1}$, for all $n \geq 2$.

Problem 22.10
Determine whether the recursively defined sequence: $a_1 = 0$ and $a_n = 2a_{n-1} + n - 1$ satisfies the recursive formula $a_n = (n-1)^2, \ n \geq 1$.

Problem 22.11
Which of the following are second-order homogeneous recurrence relations with constant coefficients?
a. $a_n = 2a_{n-1} - 5a_{n-2}$.
b. $b_n = nb_{n-1} + b_{n-2}$.
c. $c_n = 3c_{n-1} \cdot c_{n-2}^2$.
d. $d_n = 3d_{n-1} + d_{n-2}$.
e. $r_n = r_{n-1} - r_{n-2} - 2$.
f. $s_n = 10s_{n-2}$.

Problem 22.12
Let $a_0, a_1, a_2, \cdots$ be the sequence defined by the recursive formula

$$a_n = C \cdot 2^n + D, \quad n \geq 0$$

where C and D are real numbers.
a. Find C and D so that $a_0 = 1$ and $a_1 = 3$. What is a_2 in this case?
b. Find C and D so that $a_0 = 0$ and $a_1 = 2$. What is a_2 in this case?

Problem 22.13

Let $a_0, a_1, a_2, \cdots$ be the sequence defined by the recursive formula

$$a_n = C \cdot 2^n + D, \quad n \geq 0$$

where C and D are real numbers. Show that for any choice of C and D,

$$a_n = 3a_{n-1} - 2a_{n-2}, \quad n \geq 2.$$

Problem 22.14

Let $a_0, a_1, a_2, \cdots$ be the sequence defined by the recursive formula

$$\begin{aligned} a_0 =& 1, a_1 = 2 \\ a_n =& 2a_{n-1} + 3a_{n-2}, \quad n \geq 2. \end{aligned}$$

Find an explicit formula for the sequence.

Problem 22.15

Let $a_0, a_1, a_2, \cdots$ be the sequence defined by the recursive formula

$$\begin{aligned} a_0 =& 1, a_1 = 4 \\ a_n =& 2a_{n-1} - a_{n-2}, \quad n \geq 2. \end{aligned}$$

Find an explicit formula for the sequence.

Problem 22.16

The triangle inequality for absolute value states that for all real numbers a and b, $|a+b| \leq |a|+|b|$. Use the recursive definition of summation, the triangle inequality, the definition of absolute value, and mathematical induction to prove that for all positive integers n, if $a_1, a_2, \cdots, a_n$ are real numbers then

$$|\sum_{k=1}^{n} a_k| \leq \sum_{k=1}^{n} |a_k|.$$

Problem 22.17

Use the recursive definition of union and intersection to prove the following general distributive law: For all positive integers n, if A and $B_1, B_2, \cdots, B_n$ are sets then

$$A \cap (\cup_{k=1}^{n} B_k) = \cup_{k=1}^{n} (A \cap B_k).$$

Problem 22.18
Use mathematical induction to prove the following generalized De Morgan's law.

$$(\cap_{i=1}^{n} A_i)^c = \cup_{i=1}^{n} A_i^c$$

Problem 22.19
Show that the relation $F : \mathbb{N} \to \mathbf{Z}$ given by the rule

$$F(n) = \begin{cases} 1 & \text{if } n = 1. \\ F(\frac{n}{2}) & \text{if n is even} \\ 1 - F(5n - 9) & \text{if } n \text{ is odd and } n > 1 \end{cases}$$

does not define a function.