Relations and Functions

The reader is familiar with many relations which are used in mathematics and computer science, i.e. "is a subset of", " is less than" and so on.
One frequently wants to compare or contrast various members of a set, perhaps to arrange them in some appropriate order or to group together those with similar properties. The mathematical framework to describe this kind of organization of sets is the theory of relations.
There are three kinds of relations which we discuss in this chapter: (i) equivalence relations, (ii) order relations, (iii) functions.

18 Equivalence Relations

Let A be a given set. An **ordered pair** (a, b) of elements in A is defined to be the set $\{a, \{a, b\}\}$. The element a (resp. b) is called the **first** (resp. **second**) **component**.

Example 18.1
a. Show that if $a \neq b$ then $(a, b) \neq (b, a)$.
b. Show that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Solution.
a. If $a \neq b$ then $\{a, \{a, b\}\} \neq \{b, \{a, b\}\}$. That is, $(a, b) \neq (b, a)$.
b. $(a, b) = (c, d)$ if and only if $\{a, \{a, b\}\} = \{c, \{c, d\}\}$ and this is equivalent to $a = c$ and $\{a, b\} = \{c, d\}$ by the definition of equality of sets. Thus, $a = c$ and $b = d$. ■

Example 18.2
Find x and y such that $(x + y, 0) = (1, x - y)$.

Solution.
By the previous exercise we have the system

$$\begin{cases} x + y = 1 \\ x - y = 0 \end{cases}$$

Solving by the method of elimination one finds $x = \frac{1}{2}$ and $y = \frac{1}{2}$. ■

If A and B are sets, we let $A \times B$ denote the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. We call $A \times B$ the **Cartesian product** of A and B.

Example 18.3
a. Show that if A is a set with m elements and B is a set of n elements then $A \times B$ is a set of mn elements.
b. Show that if $A \times B = \emptyset$ then $A = \emptyset$ or $B = \emptyset$.

Solution.
a. Consider an ordered pair (a, b). There are m possibilities for a. For each fixed a, there are n possibilities for b. Thus, there are $m \times n$ ordered pairs (a, b). That is, $|A \times B| = mn$.
b. We use the proof by contrapositive. Suppose that $A \neq \emptyset$ and $B \neq \emptyset$. Then there is at least an $a \in A$ and an element $b \in B$. That is, $(a, b) \in A \times B$ and this shows that $A \times B \neq \emptyset$. A contradiction to the assumption that $A \times B = \emptyset$■

Example 18.4
Let $A = \{1, 2\}, B = \{1\}$. Show that $A \times B \neq B \times A$.

Solution.
We have $A \times B = \{(1, 1), (2, 1)\} \neq \{(1, 1), (1, 2)\} = B \times A$.■

A **binary relation** R from a set A to a set B is a subset of $A \times B$. If $(a, b) \in R$ we write aRb and we say that a is related to b. If a is not related to b we write $a \not{R} b$. In case $A = B$ we call R a **binary relation** on A.
The set

$$Dom(R) = \{a \in A | (a, b) \in R \text{ for some } b \in B\}$$

is called the **domain** of R. The set

$$Range(R) = \{b \in B | (a, b) \in R \text{ for some } a \in A\}$$

is called the **range** of R.

Example 18.5
a. Let $A = \{2, 3, 4\}$ and $B = \{3, 4, 5, 6, 7\}$. Define the relation R by aRb if and only if a *divides* b. Find, $R, Dom(R), Range(R)$.
b. Let $A = \{1, 2, 3, 4\}$. Define the relation R by aRb if and only if $a \leq b$. Find, $R, Dom(R), Range(R)$.

Solution.
a. $R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}, Dom(R) = \{2, 3, 4\}$, and $Range(R) = \{3, 4, 6\}$.
b. $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}, Dom(R) = A, Range(R) = A$. ■

A **function** is a special case of a relation. A function from A to B, denoted by $f : A \to B$, is a relation from A to B such that for every $x \in A$ there is a unique $y \in B$ such that $(x, y) \in f$. The element y is called the **image** of x and we write $y = f(x)$. The set A is called the **domain** of f and the set of all images of f is called the **range** of f. Functions will be discussed in more detail in Section 20.

Example 18.6
a. Show that the relation

$$f = \{(1, a), (2, b), (3, a)\}$$

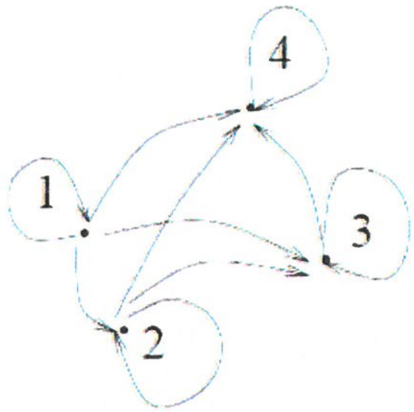
defines a function from $A = \{1, 2, 3\}$ to $B = \{a, b, c\}$. Find its range.
b. Show that the relation $f = \{(1, a), (2, b), (3, c), (1, b)\}$ does not define a function from $A = \{1, 2, 3\}$ to $B = \{a, b, c\}$.

Solution.
a. Note that each element of A has exactly one image. Hence, f is a function with domain A and range $Range(f) = \{a, b\}$.
b. The relation f does not define a function since the element 1 has two images, namely a and b. ■

An informative way to picture a relation on a set is to draw its **digraph.** To draw a digraph of a relation on a set A, we first draw dots or **vertices** to represent the elements of A. Next, if $(a, b) \in R$ we draw an arrow (called a **directed edge**) from a to b. Finally, if $(a, a) \in R$ then the directed edge is simply a **loop**.

Example 18.7
Draw the directed graph of the relation in part (b) of Problem 18.5.

Solution.



■

Next we discuss three ways of building new relations from given ones. Let R be a relation from a set A to a set B. The **inverse** of R is the relation R^{-1} from $Range(R)$ to $Dom(R)$ such that

$$R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}.$$

Example 18.8
Let $R = \{(1, y), (1, z), (3, y)\}$ be a relation from $A = \{1, 2, 3\}$ to $B = \{x, y, z\}$.
a. Find R^{-1}.
b. Compare $(R^{-1})^{-1}$ and R.

Solution.
a. $R^{-1} = \{(y, 1), (z, 1), (y, 3)\}$.
b. $(R^{-1})^{-1} = R$. ■

Let R and S be two relations from a set A to a set B. Then we define the relations $R \cup S$ and $R \cap S$ by

$$R \cup S = \{(a, b) \in A \times B | (a, b) \in R \text{ or } (a, b) \in S\},$$

and

$$R \cap S = \{(a, b) \in A \times B | (a, b) \in R \text{ and } (a, b) \in S\}.$$

Example 18.9
Given the following two relations from $A = \{1, 2, 4\}$ to $B = \{2, 6, 8, 10\}$:

aRb if and only if $a|b$.
aSb if and only if $b - 4 = a$.

List the elements of $R, S, R \cup S$, and $R \cap S$.

Solution.
We have

$$\begin{aligned} R =& \{(1,2),(1,6),(1,8),(1,10),(2,2),(2,6),(2,8),(2,10),(4,8)\} \\ S =& \{(2,6),(4,8)\} \\ R \cup S =& R \\ R \cap S =& S \blacksquare \end{aligned}$$

Now, if we have a relation R from A to B and a relation S from B to C we can define the relation $S \circ R$, called the **composition** relation, to be the relation from A to C defined by

$$S \circ R = \{(a,c)|(a,b) \in R \ and \ (b,c) \in S \ for \ some \ b \in B\}.$$

Example 18.10
Let

$$\begin{aligned} R =& \{(1,2),(1,6),(2,4),(3,4),(3,6),(3,8)\} \\ S =& \{(2,u),(4,s),(4,t),(6,t),(8,u)\} \end{aligned}$$

Find $S \circ R$.

Solution.

$$S \circ R = \{(1,u),(1,t),(2,s),(2,t),(3,s),(3,t),(3,u)\} \blacksquare$$

We next define four types of binary relations. A relation R on a set A is called **reflexive** if $(a,a) \in R$ for all $a \in A$. In this case, the digraph of R has a loop at each vertex.

Example 18.11

a. Show that the relation $a \leq b$ on the set $A = \{1, 2, 3, 4\}$ is reflexive.
b. Show that the relation on $\mathbb{R}$ defined by aRb if and only if $a < b$ is not reflexive.

Solution.

a. By Example 18.7, each vertex has a loop.
b. Indeed, for any real number a we have $a - a = 0$ and not $a - a < 0$. ∎

A relation R on A is called **symmetric** if whenever $(a, b) \in R$ then we must have $(b, a) \in R$. The digraph of a symmetric relation has the property that whenever there is a directed edge from a to b, there is also a directed edge from b to a.

Example 18.12

a. Let $A = \{a, b, c, d\}$ and $R = \{(a, a), (b, c), (c, b), (d, d)\}$. Show that R is symmetric.
b. Let $\mathbb{R}$ be the set of real numbers and R be the relation aRb if and only if $a < b$. Show that R is not symmetric.

Solution.

a. bRc and cRb so R is symmetric.
b. $2 < 4$ but $4 \not< 2$. ∎

A relation R on a set A is called **antisymmetric** if whenever $(a, b) \in R$ and $a \neq b$ then $(b, a) \notin R$. The digraph of an antisymmetric relation has the property that between any two vertices there is at most one directed edge.

Example 18.13

a. Let $\mathbb{N}$ be the set of nonnegative integers and R the relation aRb if and only if a divides b. Show that R is antisymmetric.
b. Let $A = \{a, b, c, d\}$ and $R = \{(a, a), (b, c), (c, b), (d, d)\}$. Show that R is not antisymmetric.

Solution.

a. Suppose that $a|b$ and $b|a$. We must show that $a = b$. Indeed, by the definition of division, there exist positive integers k_1 and k_2 such that $b = k_1a$ and $a = k_2b$. This implies that $a = k_2k_1a$ and hence $k_1k_2 = 1$. Since k_1 and k_2 are positive integers, we must have $k_1 = k_2 = 1$. Hence, $a = b$.

b. bRc and cRb with $b \neq c$. ■

A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$. The digraph of a transitive relation has the property that whenever there are directed edges from a to b and from b to c then there is also a directed edge from a to c.

Example 18.14
a. Let $A = \{a, b, c, d\}$ and $R = \{(a, a), (b, c), (c, b), (d, d)\}$. Show that R is not transitive.
b. Let $\mathbf{Z}$ be the set of integers and R the relation aRb if a divides b. Show that R is transitive.

Solution.
a. $(b, c) \in R$ and $(c, b) \in R$ but $(b, b) \notin R$.
b. Suppose that $a|b$ and $b|c$. Then there exist integers k_1 and k_2 such that $b = k_1 a$ and $c = k_2 b$. Thus, $c = (k_1 k_2)a$ which means that $a|c$. ■

Now, let $A_1, A_2, \cdots, A_n$ be a partition of a set A. That is, the $A_i's$ are subsets of A that satisfy
(i) $\cup_{i=1}^{n} A_i = A$
(ii) $A_i \cap A_j = \emptyset$ for $i \neq j$.
Define on A the binary relation $x\ R\ y$ if and only if x and y belongs to the same set A_i for some $1 \leq i \leq n$.

Theorem 18.1
The relation R defined above is reflexive, symmetric, and transitive.

Proof.
• R is reflexive: If $x \in A$ then by (i) $x \in A_k$ for some $1 \leq k \leq n$. Thus, x and x belong to A_k so that $x\ R\ x$.
• R is symmetric: Let $x, y \in A$ such that $x\ R\ y$. Then there is an index k such that $x, y \in A_k$. But then $y, x \in A_k$. That is, $y\ R\ x$.
• R is transitive: Let $x, y, z \in A$ such that $x\ R\ y$ and $y\ R\ z$. Then there exist indices i and j such that $x, y \in A_i$ and $y, z \in A_j$. Since $y \in A_i \cap A_j$, by (ii) we must have $i = j$. This implies that $x, y, z \in A_i$ and in particular $x, z \in A_i$. Hence, $x\ R\ z$. ■

A relation that is reflexive, symmetric, and transitive on a set A is called an **equivalence relation on A.** For example, the relation "$=$" is an equivalence relation on $\mathbb{R}$.

Example 18.15
Let $\mathbf{Z}$ be the set of integers and $n \in \mathbf{Z}$. Let R be the relation on $\mathbf{Z}$ defined by aRb if $a - b$ is a multiple of n. We denote this relation by $a \equiv b \ (mod\ n)$ read "a congruent to b modulo n." Show that R is an equivalence relation on $\mathbf{Z}$.

Solution.
$\equiv$ is reflexive: For all $a \in \mathbf{Z}$, $a - a = 0 \cdot n$. That is, $a \equiv a \ (mod\ n)$.
$\equiv$ is symmetric: Let $a, b \in \mathbf{Z}$ such that $a \equiv b \ (mod\ n)$. Then there is an integer k such that $a - b = kn$. Multiply both sides of this equality by (-1) and letting $k' = -k$ we find that $b - a = k'n$. That is $b \equiv a \ (mod\ n)$.
$\equiv$ is transitive: Let $a, b, c \in \mathbf{Z}$ be such that $a \equiv b \ (mod\ n)$ and $b \equiv c \ (mod\ n)$. Then there exist integers k_1 and k_2 such that $a - b = k_1 n$ and $b - c = k_2 n$. Adding these equalities together we find $a - c = kn$ where $k = k_1 + k_2 \in \mathbf{Z}$ which shows that $a \equiv c \ (mod\ n)$. ∎

Theorem 18.2
Let R be an equivalence relation on A. For each $a \in A$ let

$$[a] = \{x \in A | xRa\}$$

$$A/R = \{[a] | a \in A\}.$$

Then the union of all the elements of A/R is equal to A and the intersection of any two distinct members of A/R is the empty set. That is, the family A/R forms a partition of A.

Proof.
By the definition of $[a]$ we have that $[a] \subseteq A$. Hence, $\cup_{a \in A}[a] \subseteq A$. We next show that $A \subseteq \cup_{a \in A}[a]$. Indeed, let $a \in A$. Since A is reflexive, $a \in [a]$ and consequently $a \in \cup_{b \in A}[b]$. Hence, $A \subseteq \cup_{b \in A}[b]$. It follows that $A = \cup_{a \in A}[a]$. This establishes (i).
It remains to show that if $[a] \neq [b]$ then $[a] \cap [b] = \emptyset$ for $a, b \in A$. Suppose the contrary. That is, suppose $[a] \cap [b] \neq \emptyset$. Then there is an element $c \in [a] \cap [b]$. This means that $c \in [a]$ and $c \in [b]$. Hence, $a \ R \ c$ and $b \ R \ c$. Since R is symmetric and transitive, $a \ R \ b$. We will show that the conclusion $a \ R \ b$ leads to

$[a] = [b]$. The proof is by double inclusions. Let $x \in [a]$. Then $x\ R\ a$. Since $a\ R\ b$ and R is transitive, $x\ R\ b$ which means that $x \in [b]$. Thus, $[a] \subseteq [b]$. Now interchange the letters a and b to show that $[b] \subseteq [a]$. Hence, $[a] = [b]$ which contradicts our assumption that $[a] \neq [b]$. This establishes (ii). Thus, A/R is a partition of A.■

The sets $[a]$ defined in the previous exercise are called the **equivalence classes** of A given by the relation R. The element a in $[a]$ is called a **representative** of the equivalence class $[a]$.

Review Problems

Problem 18.1

Let $X = \{a, b, c\}$. Recall that $\mathcal{P}(X)$ is the power set of X. Define a binary relation $\mathcal{R}$ on $\mathcal{P}(X)$ as follows:

$$A, B \in \mathcal{P}(x), \ A \ \mathcal{R} \ B \Leftrightarrow |A| = |B|.$$

a. Is $\{a, b\}\mathcal{R}\{b, c\}$?
b. Is $\{a\}\mathcal{R}\{a, b\}$?
c. Is $\{c\}\mathcal{R}\{b\}$?

Problem 18.2

Let $\Sigma = \{a, b\}$. Then Σ^4 is the set of all strings over Σ of length 4. Define a relation R on Σ^4 as follows:

$$s, t \in \Sigma^4, \ s \ R \ t \Leftrightarrow \text{ s has the same first two characters as t.}$$

a. Is $abaa \ R \ abba$?
b. Is $aabb \ R \ bbaa$?
c. Is $aaaa \ R \ aaab$?

Problem 18.3

Let $A = \{4, 5, 6\}$ and $B = \{5, 6, 7\}$ and define the binary relations $R, S,$ and T from A to B as follows:

$$(x, y) \in A \times B, (x, y) \in R \Leftrightarrow x \geq y.$$

$$(x, y) \in A \times B, x \ S \ y \Leftrightarrow 2|(x - y).$$

$$T = \{(4, 7), (6, 5), (6, 7)\}.$$

a. Draw arrow diagrams for $R, S,$ and T.
b. Indicate whether any of the relations $S, R,$ or T are functions.

Problem 18.4

Let $A = \{3, 4, 5\}$ and $B = \{4, 5, 6\}$ and define the binary relation R as follows:

$$(x, y) \in A \times B, (x, y) \in R \Leftrightarrow x < y.$$

List the elements of the sets R and R^{-1}.

Problem 18.5
Let $A = \{2,4\}$ and $B = \{6,8,10\}$ and define the binary relations R and S from A to B as follows:

$$(x,y) \in A \times B, (x,y) \in R \Leftrightarrow x|y.$$

$$(x,y) \in A \times B, x\ S\ y \Leftrightarrow y - 4 = x.$$

List the elements of $A \times B, R, S, R \cup S$, and $R \cap S$.

Problem 18.6
Consider the binary relation on $\mathbb{R}$ defined as follows:

$$x, y \in R,\ x\ R\ y \Leftrightarrow x \geq y.$$

Is R reflexive? symmetric? transitive?

Problem 18.7
Consider the binary relation on $\mathbb{R}$ defined as follows:

$$x, y \in R,\ x\ R\ y \Leftrightarrow xy \geq 0.$$

Is R reflexive? symmetric? transitive?

Problem 18.8
Let $\Sigma = \{0,1\}$ and $A = \Sigma^*$. Consider the binary relation on A defined as follows:

$$x, y \in A,\ x\ R\ y \Leftrightarrow |x| < |y|,$$

where $|x|$ denotes the length of the string x. Is R reflexive? symmetric? transitive?

Problem 18.9
Let $A \neq \emptyset$ and $\mathcal{P}(A)$ be the power set of A. Consider the binary relation on $\mathcal{P}(A)$ defined as follows:

$$X, Y \in \mathcal{P}(A),\ X\ R\ Y \Leftrightarrow X \subseteq Y.$$

Is R reflexive? symmetric? transitive?

Problem 18.10

Let E be the binary relation on $\mathbf{Z}$ defined as follows:

$$a\ E\ b \Leftrightarrow m \equiv n\ (mod\ 2).$$

Show that E is an equivalence relation on $\mathbf{Z}$ and find the different equivalence classes.

Problem 18.11

Let I be the binary relation on $\mathbb{R}$ defined as follows:

$$a\ I\ b \Leftrightarrow a - b \in \mathbf{Z}.$$

Show that I is an equivalence relation on $\mathbb{R}$ and find the different equivalence classes.

Problem 18.12

Let A be the set all straight lines in the cartesian plane. Let $||$ be the binary relation on A defined as follows:

$$l_1||l_2 \Leftrightarrow l_1 \text{ is parallel to } l_2.$$

Show that $||$ is an equivalence relation on A and find the different equivalence classes.

Problem 18.13

Let $A = \mathbb{N} \times \mathbb{N}$. Define the binary relation R on A as follows:

$$(a, b)\ R\ (c, d) \Leftrightarrow a + d = b + c.$$

a. Show that R is reflexive.
b. Show that R is symmetric.
c. Show that R is transitive.
d. List five elements in $[(1, 1)]$.
e. List five elements in $[(3, 1)]$.
f. List five elements in $[(1, 2)]$.
g. Describe the distinct equivalence classes of R.

Problem 18.14

Let R be a binary relation on a set A and suppose that R is symmetric and transitive. Prove the following: If for every $x \in A$ there is a $y \in A$ such that $x\ R\ y$ then R is reflexive and hence an equivalence relation on A.

19 Partial Order Relations

A relation $\leq$ on a set A is called **a partial order** if $\leq$ is reflexive, antisymmetric, and transitive. In this case we call A a **poset.**

Example 19.1
Show that the set $\mathbf{Z}$ of integers together with the relation of inequality $\leq$ is a poset.

Solution.
$\leq$ is reflexive: For all $x \in \mathbf{Z}$ we have $x \leq x$ since $x = x$.
$\leq$ is antisymmetric: By the trichotomy law of real numbers, for a given pair of numbers x and y only one of the following is true: $x < y, x = y$, or $x > y$. So if $x \leq y$ and $y \leq x$ then we must have $x = y$.
$\leq$ is transitive: By the transitivity property of $<$ in $\mathbb{R}$ if $x < y$ and $y < z$ then $x < z$. Thus, if $x \leq y$ and $y \leq z$ then the definition of $\leq$ and the above property imply that $x \leq z$. ■

Example 19.2
Show that the relation $a|b$ in $\mathbb{N}^*$ is a partial order relation.

Solution.
Reflexivity: Since $a = 1 \cdot a$, we have $a|a$.
Antisymmetry: Suppose that $a|b$ and $b|a$. Then there exist positive integers k_1 and k_2 such that $b = k_1a$ and $a = k_2b$. Hence, $a = k_1k_2a$ which implies that $k_1k_2 = 1$. Since $k_1, k_2 \in \mathbb{N}^*$, we must have $k_1 = k_2 = 1$; that is, $a = b$.
Transitivity: Suppose that $a|b$ and $b|c$. Then there exist positive integers k_1 and k_2 such that $b = k_1a$ and $c = k_2b$. Thus, $c = k_1k_2a$ which means that $a|c$. ■

Example 19.3
Let $\mathcal{A}$ be a collection of subsets. Let R be the relation defined by

$$A\ R\ B \Leftrightarrow A \subseteq B.$$

Show that $\mathcal{A}$ is a poset.

Solution.
$\subseteq$ is reflexive: For any set $X \in \mathcal{A}$, $X \subseteq X$.

$\subseteq$ is antisymmetric: By the definition of $=$ if $X \subseteq Y$ and $Y \subseteq X$ then $X = Y$, where $X, Y \in \mathcal{A}$.
$\subseteq$ is transitive: We have seen in Chapter 3 that if $X \subseteq Y$ and $Y \subseteq Z$ then $X \subseteq Z$. ■

To figure out which of two words comes first in an English dictionary, one compares their letters one by one from left to right. If all the letters have been the same to a certain point and one word runs out of letters, that word comes first in the dictionary. For example, *play* comes before *playground*. If all the letters up to a certain point are the same and the next letters differ, then the word whose next letter is located earlier in the alphabet comes first in the dictionary. For example, *playground* comes before *playmate*. This type of order relation is called **lexicographic** or **dictionary** order. A general definition is the following:

Let Σ^* be the set of words with letters from an ordered set Σ. Define the relation $\leq$ on Σ^* as follows: for all $w, z \in \Sigma^*, w \leq z$ if and only if either

(a)$z = wu$ for some $u \in \Sigma^*$, or
(b)$w = xu$ and $z = xv$ where $u, v \in \Sigma^*$ such that the first letter of u precedes the first letter of v in the ordering of Σ.
Then it can be shown that $\leq$ is a partial order relation on Σ^*.

Example 19.4
Let $\Sigma = \{a, b\}$ and suppose that Σ has the partial order relation $R = \{(a, a), (a, b), (b, b)\}$. Let $\leq$ be the corresponding lexicographic order on Σ^*. Indicate which of the following statements are true.

a. $aab \leq aaba$.
b. $bbab \leq bba$.
c. $\epsilon \leq aba$.
d. $aba \leq abb$.
e. $bbab \leq bbaa$.
f. $ababa \leq ababaa$.
g. $bbaba \leq bbabb$.

Solution.
a. True since $aaba = (aab)a$.

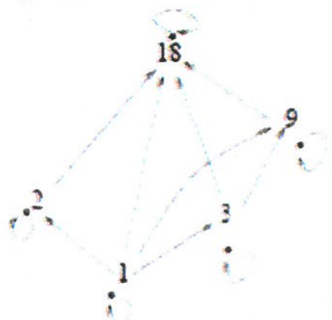
b. False since $bba \leq bbab$.
c. True since $aba = \epsilon aba$.
d. True since $aba = (ab)a, abb = (ab)b$ and $a\ R\ b$.
e. False since $bbaa \leq bbab$.
f. True since $ababaa = (ababa)a$.
g. True since $bbaba = (bbab)a, bbabb = (bbab)b$ and $a\ R\ b$. ■

Another simple pictorial representation of a partial order is the so called **Hasse diagram.** The Hasse diagram of a partial order on the set A is a drawing of the points of A and some of the arrows of the digraph of the order relation. We assume that the directed edges of the Hasse diagram point upward. There are rules to determine which arrows are drawn and which are omitted, namely,

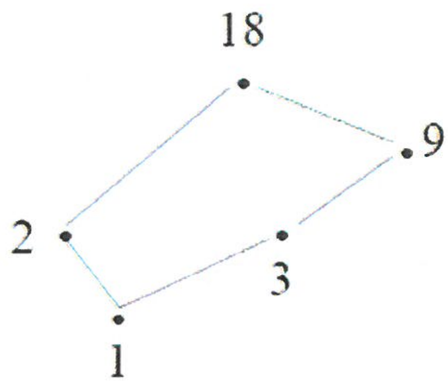
- omit all arrows that can be inferred from transitivity
- omit all loops
- draw arrows without "heads".

Example 19.5
Let $A = \{1, 2, 3, 9, 18\}$ and the "divides" relation on A. Draw the Hasse diagram of this relation.

Solution.
The directed graph of the given relation is



The corresponding Hasse diagram is given by

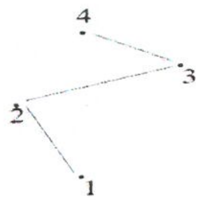


■

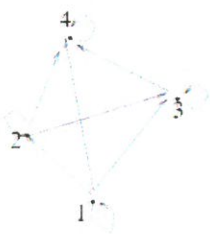
Now, given the Hasse diagram of a partial order relation one can find the digraph as follows:

- reinsert the direction markers on the arrows making all arrows point upward
- add loops at each vertex
- for each sequence of arrows from one point to a second point and from that second point to a third point, add an arrow from the first point to the third.

Example 19.6
Let $A = \{1, 2, 3, 4\}$ be a poset. Find the directed graph corresponding to the following Hasse diagram on A.



Solution.



■

Next, if A is a poset then we say that a and b are **comparable** if either $a \leq b$ or $b \leq a$. If every pair of elements of A are comparable then we call $\leq$ a **total order**.

Example 19.7
Consider the "divides" relation defined on the set $A = \{5, 15, 30\}$. Prove that this relation is a total order on A.

Solution.
The fact that the "divides" relation is a partial order is easy to verify. Since $5|15, 5|30$, and $15|30$, any pair of elements in A are comparable. Thus, the "divides" relation is a total order on A. ■

Example 19.8
Show that the "divides" relation on $\mathbb{N}^*$ is not a total order.

Solution.
A counterexample of two noncomparable numbers are 2 and 3, since 2 does not divide 3 and 3 does not divide 2. ■

Review Problems

Problem 19.1
Let $\Sigma = \{a, b\}$ and let Σ^* be the set of all strings over Σ. Define the relation R on Σ^* as follows: for all $s, t \in \Sigma^*$,

$$s\ R\ t \Leftrightarrow l(s) \leq l(t),$$

where $l(x)$ denotes the length of the word x. Is R antisymmetric? Prove or give a counterexample.

Problem 19.2
Define a relation R on $\mathbf{Z}$ as follows: for all $m, n \in \mathbf{Z}$

$$m\ R\ n \Leftrightarrow m + n \textit{ is even}.$$

Is R a partial order? Prove or give a counterexample.

Problem 19.3
Define a relation R on $\mathbb{R}$ as follows: for all $m, n \in \mathbb{R}$

$$m\ R\ n \Leftrightarrow m^2 \leq n^2.$$

Is R a partial order? Prove or give a counterexample.

Problem 19.4
Let $S = \{0, 1\}$ and consider the partial order relation R defined on $S \times S$ as follows: for all ordered pairs (a, b) and (c, d) in $S \times S$

$$(a, b)\ R\ (c, d) \Leftrightarrow a \leq c \textit{ and } b \leq d.$$

Draw the Hasse diagram for R.

Problem 19.5
Consider the "divides" relation defined on the set $A = \{1, 2, 2^2, \cdots, 2^n\}$, where n is a nonnegative integer.
a. Prove that this relation is a total order on A.
b. Draw the Hasse diagram for this relation when $n = 3$.