Övningar till kap 3

20 Functions: Definitions and Examples

A function is a special case of a relation. A **function** f from a set A to a set B is a relation from A to B such that for every $x \in A$ there is a unique $y \in B$ such that $(x, y) \in f$. For $(x, y) \in f$ we use the notation $y = f(x)$. We call y the **image** of x under f. The set A is called the **domain** of f whereas B is called the **codomain**. The collection of all images of f is called the **range** of f.

Example 20.1
Show that the relation $f = \{(1, a), (2, b), (3, a)\}$ defines a function from $A = \{1, 2, 3\}$ to $B = \{a, b, c\}$. Find its range.

Solution.
Since every element of A has a unique image, f is a function. Its range consists of the elements a and b. ■

Example 20.2
Show that the relation $f = \{(1, a), (2, b), (3, c), (1, b)\}$ does not define a function from $A = \{1, 2, 3\}$ to $B = \{a, b, c\}$.

Solution.
Indeed, since 1 has two images in B, f is not a function. ■

Example 20.3
A **sequence** of elements of a set A is a function from $\mathbb{N}^*$ to A. We write (a_n) and we call a_n the nth term of the sequence.
a. Define the sequence $a_n = n, n \geq 1$. Compute $\sum_{k=1}^{n} a_k$.
b. Define the sequence $a_n = n^2$. Compute the sum $\sum_{k=1}^{n} a_k$.

Solution.
a. Let $S_n = \sum_{k=1}^{n} a_k$. Then write S_n in two different ways, namely, $S_n = 1 + 2 + \cdots + n$ and $S_n = n + (n - 1) + \cdots + 1$. Adding, we obtain $2S_n = (n + 1) + (n + 1) + \cdots + (n + 1) = n(n + 1)$. Thus, $S_n = \frac{n(n+1)}{2}$.
b. First note that $(n + 1)^3 - n^3 = 3n^2 + 3n + 1$. From this we obtain the following chain of equalities:

$$\begin{array}{ccccccccc}
2^3 & - & 1^3 & = & 3(1)^2 & + & 3(1) & + & 1 \\
3^3 & - & 2^3 & = & 3(2)^2 & + & 3(2) & + & 1 \\
\vdots & & & & & & & & \\
(n+1)^3 & - & n^3 & = & 3n^2 & + & 3n & + & 1
\end{array}$$

Adding these equalities we find

$$3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + n = (n+1)^3 - 1.$$

Using a. we find

$$3\sum_{k=1}^{n} k^2 + \frac{3n(n+1)}{2} + n = n^3 + 3n^2 + 3n.$$

Simple arithmetic shows that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \blacksquare$$

Example 20.4
Let $A = \{a, b, c\}$. Define the function $f : \mathcal{P}(A) \to \mathbb{N}$ by $f(X) = |X|$. Find the range of f.

Solution.
By applying f to each member of $\mathcal{P}(A)$ we find $Range(f) = \{0, 1, 2, 3\}$. $\blacksquare$

Example 20.5
Consider the alphabet $\Sigma = \{a, b\}$ and the function $f : \Sigma^* \to \mathbf{Z}$ defined as follows: for any string $s \in \Sigma^*$

$$f(s) = \textit{the number of } a's \textit{ in } s.$$

Find $f(\epsilon), f(ababb)$, and $f(bbbaa)$.

Solution.
$f(\epsilon) = 0, f(ababb) = 2$, and $f(bbbaa) = 2$. $\blacksquare$

Example 20.6 *(Equality of Functions)*
Two functions f and g defined on the same domain D are said to be **equal** if and only if $f(x) = g(x)$ for all $x \in D$. Show that the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ and $g(x) = \sqrt{x^2}$ are equal.

Solution.
A simple argument by the method of proof by cases shows that $\sqrt{x^2} = |x|$. $\blacksquare$

and we define the decoding function $D : L \to \Sigma^*$ by

$$D(s) = \textit{the string obtained from } s \textit{ by replacing consecutive triple of bits of } s \textit{ by a single copy of that bit.}$$

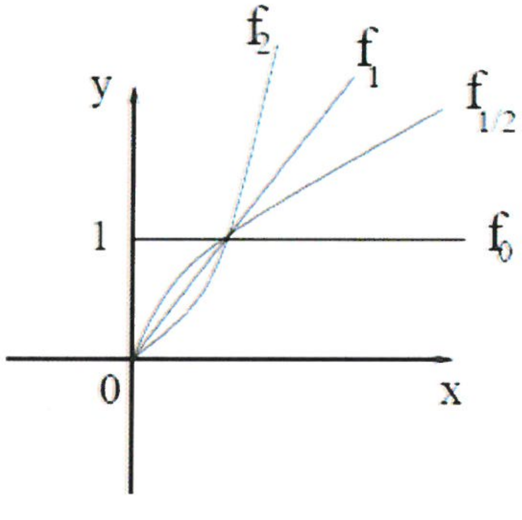
Find $E(0110)$ and $D(111111000111)$.

Solution.
We have $E(0110) = 000111111000$ and $D(111111000111) = 1101$. ■

Now, let A and B be subsets of $\mathbb{R}$. A function $f : A \to B$ is called a **real-valued function of a real variable.** In this case, each ordered pair $(x, f(x))$ can be represented by a point in the Cartesian plane. The collection of all such points is called the **graph** of f.

Example 20.10
Consider the power function $f_a(x) = x^a$, where $a, x \in \mathbb{R}^+ \cup \{0\}$. Graph on the same Cartesian plane the functions $f_0(x), f_1(x), f_{\frac{1}{2}}(x)$, and $f_2(x)$.

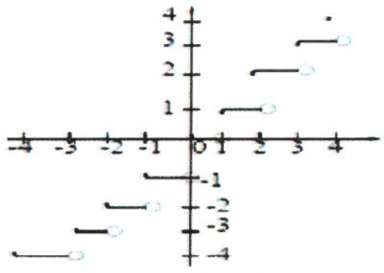
Solution.



■

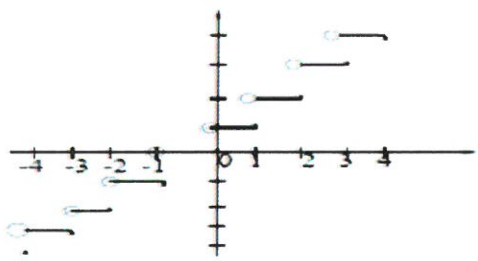
Example 20.11
Graph the functions $f(x) = \lfloor x \rfloor$ and $g(x) = \lceil x \rceil$ on the closed interval $[-4, 4]$.

Solution.



$\lfloor x \rfloor$

$\lceil x \rceil$



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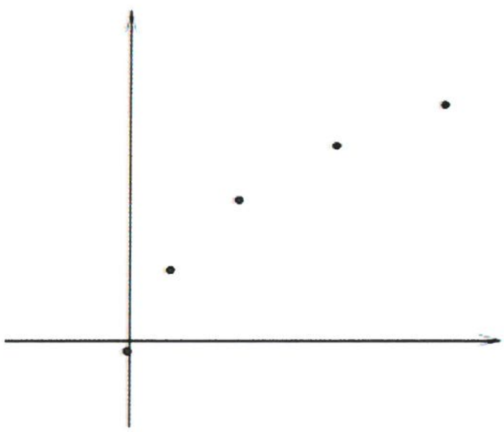
Example 20.12
Graph the function $f : \mathbb{N} \to \mathbb{R}$ defined by $f(n) = \sqrt{n}$.

Solution.



■

Example 20.13

Let D_f be the domain of a function f and $S \subseteq D_f$. We say that f is **increasing** on S if and only if, for all $x_1, x_2 \in S$, if $x_1 < x_2$ then $f(x_1) < f(x_2)$. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x - 3$ is increasing on $\mathbb{R}$.

Solution.

Indeed, for any real numbers x_1 and x_2 such that $x_1 < x_2$, we have $2x_1 - 3 < 2x_2 - 3$. That is, $f(x_1) < f(x_2)$ so that f is increasing. ■

Example 20.14

Let D_f be the domain of a function f and $S \subseteq D_f$. We say that f is **decreasing** on S if and only if, for all $x_1, x_2 \in S$, if $x_1 < x_2$ then $f(x_1) > f(x_2)$. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{x+2}{x+1}$ is decreasing on $(-\infty, -1)$ and $(-1, \infty)$.

Solution.

Indeed, for any real numbers $x_1, x_2 \in (-\infty, -1)$ or $x_1, x_2 \in (-1, \infty)$ such that $x_1 < x_2$, we have $(x_1 + 1)(x_2 + 1) > 0$. This implies, that $f(x_1) - f(x_2) = \frac{x_2 - x_1}{(x_1+1)(x_2+1)} > 0$. Thus, f is decreasing on the given intervals. ■

Review Problems

Problem 20.1
Let $f, g : \mathbb{R} \to \mathbb{R}$ be the functions $f(x) = 2x$ and $g(x) = \frac{2x^3+2x}{x^2+1}$. Show that $f = g$.

Problem 20.2
Let $H, K : \mathbb{R} \to \mathbb{R}$ be the functions $H(x) = \lfloor x \rfloor + 1$ and $K(x) = \lceil x \rceil$. Does $H = K$? Explain.

Problem 20.3
Find functions defined on the set of nonnegative integers that define the sequences whose first six terms are given below.
a. $1, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, \frac{1}{9}, -\frac{1}{11}$.
b. $0, -2, 4, -6, 8, -10$.

Problem 20.4
Let $A = \{1, 2, 3, 4, 5\}$ and let $F : \mathcal{P}(A) \to \mathbf{Z}$ be defined as follows:

$$F(X) = \begin{cases} 0 & if\ X\ has\ an\ even\ number\ of\ elements \\ 1 & if\ X\ has\ an\ odd\ number\ of\ elements \end{cases}$$

Find the following

a. $F(\{1, 3, 4\})$
b. $F(\emptyset)$.
c. $F(\{2, 3\})$.
d. $F(\{2, 3, 4, 5\})$.

Problem 20.5
Let $\Sigma = \{a, b\}$ and Σ^* be the set of all strings over Σ.
a. Define $f : \Sigma^* \to \mathbf{Z}$ as follows:

$$f(s) = \begin{cases} the\ number\ of\ b's\ to\ the\ left\ of\ the\ leftmost\ a\ in\ s \\ 0 \quad if\ s\ contains\ no\ a's \end{cases}$$

Find $f(aba), f(bbab)$, and $f(b)$. What is the range of f?
b. Define $g : \Sigma^* \to \Sigma^*$ as follows:

$g(s) =$ *the string obtained by writing the characters of s in reverse order.*

Find $g(aba), g(bbab)$, and $g(b)$. What is the range of g?

Problem 20.6
Let E and D be the encoding and decoding functions.
a. Find $E(0110)$ and $D(111111000111)$.
b. Find $E(1010)$ and $D(000000111111)$.

Problem 20.7
Let H denote the Hamming distance function on Σ^5.
a. Find $H(10101, 00011)$.
b. Find $H(00110, 10111)$.

Problem 20.8
Consider the three-place Boolean function $f : \{0,1\}^3 \to \{0,1\}$ defined as follows:

$$f(x_1, x_2, x_3) = (3x_1 + x_2 + 2x_3) \ mod\ 2$$

a. Find $f(1,1,1)$ and $f(0,1,1)$.
b. Describe f using an input/output table.

Problem 20.9
Draw the graphs of the power functions $f_{\frac{1}{3}}(x)$ and $f_{\frac{1}{4}}(x)$ on the same set of axes. When, $0 < x < 1$, which is greater: $x^{\frac{1}{3}}$ or $x^{\frac{1}{4}}$? When $x > 1$, which is greater $x^{\frac{1}{3}}$ or $x^{\frac{1}{4}}$?

Problem 20.10
Graph the function $f(x) = \lceil x \rceil - \lfloor x \rfloor$ on the interval $(-\infty, \infty)$.

Problem 20.11
Graph the function $f(x) = x - \lfloor x \rfloor$ on the interval $(-\infty, \infty)$.

Problem 20.12
Graph the function $h : \mathbb{N} \to \mathbb{R}$ defined by $h(n) = \lfloor \frac{n}{2} \rfloor$.

Problem 20.13
Let $k : \mathbb{R} \to \mathbb{R}$ be the function defined by the formula $k(x) = \frac{x-1}{x}$ for all nonzero real numbers x.
a. Show that k is increasing on $(0, \infty)$.
b. Is k increasing or decreasing on $(-\infty, 0)$? Prove your answer.

21 Bijective and Inverse Functions

Let $f : A \rightarrow B$ be a function. We say that f is **injective** or **one-to-one** if and only if for all $x, y \in A$, if $f(x) = f(y)$ then $x = y$. Using the concept of contrapositive, a function f is injective if and only if for all $x, y \in A$, if $x \neq y$ then $f(x) \neq f(y)$. Taking the negation of this last conditional implication we see that f is not injective if and only if there exist two distinct elements a and b of A such that $f(a) = f(b)$.

Example 21.1
a. Show that the identity function I_A on a set A is injective.
b. Show that the function $f : \mathbf{Z} \rightarrow \mathbf{Z}$ defined by $f(n) = n^2$ is not injective.

Solution.
a. Let $x, y \in A$. If $I_A(x) = I_A(y)$ then $x = y$ by the definition of I_A. This shows that I_A is injective.
b. Since $1^2 = (-1)^2$ and $1 \neq -1$, f is not injective.■

Example 21.2 *(Hash Functions)*
Let $m > 1$ be a positive integer . Show that the function $h : \mathbf{Z} \rightarrow \mathbf{Z}$ defined by $h(n) = n \ mod \ m$ is not injective.

Solution.
Indeed, since $m > 1$, we have $2m+1 \neq m+1$ and $h(m+1) = h(2m+1) = 1$. So h is not injective. ■

Example 21.3
Show that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is increasing then f is one-to-one.

Solution.
Suppose that $x_1 \neq x_2$. Then without loss of generality we can assume that $x_1 < x_2$. Since f is increasing, $f(x_1) < f(x_2)$. That is, $f(x_1) \neq f(x_2)$. Hence, f is one-to-one. ■

Example 21.4
Show that the composition of two injective functions is also injective.

Solution.
Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two injective functions. We will show that $g \circ f : A \rightarrow C$ is also injective. Indeed, suppose that $(g \circ f)(x_1) = (g \circ f)(x_2)$

for $x_1, x_2 \in A$. Then $g(f(x_1)) = g(f(x_2))$. Since g is injective, $f(x_1) = f(x_2)$. Now, since f is injective, $x_1 = x_2$. This completes the proof that $g \circ f$ is injective. ■

Now, for any function $f : A \to B$ we have $Range(f) \subseteq B$. If equality holds then we say that f is **surjective** or **onto.** It follows from this definition that a function f is surjective if and only if for each $y \in B$ there is an $x \in A$ such that $f(x) = y$. By taking the negation of this we see that f is not onto if there is a $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Example 21.5
a. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x - 5$ is surjective.
b. Show that the function $f : \mathbf{Z} \to \mathbf{Z}$ defined by $f(n) = 3n - 5$ is not surjective.

Solution.
a. Let $y \in \mathbb{R}$. Is there an $x \in \mathbb{R}$ such that $f(x) = y$? That is, $3x - 5 = y$. But solving for x we find $x = \frac{y+5}{3} \in \mathbb{R}$ and $f(x) = y$. Thus, f is onto.
b. Take $m = 3$. If f is onto then there should be an $n \in \mathbf{Z}$ such that $f(n) = 3$. That is, $3n - 5 = 3$. Solving for n we find $n = \frac{8}{3}$ which is not an integer. Hence, f is not onto. ■

Example 21.6 *(Projection Functions)*
Let A and B be two nonempty sets. The functions $pr_A : A \times B \to A$ defined by $pr_A(a, b) = a$ and $pr_B : A \times B \to B$ defined by $pr_B(a, b) = b$ are called **projection** functions. Show that pr_A and pr_B are surjective functions.

Solution.
We prove that pr_A is surjective. Indeed, let $a \in A$. Since B is not empty, there is a $b \in B$. But then $(a, b) \in A \times B$ and $pr_A(a, b) = a$. Hence, pr_A is surjective. The proof that pr_B is surjective is similar. ■

Example 21.7
Show that the composition of two surjective functions is also surjective.

Solution.
Let $f : A \to B$ and $g : B \to C$, where $Range(f) \subseteq C$, be two surjective functions. We will show that $g \circ f : A \to D$ is also surjective. Indeed, let $z \in D$. Since g is surjective, there is a $y \in B$ such that $g(y) = z$. Since f is

surjective, then there is an $x \in A$ such that $f(x) = y$. Thus, $g(f(x)) = z$. This shows that $g \circ f$ is surjective. ■

Now, we say that a function f is **bijective** or **one-to-one correspondence** if and only if f is both injective and surjective. A bijective function on a set A is called a **permutation.**

Example 21.8
a. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x - 5$ is a bijective function.
b. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not bijective.

Solution.
a. First we show that f is injective. Indeed, suppose that $f(x_1) = f(x_2)$. Then $3x_1 - 5 = 3x_2 - 5$ and this implies that $x_1 = x_2$. Hence, f is injective. f is surjective by Example 21.5 (a).
b. f is not injective since $f(-1) = f(1)$ but $-1 \neq 1$. Hence, f is not bijective. ■

Example 21.9
Show that the composition of two bijective functions is also bijective.

Solution.
This follows from Example 21.4 and Example 21.7 ■

Theorem 21.1
Let $f : X \to Y$ be a bijective function. Then there is a function $f^{-1} : Y \to X$ with the following properties:

a. $f^{-1}(y) = x$ if and only if $f(x) = y$.
b. $f^{-1} \circ f = I_X$ and $f \circ f^{-1} = I_Y$ where I_X denotes the identity function on X.
c. f^{-1} is bijective.

Proof.
For each $y \in Y$ there is a unique $x \in X$ such that $f(x) = y$ since f is bijective. Thus, we can define a function $f^{-1} : Y \to X$ by $f^{-1}(y) = x$ where $f(x) = y$.
a. Follows from the definition of f^{-1}.

b. Indeed, let $x \in X$ such that $f(x) = y$. Then $f^{-1}(y) = x$ and $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x = I_X(x)$. Since x was arbitrary, $f^{-1} \circ f = I_X$. The proof that $f \circ f^{-1} = I_Y$ is similar.
c. We show first that f^{-1} is injective. Indeed, suppose $f^{-1}(y_1) = f^{-1}(y_2)$. Then $f(f^{-1}(y_1)) = f(f^{-1}(y_2))$; that is, $(f \circ f^{-1})(y_1) = (f \circ f^{-1})(y_2)$. By b. we have $I_Y(y_1) = I_Y(y_2)$. From the definition of I_Y we obtain $y_1 = y_2$. Hence, f^{-1} is injective. We next show that f^{-1} is surjective. Indeed, let $y \in Y$. Since f is onto there is a unique $x \in X$ such that $f(x) = y$. By the definition of f^{-1}, $f^{-1}(y) = x$. Thus, for every element $y \in Y$ there is an element $x \in X$ such that $f^{-1}(y) = x$. This says that f^{-1} is surjective and completes a proof of the theorem ■

Example 21.10
Show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x - 5$ is bijective and find a formula for its inverse function.

Solution.
We have already proved that f is bijective. We will just find the formula for its inverse function f^{-1}. Indeed, if $y \in Y$ we want to find $x \in X$ such that $f^{-1}(y) = x$, or equivalently, $f(x) = y$. This implies that $3x - 5 = y$ and solving for x we find $x = \frac{y+5}{3}$. Thus, $f^{-1}(y) = \frac{y+5}{3}$ ■

Review Problems

Problem 21.1
a. Define $g : \mathbf{Z} \to \mathbf{Z}$ by $g(n) = 3n - 2$.
(i) Is g one-to-one? Prove or give a counterexample.
(ii) Is g onto? Prove or give a counterexample.
b. Define $G : \mathbb{R} \to \mathbb{R}$ by $G(x) = 3x - 2$. Is G onto? Prove or give a counterexample.

Problem 21.2
Determine whether the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{x+1}{x}$ is one-to-one or not.

Problem 21.3
Determine whether the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{x}{x^2+1}$ is one-to-one or not.

Problem 21.4
Let $f : \mathbb{R} \to \mathbf{Z}$ be the floor function $f(x) = \lfloor x \rfloor$.
a. Is f one-to-one? Prove or give a counterexample.
b. Is f onto? Prove or give a counterexample.

Problem 21.5
Let $\Sigma = \{0, 1\}$ and let $l : \Sigma^* \to \mathbb{N}$ denote the length function.
a. Is l one-to-one? Prove or give a counterexample.
b. Is l onto? Prove or give a counterexample.

Problem 21.6
If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are one-to-one functions, is $f + g$ also one-to-one? Justify your answer.

Problem 21.7
Define $F : \mathcal{P}\{a, b, c\} \to \mathbb{N}$ to be the number of elements of a subset of $\mathcal{P}\{a, b, c\}$.
a. Is F one-to-one? Prove or give a counterexample.
b. Is F onto? Prove or give a counterexample.

Problem 21.8
If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are onto functions, is $f + g$ also onto? Justify your answer.

Problem 21.9
Let $\Sigma = \{a, b\}$ and let $l : \Sigma^* \to \mathbb{N}$ be the length function. Let $f : \mathbb{N} \to \{0, 1, 2\}$ be the hash function $f(n) = n \, mod \, 3$. Find $(f \circ l)(abaa)$, $(f \circ l)(baaab)$, and $(f \circ l)(aaa)$.

Problem 21.10
Show that the function $F^{-1} : \mathbb{R} \to \mathbb{R}$ given by $F^{-1}(y) = \frac{y-2}{3}$ is the inverse of the function $F(x) = 3x + 2$.

Problem 21.11
If $f : X \to Y$ and $g : Y \to Z$ are functions and $g \circ f : X \to Z$ is one-to-one, must both f and g be one-to-one? Prove or give a counterexample.

Problem 21.12
If $f : X \to Y$ and $g : Y \to Z$ are functions and $g \circ f : X \to Z$ is onto, must both f and g be onto? Prove or give a counterexample.

Problem 21.13
If $f : X \to Y$ and $g : Y \to Z$ are functions and $g \circ f : X \to Z$ is one-to-one, must f be one-to-one? Prove or give a counterexample.

Problem 21.14
If $f : X \to Y$ and $g : Y \to Z$ are functions and $g \circ f : X \to Z$ is onto, must g be onto? Prove or give a counterexample.

Problem 21.15
Let $f : W \to X, g : X \to Y$ and $h : Y \to Z$ be functions. Must $h \circ (g \circ f) = (h \circ g) \circ f$? Prove or give a counterexample.

Problem 21.16
Let $f : X \to Y$ and $g : Y \to Z$ be two bijective functions. Show that $(g \circ f)^{-1}$ exists and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.