5. Permutations

In this chapter we are going to study permutations, that is, reorderings of elements. There are aspects on permutations pertinent to both algebra, computer science, and combinatorics.

The algebraic ones derive from the fact that permutations of a set of elements form a group, the symmetric group, which we studied in chapter 2.

Computer science is of course relevant when sorting, which is a special kind of permutation. Some of the sorting algorithms that turned up in chapter 4 are performing a new role here.

Using the theory of permutations, we'll also treat some *combinatorial* problems – such as solving the fifteen puzzle – and determine how many different necklaces you can make from black and white beads.

Highlights from this chapter

- Different ways of describing a permutation: *one-line form* $\begin{bmatrix} 3 & 4 & 5 & 2 & 1 \end{bmatrix}$, *two-line form* $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{bmatrix}$, *cycle form* $\begin{pmatrix} 1 & 3 & 5 \end{pmatrix}\begin{pmatrix} 2 & 4 \end{pmatrix}$ and matrix form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

- Permutations are multiplied using function composition. The set of permutations of n elements forms the *symmetric group* S_n.
- The *type* of a permutation is its multiset of cycle lengths, so $\begin{pmatrix} 1 & 3 & 5 \end{pmatrix}\begin{pmatrix} 4 & 2 \end{pmatrix}$ has the type $\{3, 2\}$.
- The *order* of a permutation is the number of times the permutation has to be carried out before the original order is restored.
- A *transposition* is letting two elements switch places, that is, a cycle of length 2. Every permutation can be made up by a sequence of transpositions. If an even number of transpositions is needed the permutation is called *even*, otherwise it's *odd*. For instance, $\begin{bmatrix} 3 & 4 & 5 & 2 & 1 \end{bmatrix} = \begin{pmatrix} 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 3 \end{pmatrix}\begin{pmatrix} 4 & 2 \end{pmatrix}$ is an odd permutation.
- In the *Fifteen puzzle* all even permutations can be solved, but no odd permutations.
- Using *Burnside's lemma*, combinatorial problems concerning symmetries can be solved.

5.1 Representation of Permutations

A permutation is an ordering of a set of elements. Since it's always possible to start by assigning numbers to the elements, studies frequently focus on permutations of numbers, but the conclusions apply to all finite sets.

There exist several ways of representing a permutation. The most simple is of course just to write down the elements in the order given by the permutation. For instance

$$\begin{bmatrix}3 & 4 & 5 & 2 & 1\end{bmatrix} \qquad \text{and} \qquad \text{cdeba}$$

are permutations of the numbers 12345 and the letters abcde, respecively. This way of writing a permutation is called **one-line form** and is suitable when regarding the permutation as an *sequence*. You can for instance calculate that there are $5! = 120$ five-letter words using the letters abcde.

In other cases it's prefered to regard the permutation as the *mapping* that transforms 12345 into 34521. Then **two-line notation** is used:

$$\begin{bmatrix}1 & 2 & 3 & 4 & 5\\ 3 & 4 & 5 & 2 & 1\end{bmatrix}$$

The interpretation is that 1 is mapped onto 2, that 2 is mapped onto 4, etc. On two-line form we can thus rearrange the columns and still describe the same permutation:

$$\begin{bmatrix}1 & 2 & 3 & 4 & 5\\ 3 & 4 & 5 & 2 & 1\end{bmatrix} = \begin{bmatrix}3 & 1 & 4 & 5 & 2\\ 5 & 3 & 2 & 1 & 4\end{bmatrix}$$

Another way of describing the relationship is to draw arrows like this:

$$\begin{aligned}1 &\mapsto 3\\ 2 &\mapsto 4\\ 3 &\mapsto 5\\ 4 &\mapsto 2\\ 5 &\mapsto 1\end{aligned}$$

This diagram describes the permutation as a **mapping** where every element is mapped on another one. The mapping is an bijection from the set onto itself, so we can also describe the mapping in a graphical way like this:

$$\begin{tikzcd}
& 5 \arrow[dl] & \\
1 \arrow[rr] & & 3 \arrow[ul] \\
& 2 \arrow[r, shift left] & 4 \arrow[l, shift left]
\end{tikzcd}$$

This is called the **cycle diagram** of the permutation, since the mapping gives a number of closed paths, so-called **cycles**. We can describe each cycle by choosing an element to use as a starting point and then follow the arrows. If we write down each cycle surrounded by parentheses we get the permutation on **cycle form**:

$$(1 \quad 3 \quad 5)(2 \quad 4)$$

which is interpreted as "1 is mapped onto 3, which is mapped onto 5, which is mapped onto the start of the cycle, 1. And 2 is mapped onto 4 which is mapped back onto 2". Note that the cycle form isn't *unique*, which means that there are several different ways of describing one and the same permutation on cycle form. (For the rest, see exercise 5.4.)

Exercise 5.1 What does the permutation

$$\begin{bmatrix} 5 & 2 & 1 & 7 & 4 & 3 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{bmatrix}$$

look like written on one-line form and cycle form, respecively?

Exercise 5.2 Write the permutation

$$\begin{bmatrix} 5 & 2 & 1 & 7 & 4 & 3 & 6 \end{bmatrix}$$

on cycle form.

Exercise 5.3 Write the permutation

$$(5 \quad 2)\,(1)\,(7 \quad 4 \quad 3 \quad 6)$$

on one-line form.

Exercise 5.4 In the cycle form of a permutation the cycles can come in an arbitrary order and in each cycle an abritrary element can be placed first. In how many ways can

$$(5 \quad 2)\,(1)\,(7 \quad 4 \quad 3 \quad 6)$$

be written on cycle form?

Exercise 5.5: ***Important!*** A cycle of the length k is called a k-cycle. If a permutation has e_1 1-cycles, e_2 2-cycles, etc., in how many ways can it then be written on cycle form?

5.1.1 Multiplication and Inversion of Permutations

There is a natural way of defining a kind of composition, we can call it **multiplication**, of two permutations. If you regard permutations as mappings of a set onto itself you can simply combine the mappings one after the other.

Assume that we have two permutations, which we denote by the Greek letters π and τ. We let $\pi(i)$ denote the value π has on place i. In other words, the permutation π maps the element i onto the element $\pi(i)$. Using composition of functions, we now get a new permutation by first performing π and then τ, and this mapping is denoted $\tau\pi$ (or $\tau \circ \pi$ if you want to really emphasise that this is a case of composition of functions). Note that the permutations are carried out right to left – that is the way with composition of functions – but are usually drawn left to right, an inconsequence that mathematics has to live with.

Example 5.1 Composition of two mappings π and σ is done like this:

$$\begin{aligned} 1 &\overset{\sigma}{\mapsto} 3 \overset{\pi}{\mapsto} 5 \\ 2 &\mapsto 4 \mapsto 3 \\ 3 &\mapsto 5 \mapsto 2 \\ 4 &\mapsto 2 \mapsto 1 \\ 5 &\mapsto 1 \mapsto 4 \end{aligned}$$

This corresponds to the following multiplication of permutations:

$$\pi \circ \sigma = \begin{bmatrix} 3 & 4 & 5 & 2 & 1 \\ 5 & 3 & 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{bmatrix}. \qquad \blacksquare$$

Exercise 5.6 Carry out the following multiplications of permutations:

(a) two-line form:

$$\begin{bmatrix} 2 & 3 & 4 & 5 & 1 \\ 5 & 3 & 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}$$

(b) two-line form:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 5 & 3 & 4 \\ 3 & 2 & 1 & 4 & 5 \end{bmatrix}$$

(c) one-line form:

$$\begin{bmatrix} 1 & 5 & 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 & 4 & 5 \end{bmatrix}$$

(d) cycle form:

$$\begin{pmatrix} 5 & 2 & 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} 5 & 2 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 4 & 3 \end{pmatrix}$$

As we saw in chapter 2, multiplication of permutations is a group operation. The set of permutations of a set A is denoted S_A, but usually A is the set $\{1, 2, \ldots, n\}$, in which case we use the notatation S_n and the name the **symmetric group of n elements.** S_n has $n!$ elements since there are $n!$ permutations of n elements. For instance we have

$$S_3 = \left\{ \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \right\}$$

Observe that in the group S_n, the identity element is the permutation id which maps every number onto itself:

$$\mathrm{id} = \begin{bmatrix} 1 & 2 & \ldots & n \\ 1 & 2 & \ldots & n \end{bmatrix}$$

The inverse π^{-1} of a permutation π in S_n we get by swapping the upper and lower rows in the two-line form:

$$\pi = \begin{bmatrix} 1 & 2 & \ldots & n \\ \pi(1) & \pi(2) & \ldots & \pi(n) \end{bmatrix} \quad \text{gives } \pi^{-1} = \begin{bmatrix} \pi(1) & \pi(2) & \ldots & \pi(n) \\ 1 & 2 & \ldots & n \end{bmatrix}$$

Exercise 5.7 Determine on cycle form the inverses of the following permutations:

(a) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 2 & 1 & 4 & 5 \end{bmatrix}$

(c) $\begin{pmatrix} 5 & 2 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 4 & 3 \end{pmatrix}$

(d) $\begin{pmatrix} 5 & 2 & 1 & 4 & 3 \end{pmatrix}$

Exercise 5.8: ***Important!*** How do you in a simple way invert a permutation written on cycle form?

Exercise 5.9: ***Important!*** What does the identity permutation id $\in S_n$ look like on one-line form and cycle form?

5.1.2 Permutation Matrices

Another way of representing permutations and multiplication of permutations is with the help of matrices. A **permutation matrix** is a square matrix where all the elements are zeros except for exactly one element in each row and column, which instead is one. There are six permutation matrices of size 3×3:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The permutation $\pi = \begin{bmatrix}\pi_1 & \pi_2 & \dots & \pi_n\end{bmatrix}$ is now represented by the $n \times n$-matrix M_π, where the one in the first column is on row π_1, the one in the second column is on row π_2, and so on. Multiplication of permutations does now correspond to ordinary matrix multiplication of permutation matrices. For instance, the permutation multiplication

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{bmatrix}$$

corresponds to the matrix multiplication

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Exercise 5.10: ***Important!***

(a) Show that there are the same number of permutations of n elements as there are permutation matrices of size $n \times n$.

(b) Show that multiplication of permutations corresponds exactly to matrix multiplication of the corresponding permutation matrices.

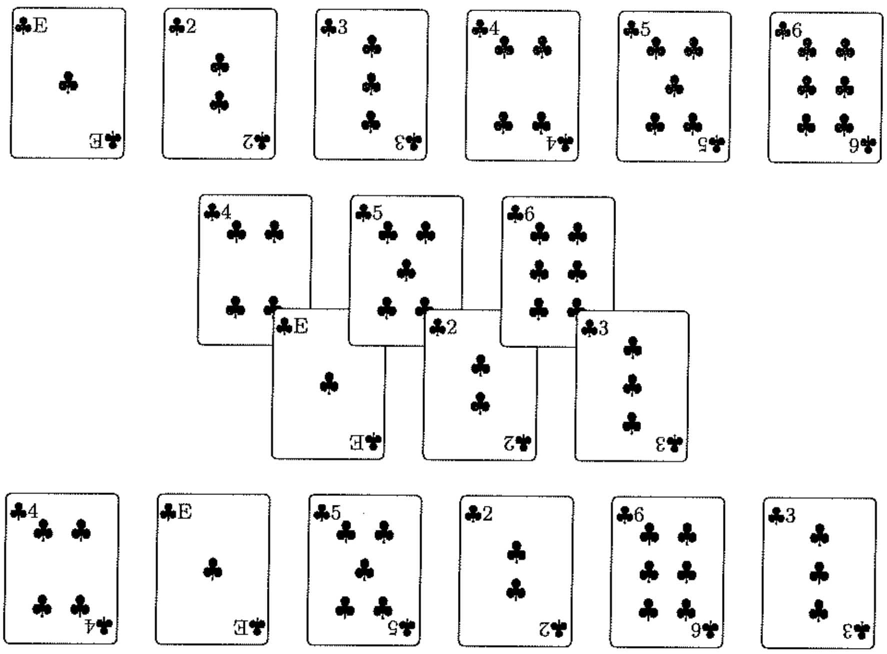
(c) Show that the inversion of a permutation corresponds to the transposition of the permutation matrix.

5.1.3 Order and Type of Permutations

Card masters learn to perform a shuffle of the pack of cards in exactly the same way each time. One advantage of this is that they then can be sure that the cards after a certain number of shuffles are in exactly the same order as they were at the start. We'll now see how it's possible – by writing the shuffle as a permutation on cycle form – to determine how many shuffles that are needed.

Example 5.2 A small pack with six cards numbered 1 to 6 is shuffled in the following way:

1. The pack is divided into two halves, first and second half.
2. The two halves are riffle shuffled, that is, the cards are taken alternatly from the second and the first half. Thus, in the first position a card from the second half will end up.



The figure above shows how the shuffle is done. Apparently it can be summarised as "the card in place 1 is moved to place 2, the card in place 2 is moved to place 4, ...", which in two-line form is represented as

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 1 & 3 & 5 \end{bmatrix}.$$

Note that a card shuffle switches the places (and not the values) of the cards, so it's the way in which the places are changed that the permutation is to describe. If we make the shuffle π several times it's easiest to see what happens when looking at the cycle form:

$$\pi = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 3 & 6 & 5 \end{pmatrix}.$$

How many shuffles of this kind are needed before the cards are back in their original order? Start by following what happens with the card on place 1. The cycle $\begin{pmatrix} 1 & 2 & 4 \end{pmatrix}$ ensures that it gets back to place 1 after three shuffles.

Using the same line of reasoning we can note that every element in a cycle of length three will have returned to its original position after three shuffles but not before that point. Since all the cards are included in one of the 3-cycles, the whole pack will be in its original order after three shuffles. ■

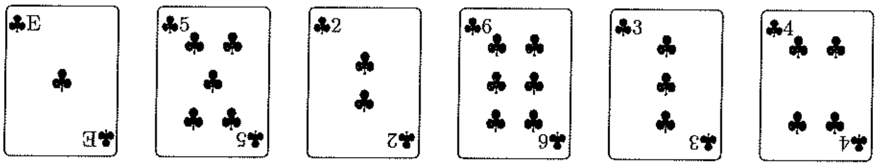
The number of times that a permutation has to be repeated before all elements are at their original places (the identity permutation) is called the **order** of the permutation. (Note that this is exactly the same concept *order* as the one we introduced for group elements in chapter 2.)

We found in the example that the order of a cycle is equal to the length of the cycle. If there are several cycles but all of them have the same length the order is the same as well. But what is the order of a permutation with cycles of different lengths?

Example 5.3 Now we make a luxury version of the shuffle in the previous example:

1. The pack is divided into two halves.
2. The two halves are riffle shuffled.

3. Then the first card is placed last.



This version of the shuffle is represented by the permutation

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 5 & 6 & 2 & 4 \end{bmatrix}, \text{ which on cycle form is } \begin{pmatrix}1\end{pmatrix}\begin{pmatrix}2 & 3 & 5\end{pmatrix}\begin{pmatrix}4 & 6\end{pmatrix}.$$

How many shuffles of this kind are needed before the card pack returns to its original order? We have one 1-cycle, one 2-cycle, and one 3-cycle. The card in the 1-cycle remains in the same place during the whole process. The cards in the 2-cycle are at their right places after every second shuffle. The cards in the 3-cycle are in their right places after every third shuffle. The first time all the cards are back at their right places at the same time is after six shuffles, when the 2-cycle has made three revolutions and the 3-cycle has made two revolutions. ■

In general, if a permutation π has a cycle of length ℓ then the order of π has to be a multiple of ℓ, since the elements in the cycle are in their right places only every ℓth time the permutation is repeated. Thus the order of π is the least common multiple of all the cycle lengths:

$$\text{order}(\pi) = \text{lcm}\{\ell \mid \pi \text{ has a cycle of length } \ell\}.$$

In the two examples above we had

$$\text{order}\big(\begin{pmatrix}1 & 4 & 2\end{pmatrix}\begin{pmatrix}3 & 5 & 6\end{pmatrix}\big) = \text{lcm}\{3,3\} = 3$$

and

$$\text{order}\big(\begin{pmatrix}1\end{pmatrix}\begin{pmatrix}2 & 5 & 3\end{pmatrix}\begin{pmatrix}4 & 6\end{pmatrix}\big) = \text{lcm}\{1,3,2\} = 6.$$

Here we have listed the lengths of all the cycles, even when there are several ones of the same length, as in $\{3,3\}$. This multiset of cycle lengths is called the **type** of the permutation. The sum of the cycle lengths is of course the total number of elements in the permutation. Using a concept that is discussed more in detail in chapter 6, the type of a permutation in S_n is is an **integer partition** of n.

Exercise 5.11 Determine the type and order of the following permutations:

(a)

$$\begin{pmatrix}1 & 7\end{pmatrix}\begin{pmatrix}2 & 5\end{pmatrix}\begin{pmatrix}3\end{pmatrix}\begin{pmatrix}4 & 6\end{pmatrix}$$

(b)

$$\begin{pmatrix}1 & 7 & 6 & 4\end{pmatrix}\begin{pmatrix}2 & 5 & 3\end{pmatrix}$$

(c)

$$\begin{bmatrix}1 & 7 & 6 & 4 & 2 & 5 & 3\end{bmatrix}$$

Exercise 5.12 How many times do you have to riffle shuffle an ordinary pack of cards containing 52 cards before the original order is restored?

Exercise 5.13: ***Important!*** Prove that two permutations of the same type always have the same order.

Exercise 5.14 Find two permutations in S_6 that have the same order but not the same type.

Exercise 5.15: ***Important!*** A common partitioning of groups is into equivalence classes that are called **conjugacy classes**. The conjugacy class of a permutation $\pi \in S_n$ consists of all permutations on the form $\sigma^{-1}\pi\sigma$. Intuitively conjugation is to be percieved in the following way: First σ changes the names of the symbols that are to be permuted. Then the symbols are permuted by π. Finally σ^{-1} changes the names of the symbols back to the original state. If for instance

$$\sigma = \begin{bmatrix} 1 & 3 & 4 & 2 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \qquad \text{and} \qquad \pi = \begin{pmatrix} 1 & 4 \end{pmatrix} \begin{pmatrix} 2 & 5 & 3 \end{pmatrix}$$

we get $\sigma^{-1}\pi\sigma$ by:

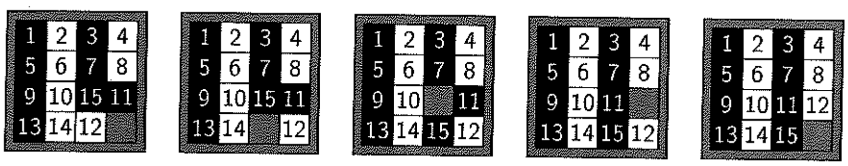
$$\begin{aligned} &1 \overset{\sigma}{\mapsto} 1 \overset{\pi}{\mapsto} 4 \overset{\sigma^{-1}}{\mapsto} 2 \\ &2 \mapsto 4 \mapsto 1 \mapsto 1 \\ &3 \mapsto 2 \mapsto 5 \mapsto 5 \\ &4 \mapsto 3 \mapsto 2 \mapsto 3 \\ &5 \mapsto 5 \mapsto 3 \mapsto 4 \end{aligned}$$

Thus $\sigma^{-1}\pi\sigma = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 5 & 4 \end{pmatrix}$. The type of $\sigma^{-1}\pi\sigma$ is thereby $\{2, 3\}$, just like the type of π. Prove that all the elements in a conjugacy class have to be of the same type!

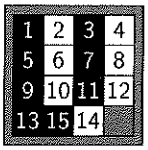
5.1.4 Even and Odd Permutations and Factorisation into 2-cycles

Single-person games where the point is first to mix up something so that it looks completely unordered and then try to put it back into order are popular. **Rubik's cube** and the **Fifteen puzzle** are two well-known examples. Both these games can be studied using the theory of permutations. Here, we'll take a closer look at the Fifteen puzzle.

Example 5.4: ***The Fifteen Puzzle*** The Fifteen puzzle is played within a 4×4-frame that contains fifteen tiles numbered $1, 2, \ldots, 15$ and an empty space. A move in the game consists of sliding a tile into the empty space (which thereby moves to previous position of the tile). The goal is to get the tiles placed in numerical order. The figure shows a starting position and a sequence of moves that results in the final position.



A common programming exercise is to write a program that solves the Fifteen puzzle from arbitrary starting positions. Anyone who succeeds in writing such a program will find while testing that some starting positions can't be solved. For instance you won't find any solution to the position where just 14 and 15 have exchanged places.



But, as one programming teacher asked, how can you know that it isn't just the program that is deficient? Is it possible to prove mathematically that some starting positions in the Fifteen puzzle can't be solved? ■

A position in the Fifteen puzzle can be modeled as a permutation of sixteen elements, namely the fifteen numbered tiles and the empty space. Making a move means exchanging the places of the empty space and a neighbouring tile. A permutation that just switches the places of two elements is usually called a **transposition**, and a transposition is thus the same thing as a 2-cycle. A move in the Fifteen puzzle can thus be modelled by a transposition, and a sequence of moves is then a product of several transpositions.

A first question to pose is whether all permutations can be factorised into transpositions – and the answer is yes, since that follows from the fact that all permutations can be sorted using Bubble-Sort. Let's look at an example.

Example 5.5 We are going to factorise $\begin{bmatrix}4 & 2 & 1 & 3\end{bmatrix}$ into transpositions. We start by checking how Bubble-Sort sorts this permutation and note which transpositions it carries out, that is, which places it is that exchange their elements. (We don't bother to draw steps where nothing is moved.)

$$
\begin{array}{ll}
\begin{bmatrix}4 & 2 & 1 & 3\end{bmatrix} & \\
\begin{bmatrix}4 & 1 & 2 & 3\end{bmatrix} & \begin{pmatrix}2 & 3\end{pmatrix} \\
\begin{bmatrix}1 & 4 & 2 & 3\end{bmatrix} & \begin{pmatrix}1 & 2\end{pmatrix} \\
\begin{bmatrix}1 & 2 & 4 & 3\end{bmatrix} & \begin{pmatrix}2 & 3\end{pmatrix} \\
\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix} & \begin{pmatrix}3 & 4\end{pmatrix}
\end{array}
$$

So we have the following factorisation into transpositions:

$$\begin{bmatrix}4 & 2 & 1 & 3\end{bmatrix} = \begin{pmatrix}2 & 3\end{pmatrix}\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}2 & 3\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix}.$$

But this factorisation is not unique. A shorter one can be found using Straight-Selection:

$$
\begin{array}{ll}
\begin{bmatrix}4 & 2 & 1 & 3\end{bmatrix} & \\
\begin{bmatrix}1 & 2 & 4 & 3\end{bmatrix} & \begin{pmatrix}1 & 3\end{pmatrix} \\
\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix} & \begin{pmatrix}3 & 4\end{pmatrix}
\end{array}
$$

which gives

$$\begin{bmatrix}4 & 2 & 1 & 3\end{bmatrix} = \begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix}.$$ ■

Exercise 5.16 Factorise $\begin{bmatrix}4 & 1 & 5 & 3 & 2\end{bmatrix}$ into transpositions using Bubble-Sort.

Exercise 5.17 Try to find the shortest possible factorisation of $\begin{bmatrix}4 & 1 & 5 & 3 & 2\end{bmatrix}$ into transpositions.

So there are several different ways to factorise one and the same permutation into transpositions. In the above example, two different factorisations had four and two transpositions, repectively. As a matter of fact, all factorisations of this permutaiton consist of an even number of transpositions. This is a consequence of a general theorem:

Theorem 5.1 The factorisations of a permutation into transpositions will either all be of even length or all of odd lenght. ■

The proof is given as an exercise below. It's based on the concept **inversion**, which means that a larger number is placed before a smaller one in the permutatiton. In 41532 there are for instance six inversions, 41, 43, 42, 53, 52, and 32. An **even permutation** is a permutation with an even number of inversions. The concept **odd permutation** is defined in a corresponding way.

Exercise 5.18: *Important!* Prove that the number of inversion in a permutation will always change by an odd number when a transposition is performed. Prove using this fact theorem 5.1.

Project excercise 5.19 Prove that the number of cycles in a permutation is always changed by exactly 1 when you multiply it by a transposition! More precisely, if $c(\pi)$ denotes the number of cycles in the permutation π and if $\tau = \begin{pmatrix}x & y\end{pmatrix}$ is a transposition you are to prove that

- $c(\pi\tau) = c(\pi) + 1$ if x and y belong to the same cycle in π;
- $c(\pi\tau) = c(\pi) - 1$ if x and y belong to different cycles in π.

After that, show that the shortest factorisation of a permutation $\pi \in S_n$ has the length $n - c(\pi)$. *

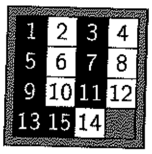
Example 5.6 Is $\pi = \begin{bmatrix}4 & 6 & 5 & 2 & 7 & 1 & 3\end{bmatrix}$ an even or an odd permutation?

We count the inversions: 42, 41, 43, 65, 62, 61, 63, 52, 51, 53, 21, 71, 73, that is, thirteen. Thus π is an odd permutation. ■

Example 5.7 Is $\pi = \begin{pmatrix}1 & 4 & 2 & 6\end{pmatrix}\begin{pmatrix}3 & 5 & 7\end{pmatrix}$ an even or an odd permutation?

One method is to go from cycle form to one-line form and then count the inversions as in the example above. An alternative method is to count the cycles and use the result of project exercise 5.19. The number of cycles in π is $c(\pi) = 2$. The shortest factorisation into transposition will then have the length $n - c(\pi) = 7 - 2 = 5$ which is odd, so π is an odd permutation. ■

Example 5.8: *The Fifteen puzzle once more* The theory about even and odd permutations can be used to prove that some positions in the Fifteen puzzle can't be solved, for instance the position where just 14 and 15 have exchanged places.



Assume that in the starting position, the empty space is placed bottom right, just as in the desired final position. Every move means that the space is moved one step heightwise or sidewise. If the empty space is to end up in the place where it was at start, it must have been moved the same number of steps upwards as downwards (in total an even number of steps) and the same number of steps to the left as to the right (also an even number of steps), so there must be an even number of moves from the starting position to the final position.

Since each move is a transposition, it has to be possible to get to the starting position using a sequence consisting of an even number of transpositions, so the starting position has to be an even permutation of the final position. In other words, only even permutations can be solved in the Fifteen puzzle!

Since the position where just 14 and 15 have exchanged places can be written as a single transposition, it is an odd permutation of the final position and can thus not be solved. ■

Exercise 5.20 Determine whether the following permutations are odd or even:

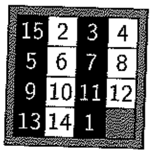
(a) $[3 \quad 5 \quad 2 \quad 1 \quad 4]$

(b) $[3 \quad 6 \quad 5 \quad 2 \quad 1 \quad 4]$

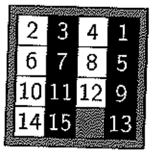
(c) $[3 \quad 6 \quad 5 \quad 2 \quad 7 \quad 1 \quad 4]$

Exercise 5.21 Can the Fifteen puzzle be solved from the following positions?

(a)



(b)



Project excercise 5.22 In example 5.8 we showed that only even permutations of the tiles can be solved (given that the empty space is placed bottom right). But can all even permutations be solved? The answer is yes. Try to prove this in the following way:

(a) Prove that you using the Fifteen puzzle can permute three tiles that are bordering to each other in an L-formation in a 3-cycle.

(b) Prove using induction (over the total distance between the three tiles) that you using the Fifteen puzzle can permute any three tiles in a 3-cycle. The previous exercise was the base case.

(c) Prove that every even permutation can be factorised into 3-cycles and thereby be solved in the Fifteen puzzle. *