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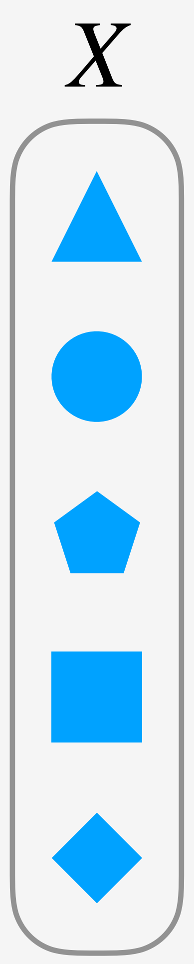

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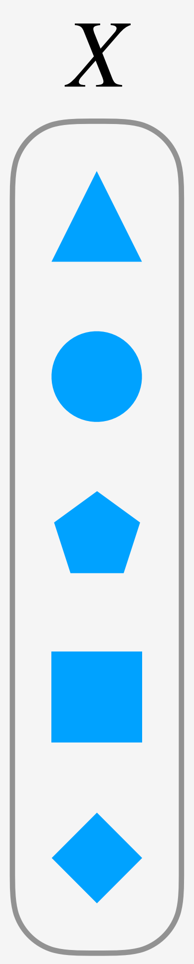
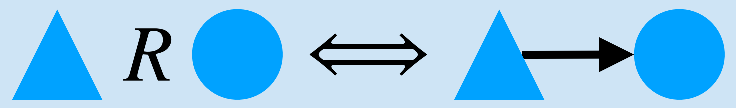


Note: 

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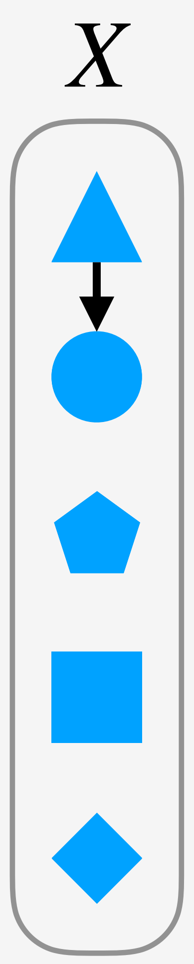
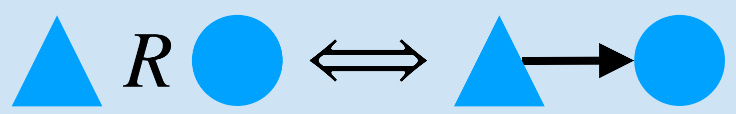


Note: 

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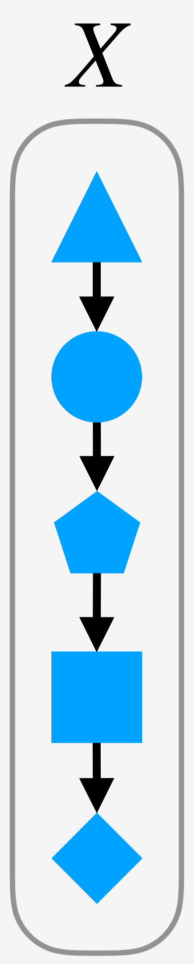
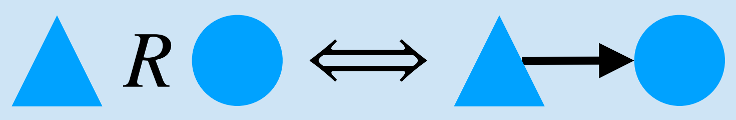


Note: 

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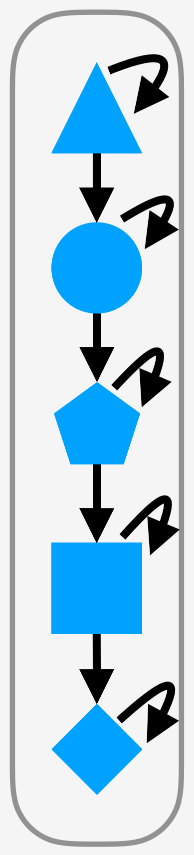
X



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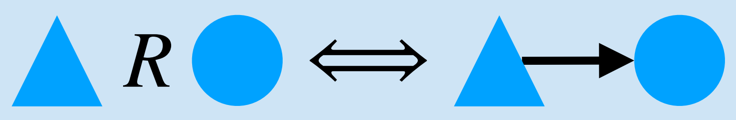

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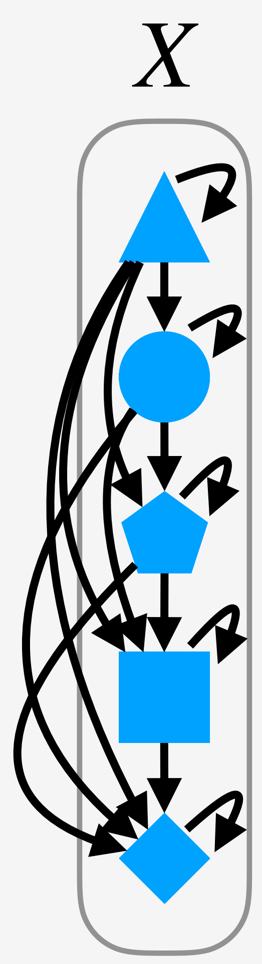

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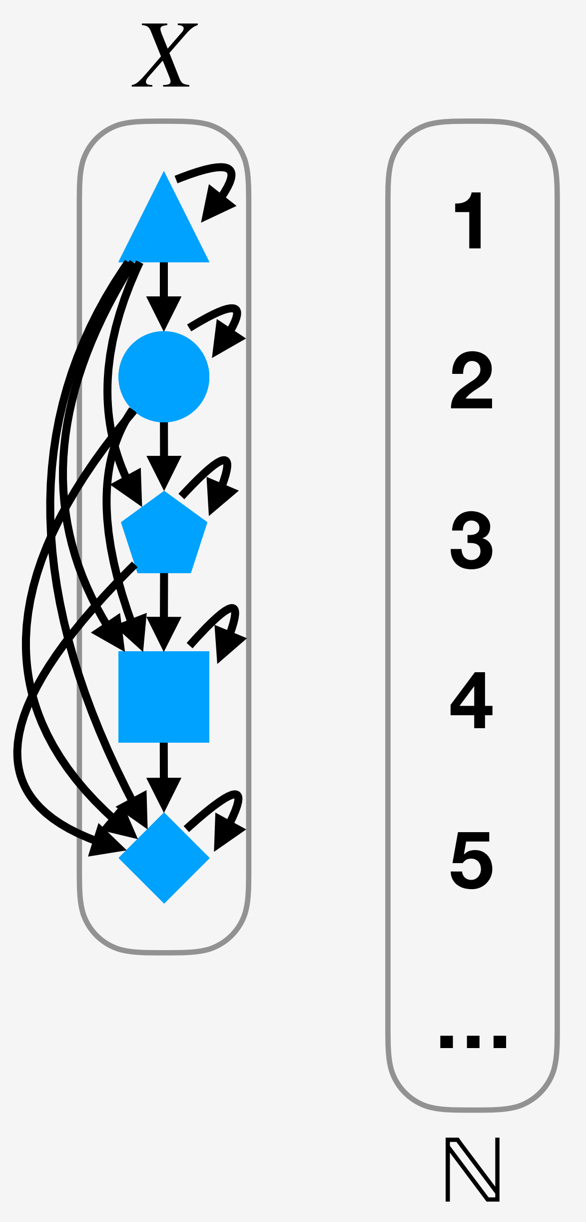
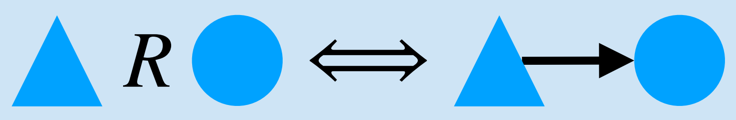


Note: 

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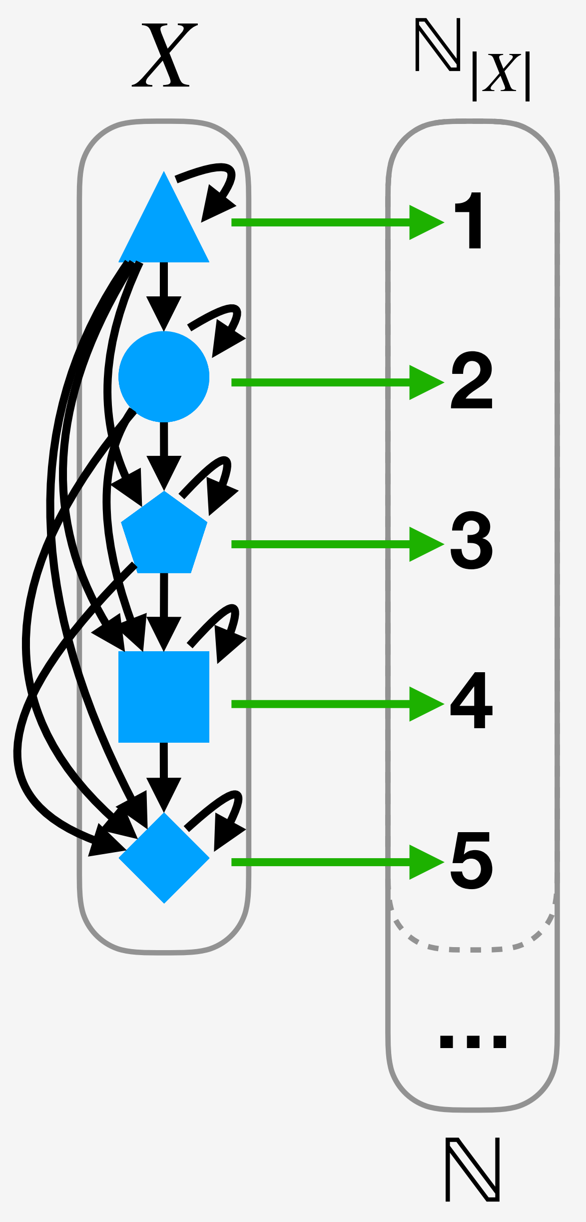

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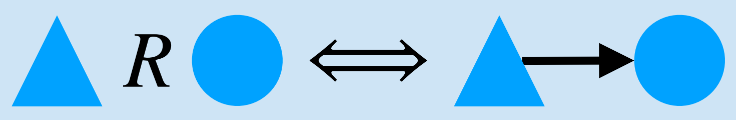

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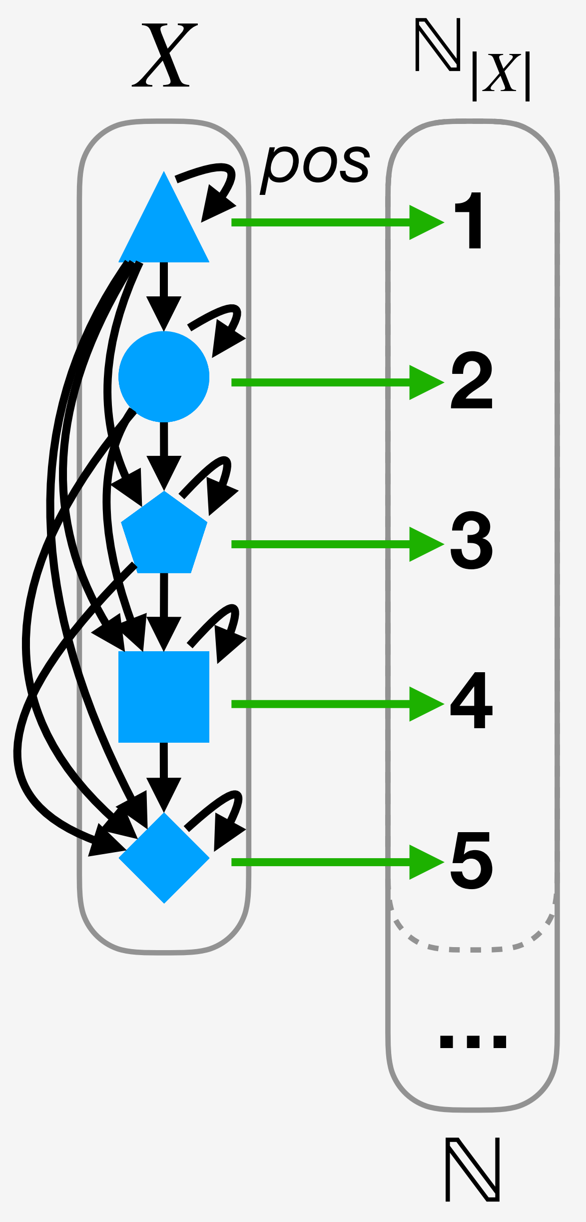

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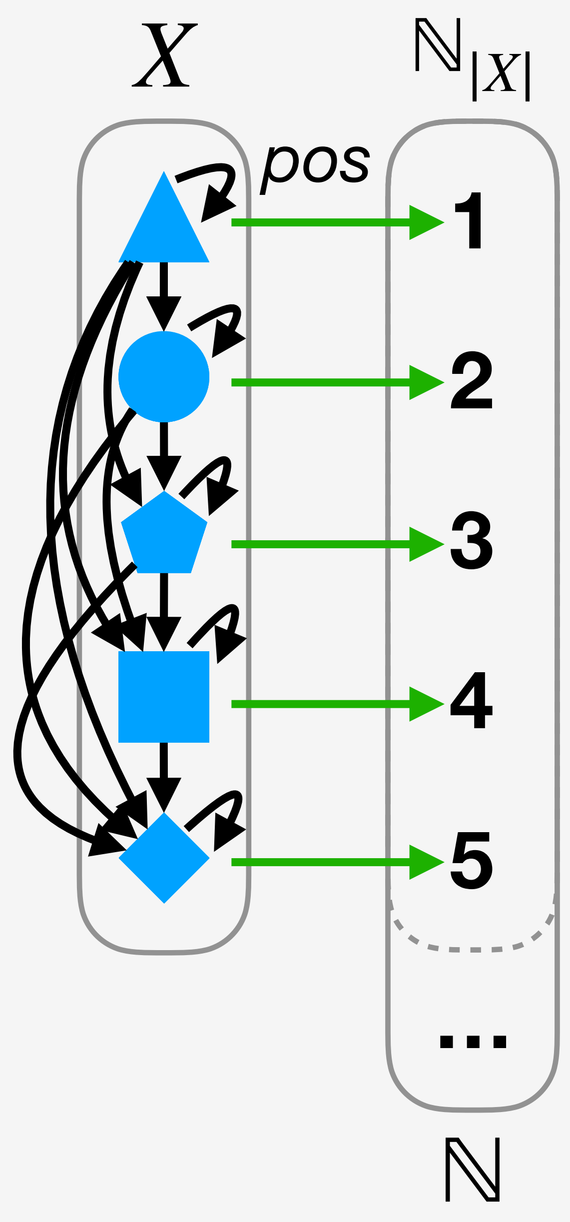

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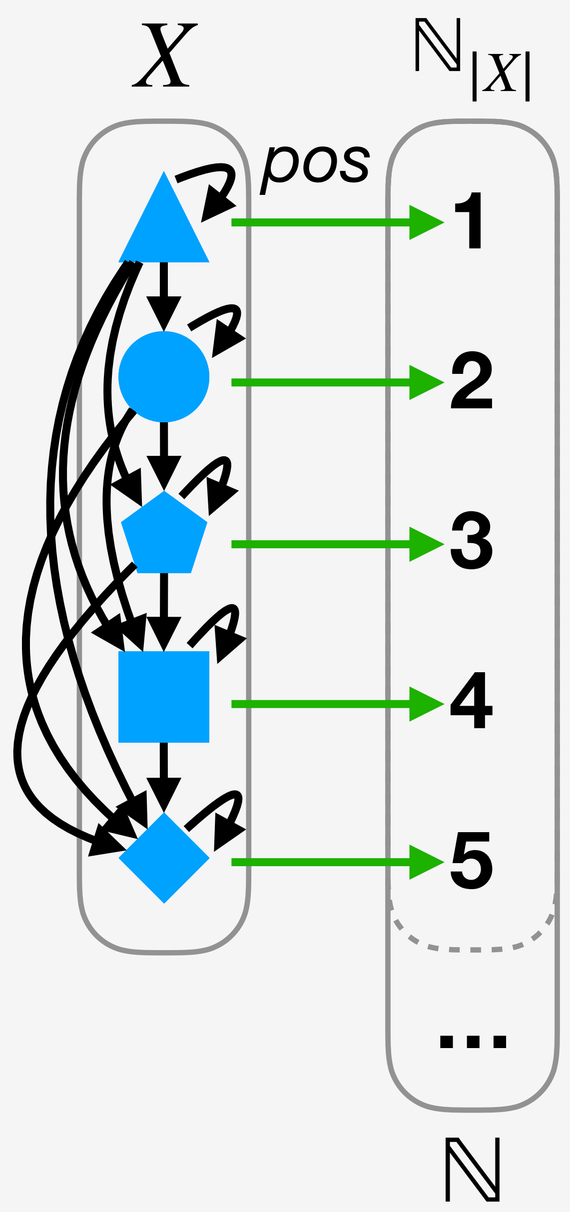
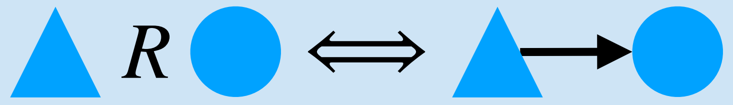


NOTE: 

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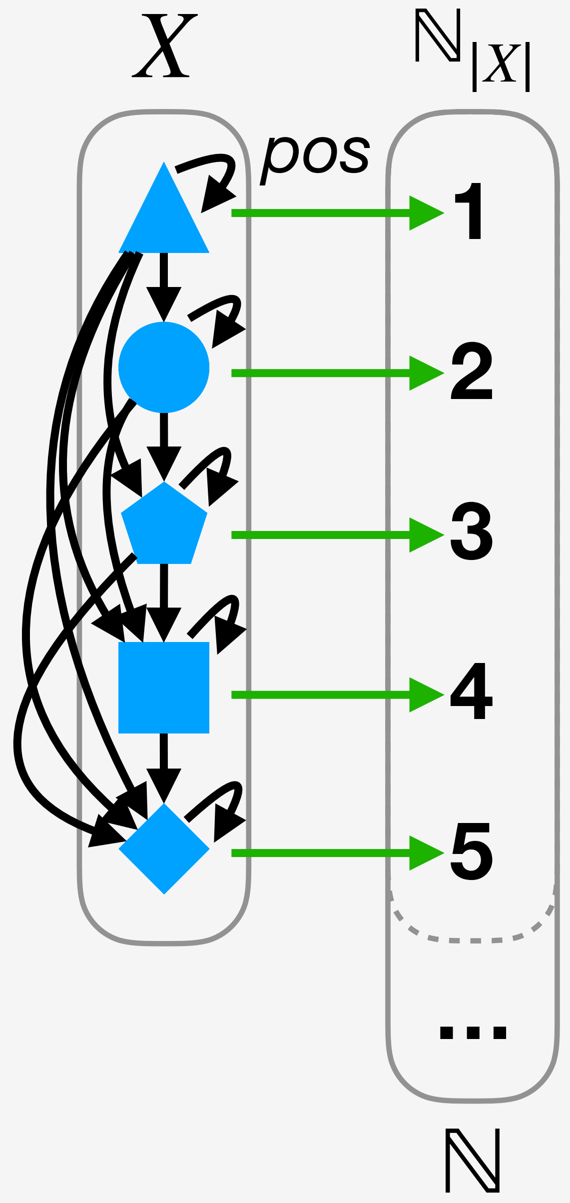

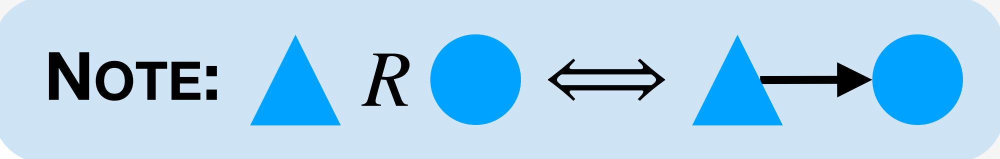

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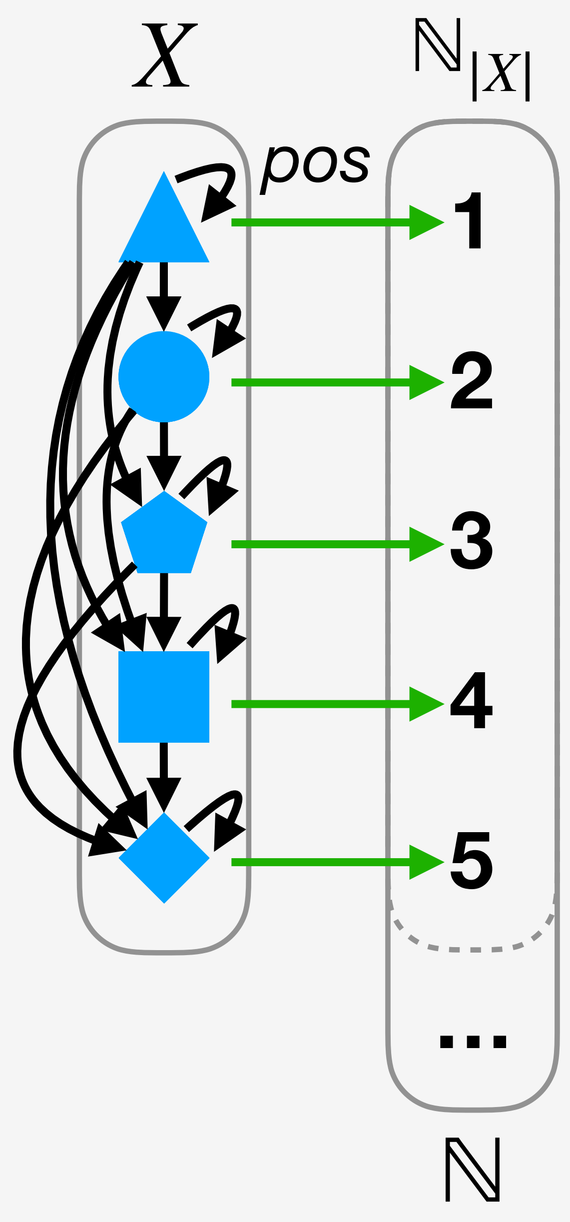

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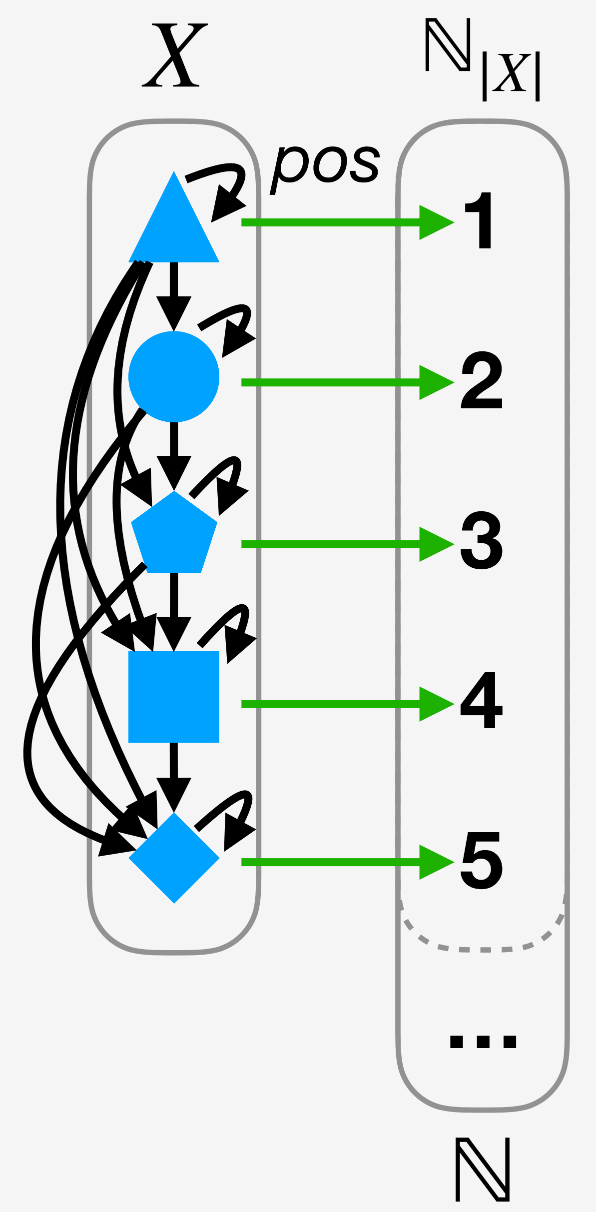

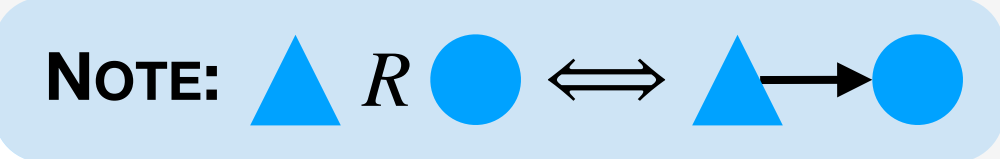

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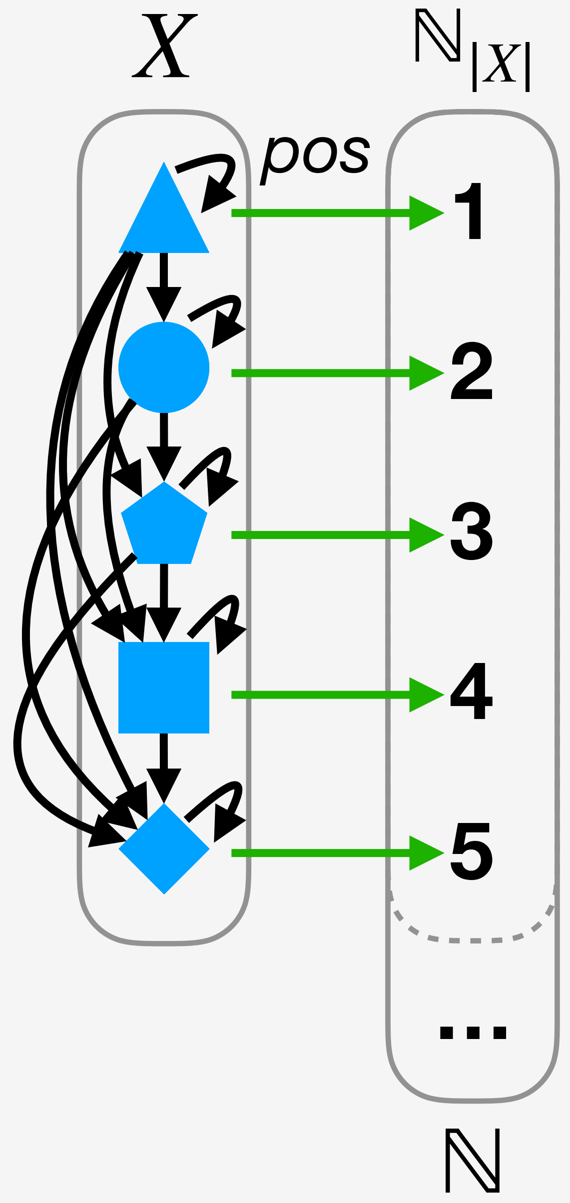

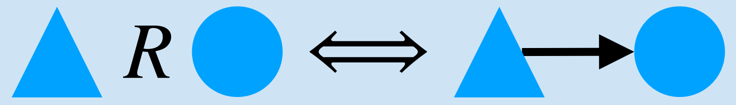

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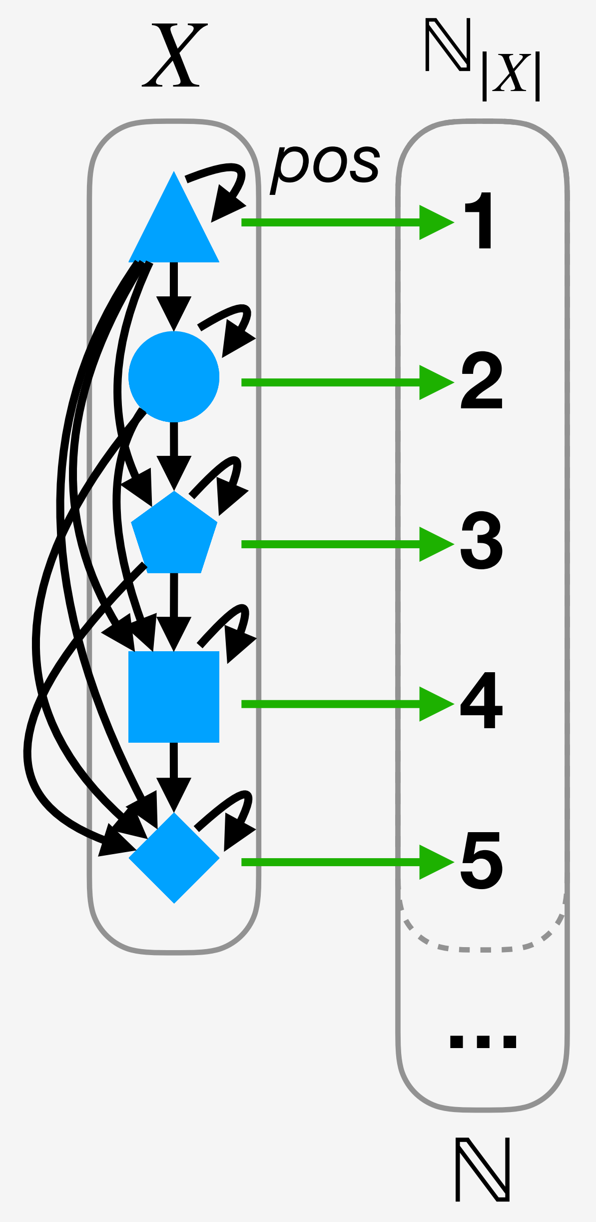

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For every non-empty, ***finite*** set X with a ***total order*** $R \subseteq X^2$, there ***exists exactly one*** $a \in X$ such that $\forall_{x \in X}(a, x) \in R$.

$\Longrightarrow$

Definitions

Reflexivity: $\forall x \in X : xRx$

Totality: $\forall x, y \in X : xRy \vee yRx$

Transitivity: $\forall x, y, z \in X : xRy \wedge yRz \implies xRz$

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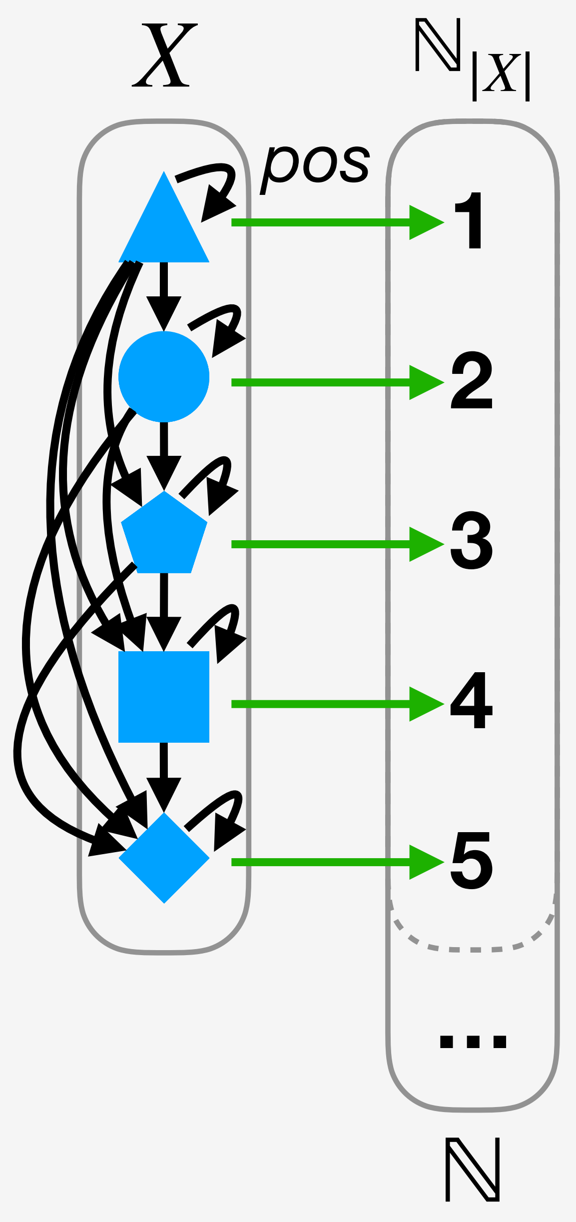
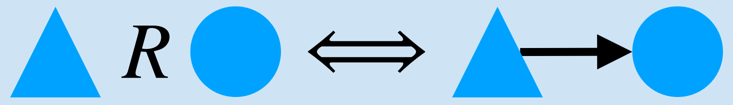


NOTE: 

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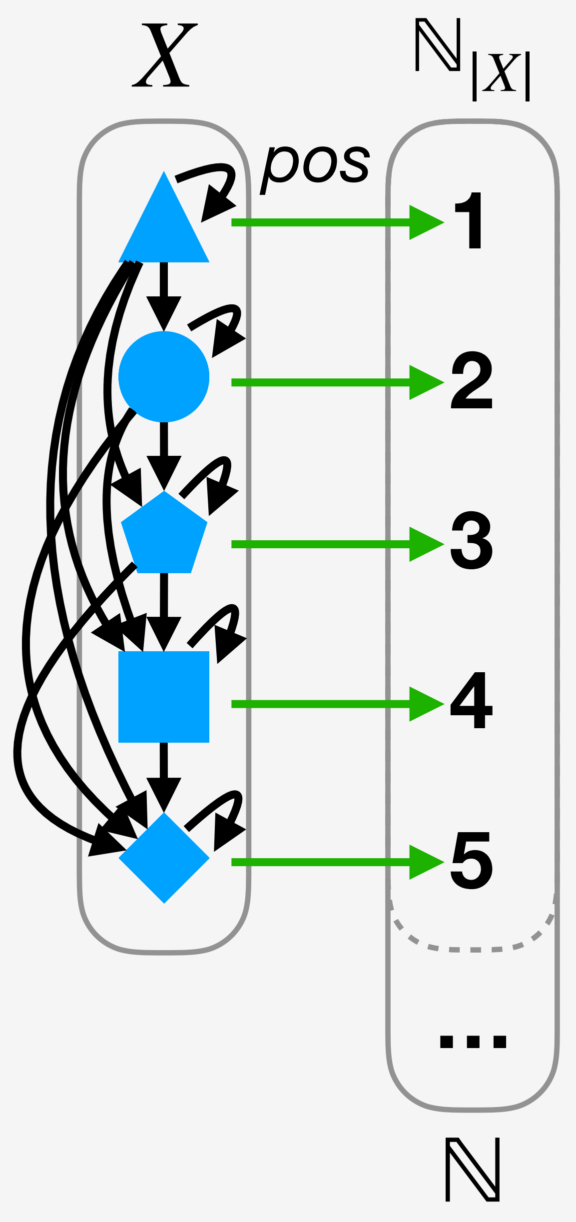
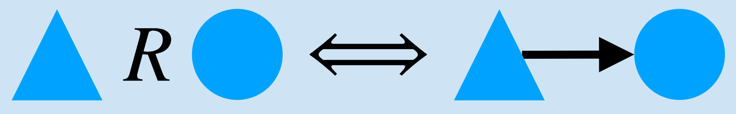


NOTE: 

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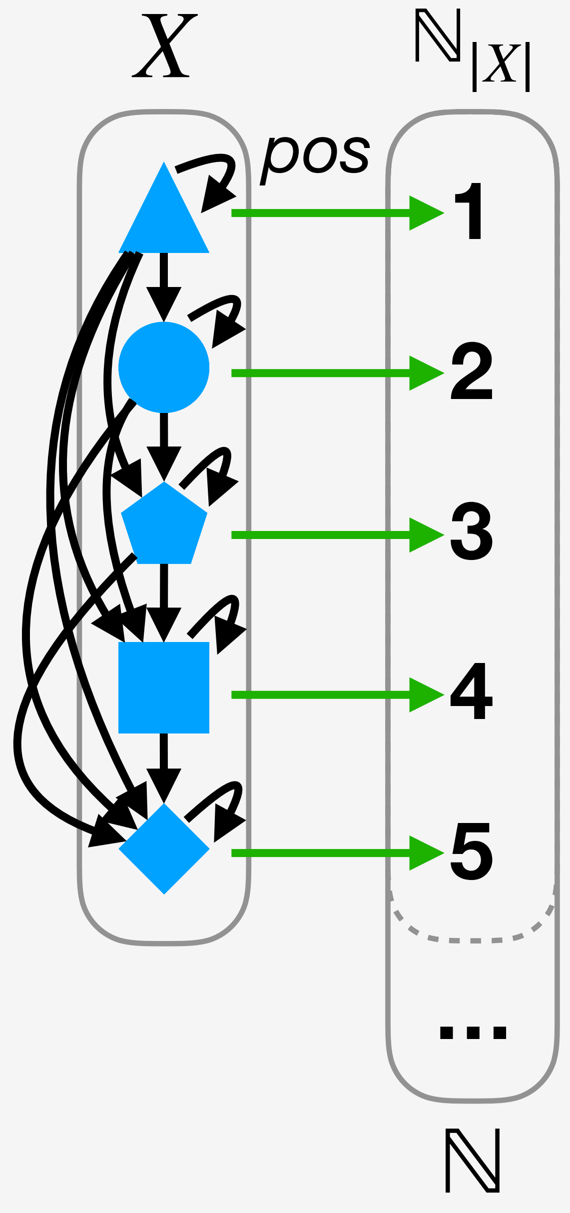
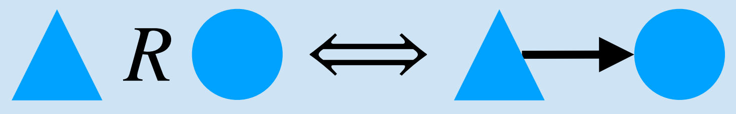
Bijective
(Injective and surjective)
Every $\mathbb{N}_{|X|}$ has exactly one X



Note: 

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Injective
(General)
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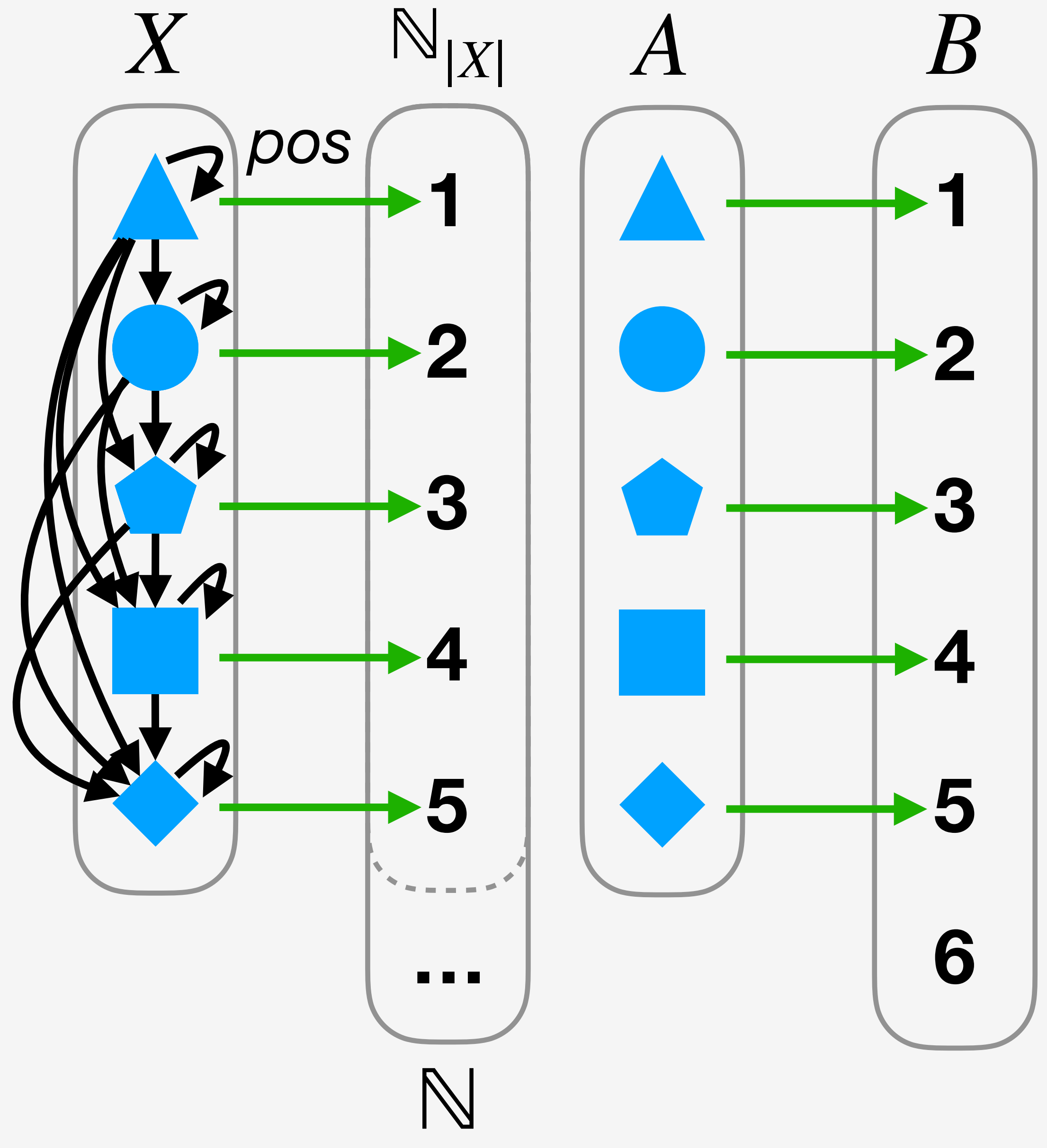
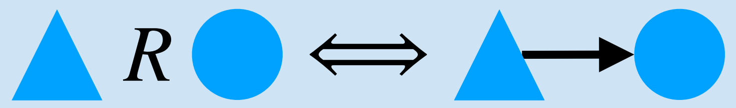


Note: 

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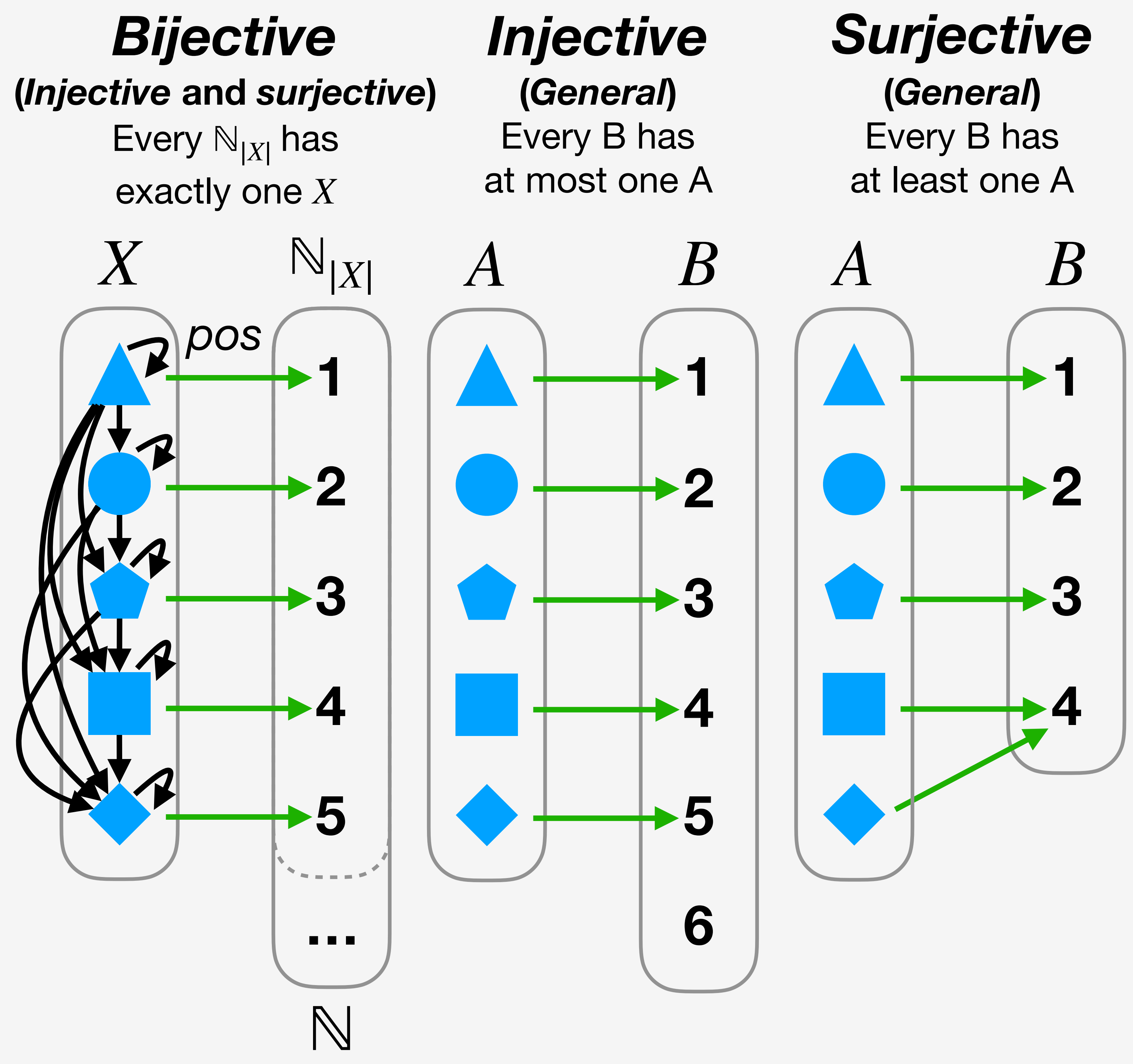
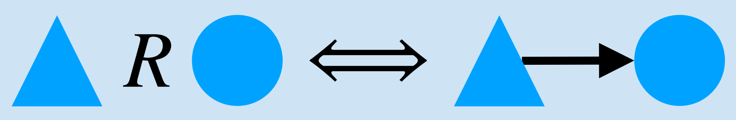


NOTE: 

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Injective
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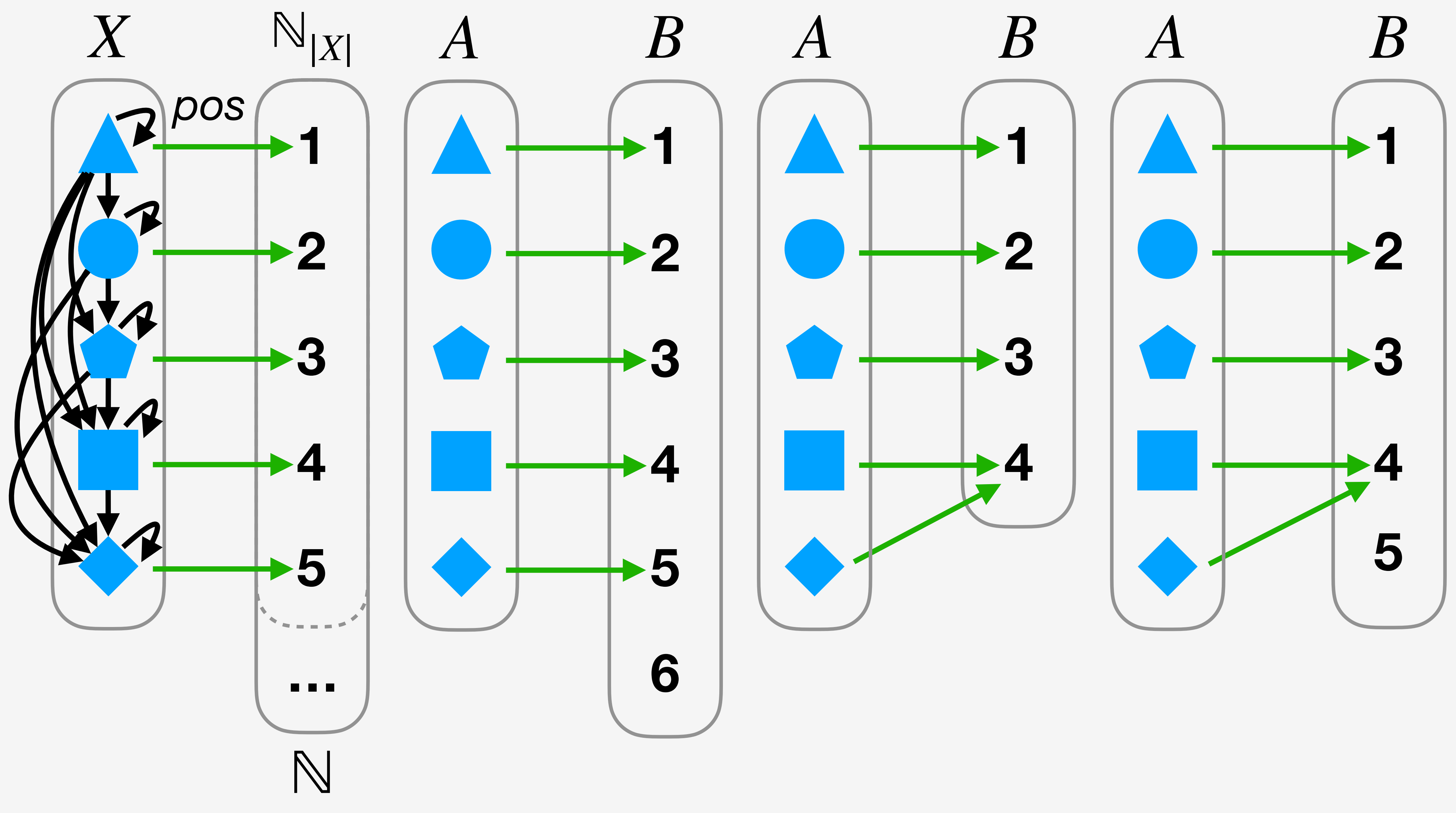
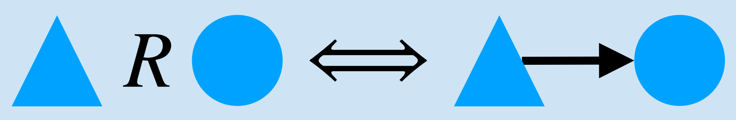


NOTE: 

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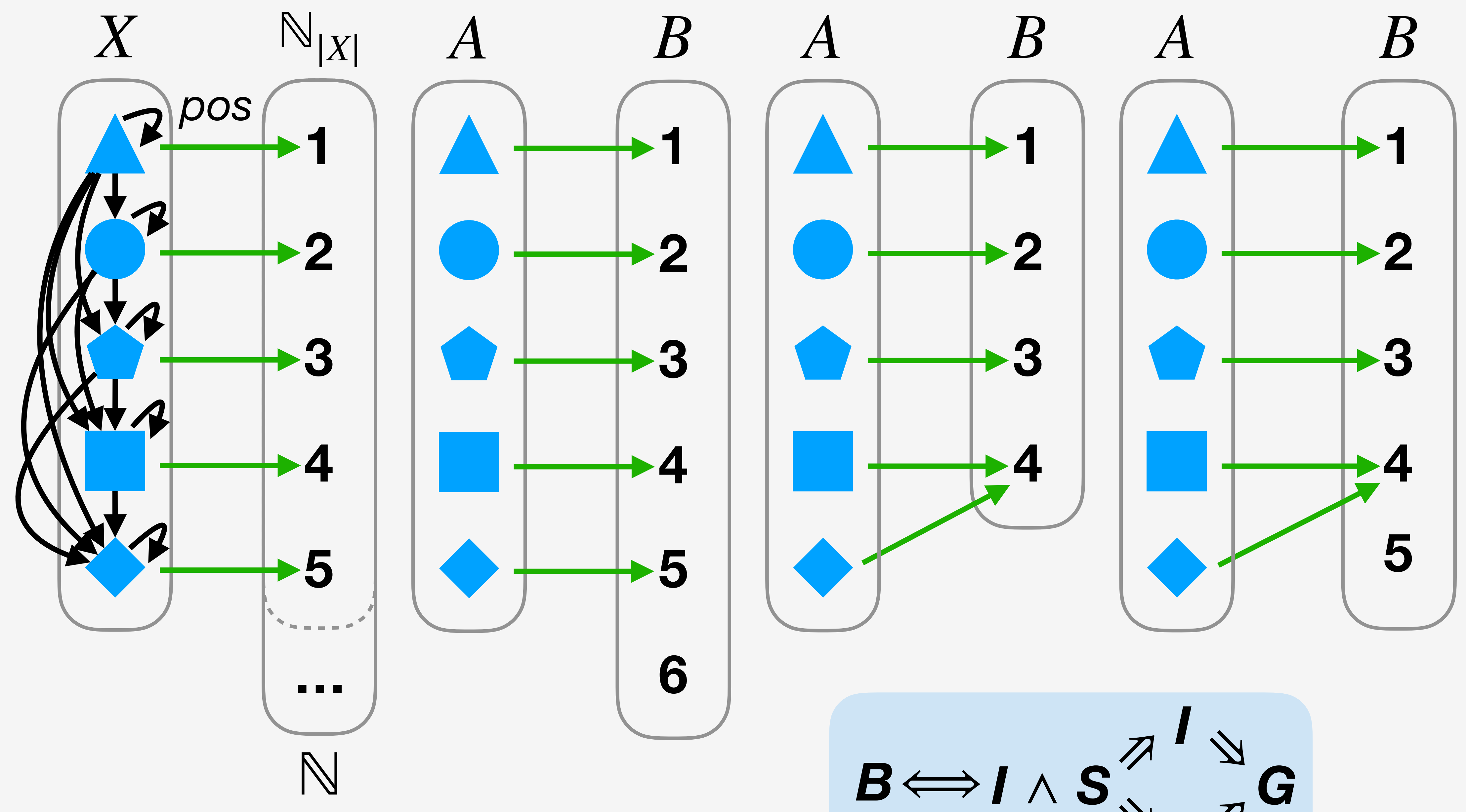

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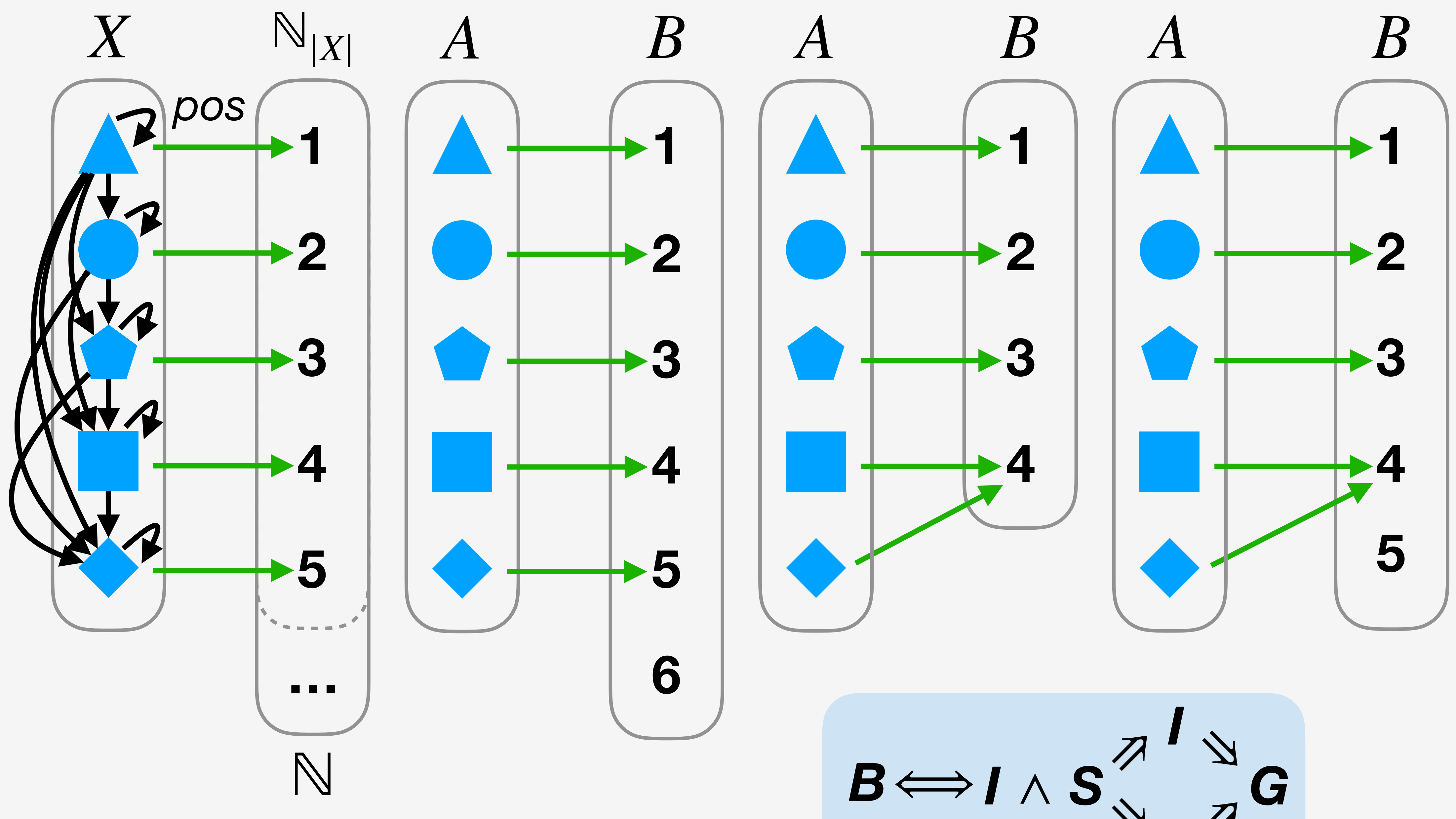

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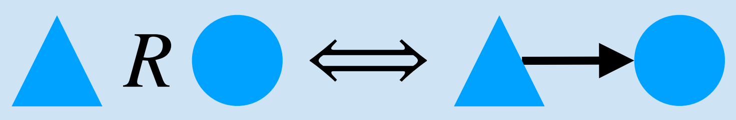

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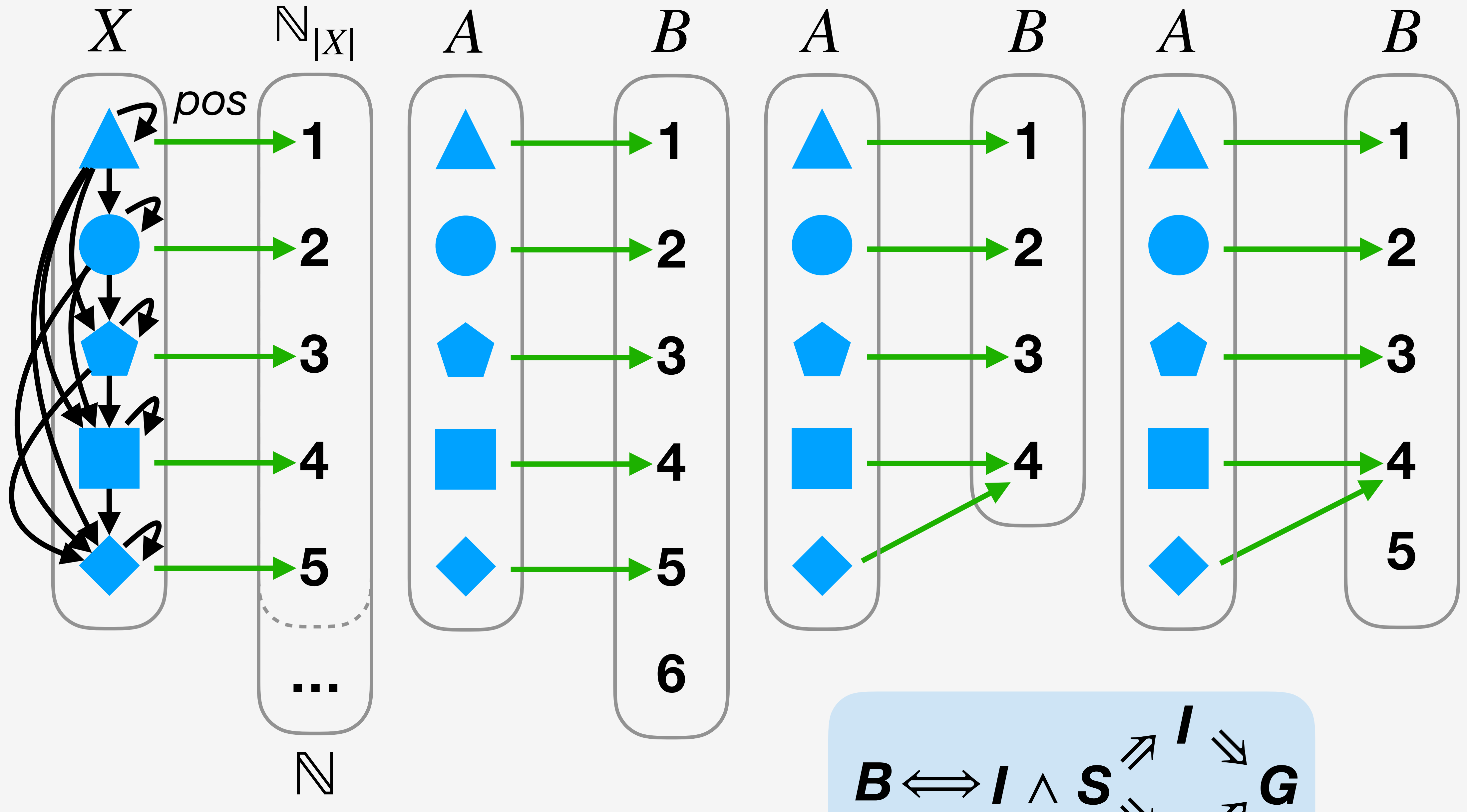

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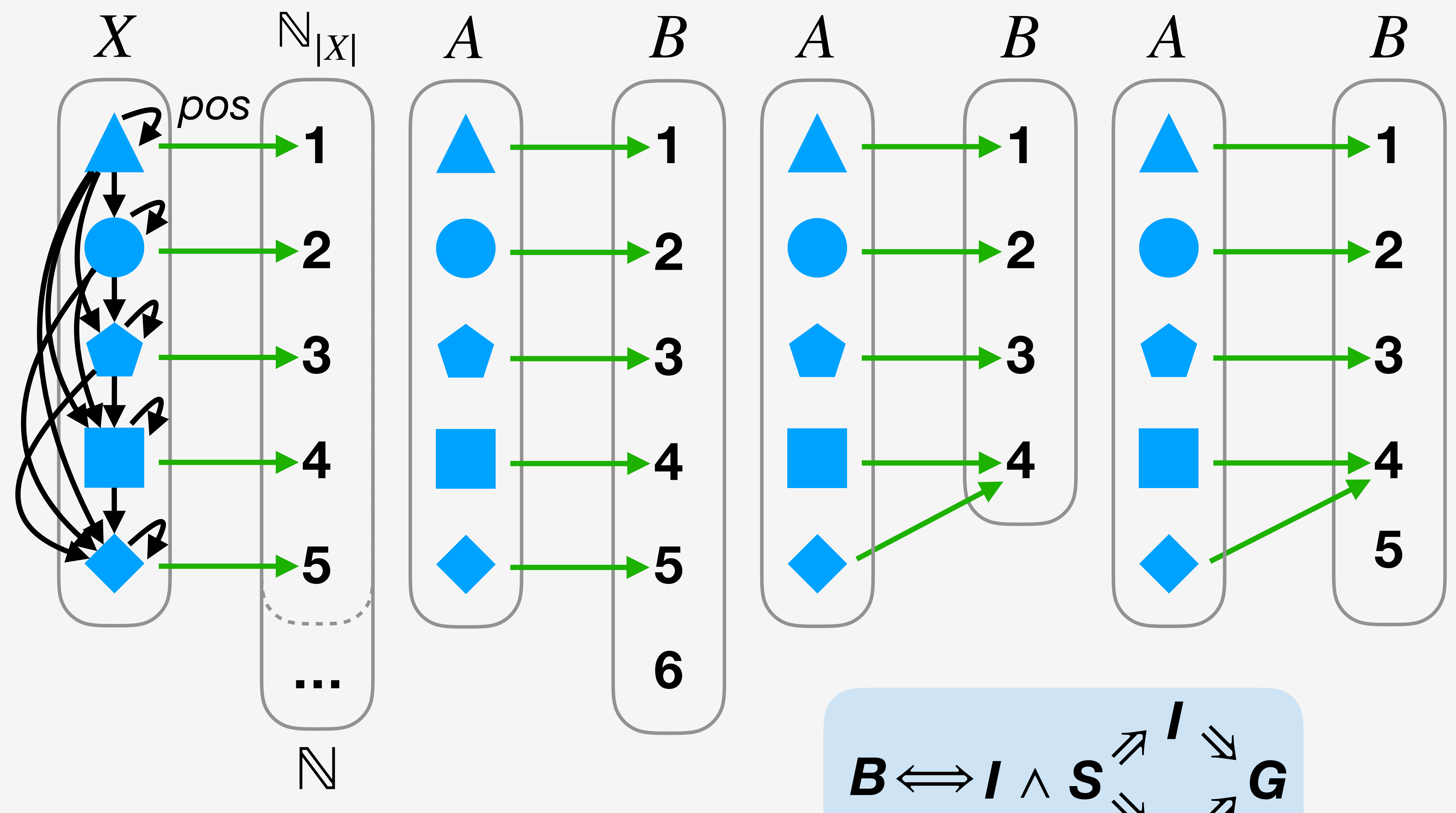

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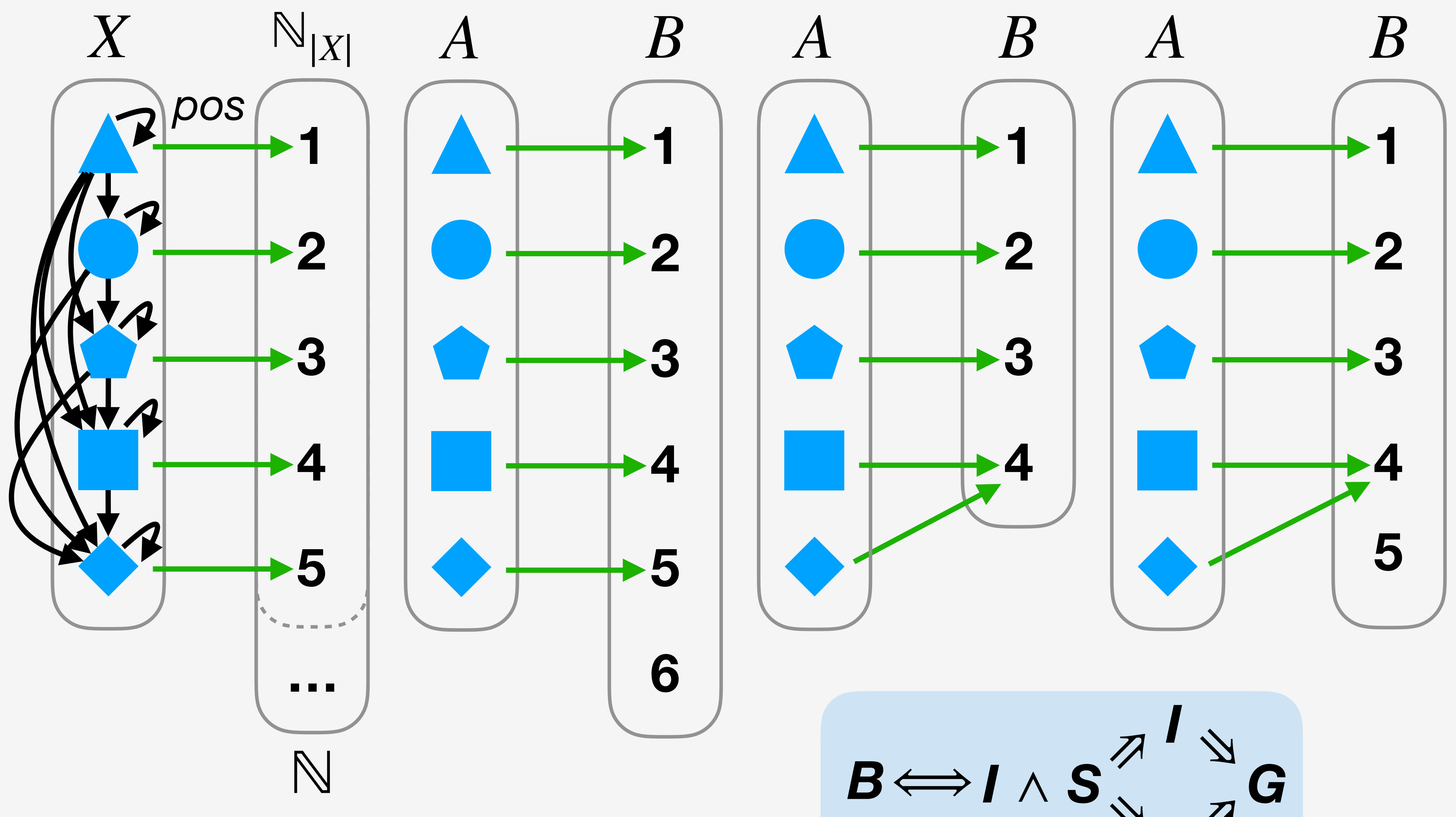

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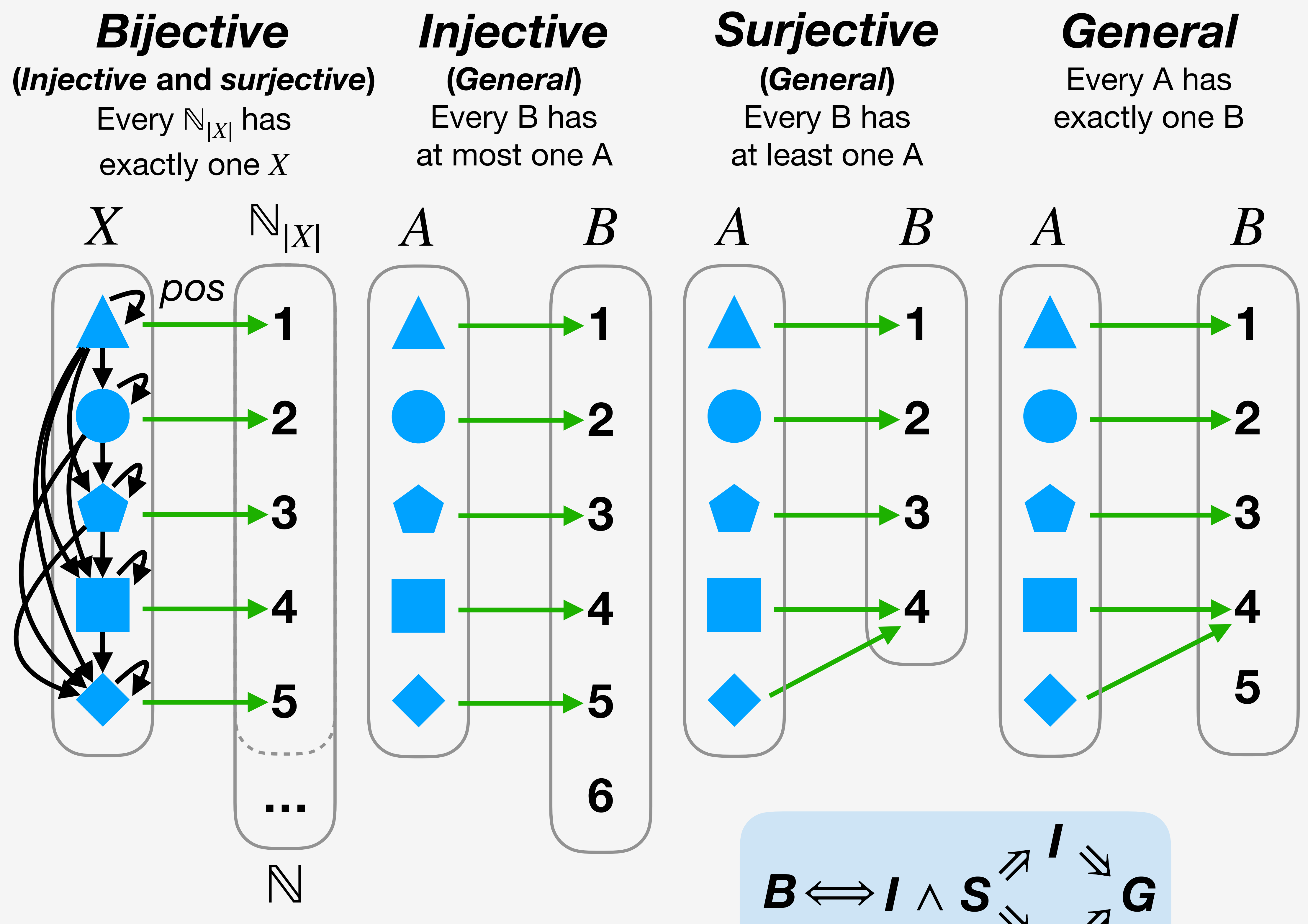

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For every non-empty, ***finite*** set X with a ***total order*** $R \subseteq X^2$, there ***exists exactly one*** $a \in X$ such that $\forall_{x \in X} (a, x) \in R$.

PROOF claim is true:

Let $\mathbb{N}_{|X|} = \{1, \ldots, |X|\} \subset \mathbb{N}$

Let *pos* $: X \to \mathbb{N}_{|X|},\ x \mapsto |\{iRx \mid i \in X\}|$

Since R is ***reflexive*** and X is ***finite***, $1 \leq pos(x) \leq |X|$

Let $a \in X : pos(a) = 1$

By ***reflexivity***, $\{iRa \mid i \in X\} = \{aRa\}$

By ***totality***, $\forall_{x \in X} aRx$

If *pos* is ***bijective***, then we know a ***exists (surjectivity)*** and is ***unique (injectivity)***

PROOF *pos* is ***injective***:

Assume $x, y \in X : pos(x) = pos(y) \implies x = y$

By ***totality***, $xRy \vee yRx$

CASE 1: $xRy \wedge yRx$, then by ***antisymmetry***, $x = y$

CASE 2: $xRy \wedge y\overline{R}x$, then by ***reflexivity*** $x \neq y$

By ***transitivity***, $\forall_{z \in X} zRx \implies zRy$ and $pos(x) < pos(y)$

$\Longrightarrow$

Definitions

NOTE: $xRy \iff (x, y) \in R$

$x\overline{R}y \iff (x, y) \notin R$

Reflexivity: $\forall x \in X : xRx$

Totality: $\forall x, y \in X : xRy \vee yRx$

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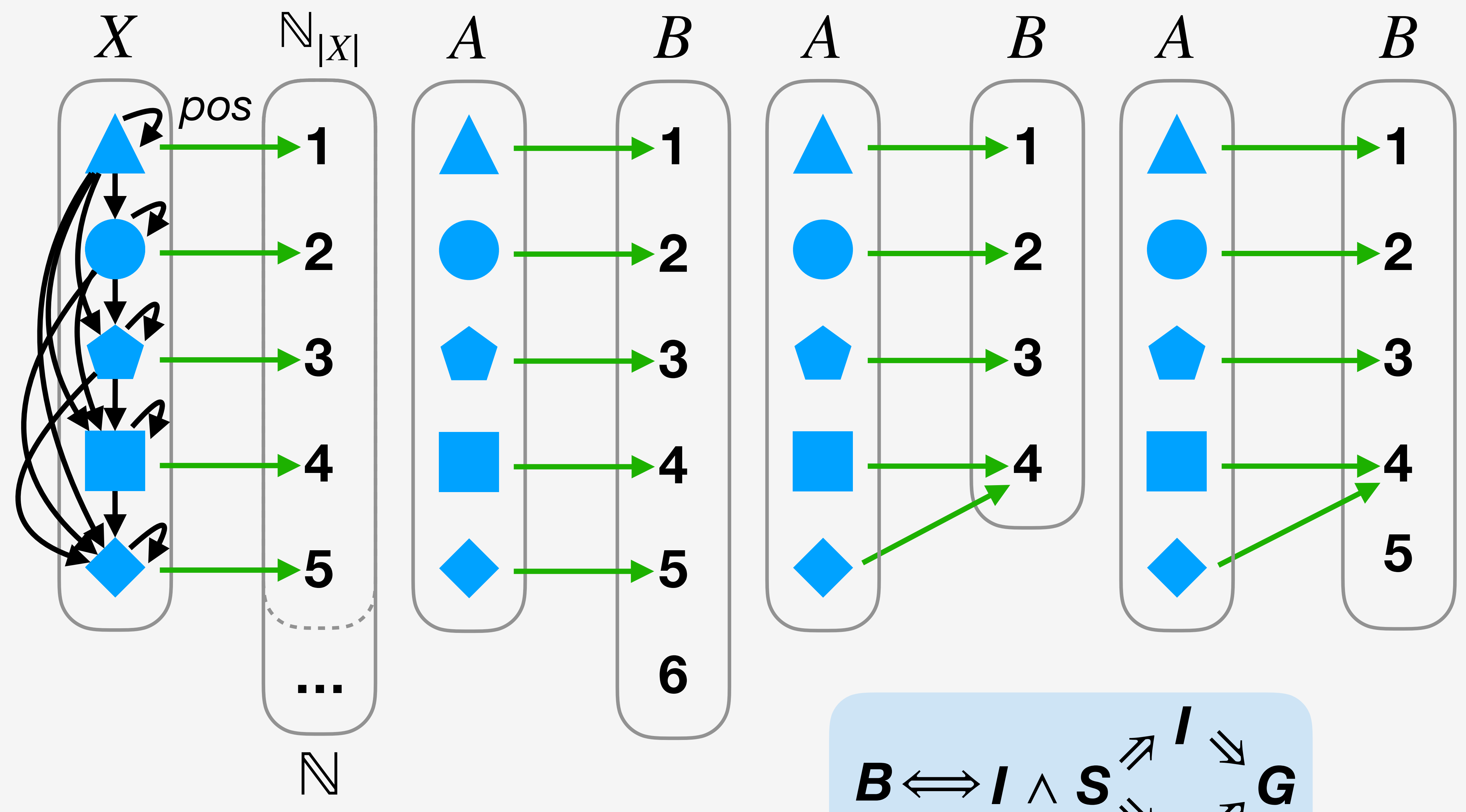
Antisymmetry: $\forall x, y \in X : xRy \wedge yRx \implies x = y$

Bijective	***Injective***	***Surjective***	***General***
(Injective and surjective) Every $\mathbb{N}_{|X|}$ has exactly one X	***(General)*** Every B has at most one A	***(General)*** Every B has at least one A	Every A has exactly one B



B $\iff$ ***I*** $\wedge$ ***S***; ***S*** $\nearrow$ ***I*** $\searrow$ ***G***; ***S*** $\searrow$ ***S*** $\nearrow$ ***G***

NOTE: ▲ R ● $\iff$ ▲→●

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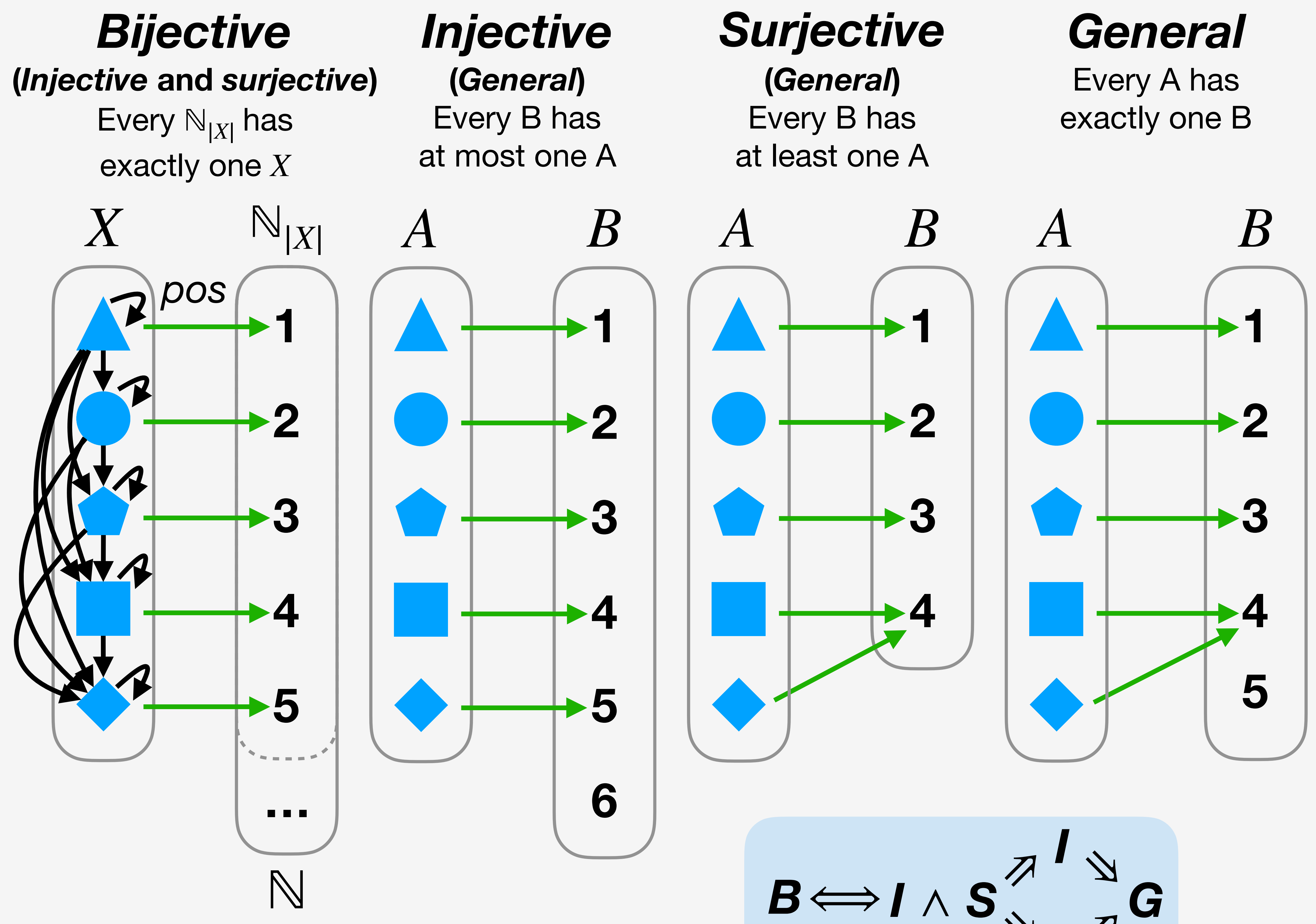
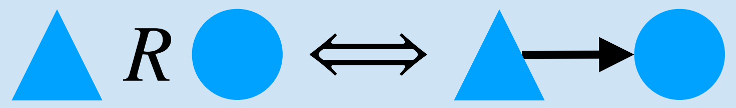


$B \iff I \wedge S$, $I \Rightarrow G$, $S \Rightarrow G$

NOTE: 

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Case 3: $x\bar{R}y \wedge yRx \implies$ Isomorphic to **Case 2**

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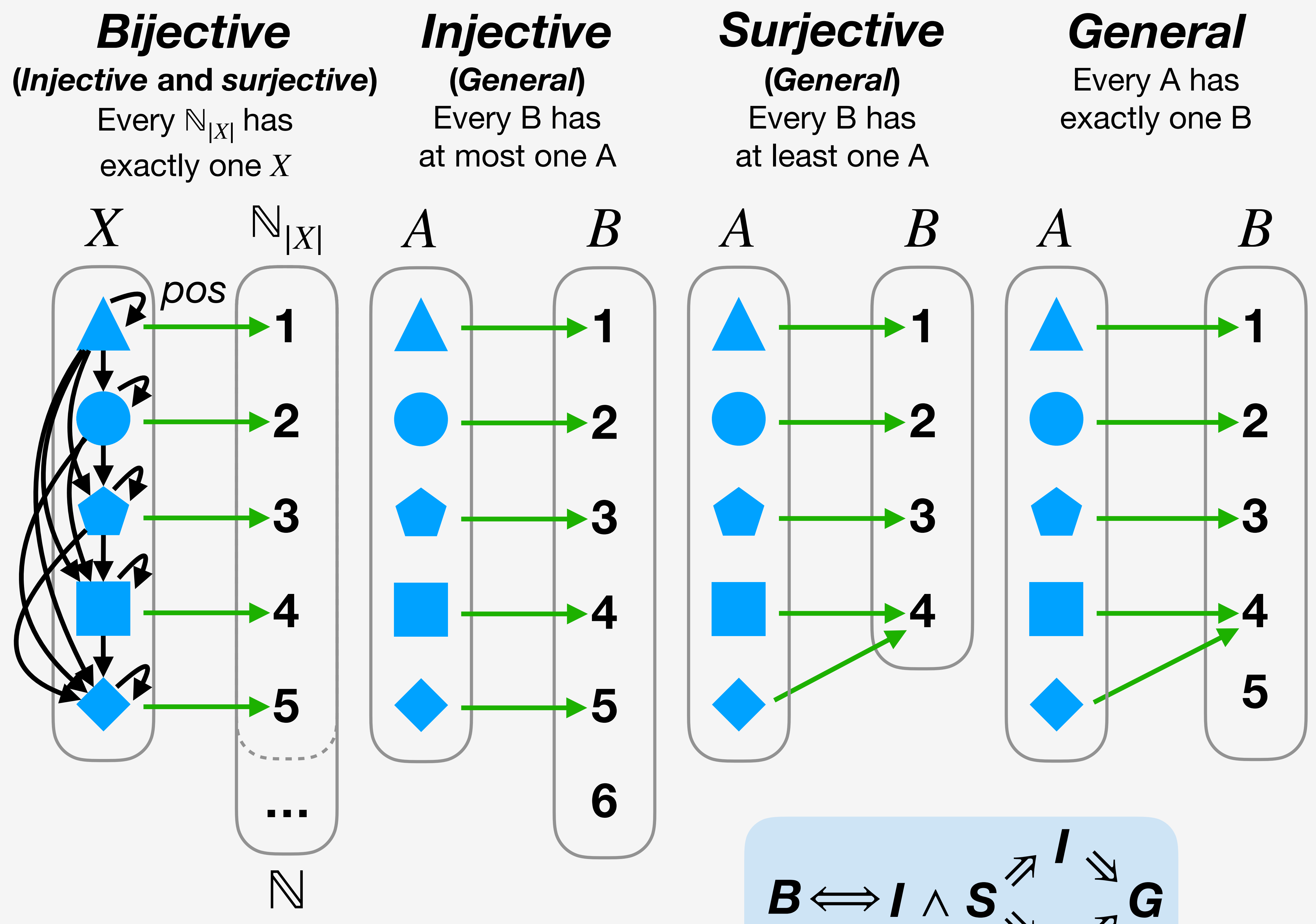

$B \iff I \wedge S \nearrow I \searrow G$, $S \searrow S \nearrow G$

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Proof *pos* is ***surjective (existence)***:

Definitions

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Totality: $\forall x, y \in X : xRy \vee yRx$

Transitivity: $\forall x, y, z \in X : xRy \wedge yRz \implies xRz$

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Note: $xRy \iff (x, y) \in R$
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Bijective
(Injective and surjective)
Every $\mathbb{N}_{|X|}$ has exactly one X

Injective
(General)
Every B has at most one A

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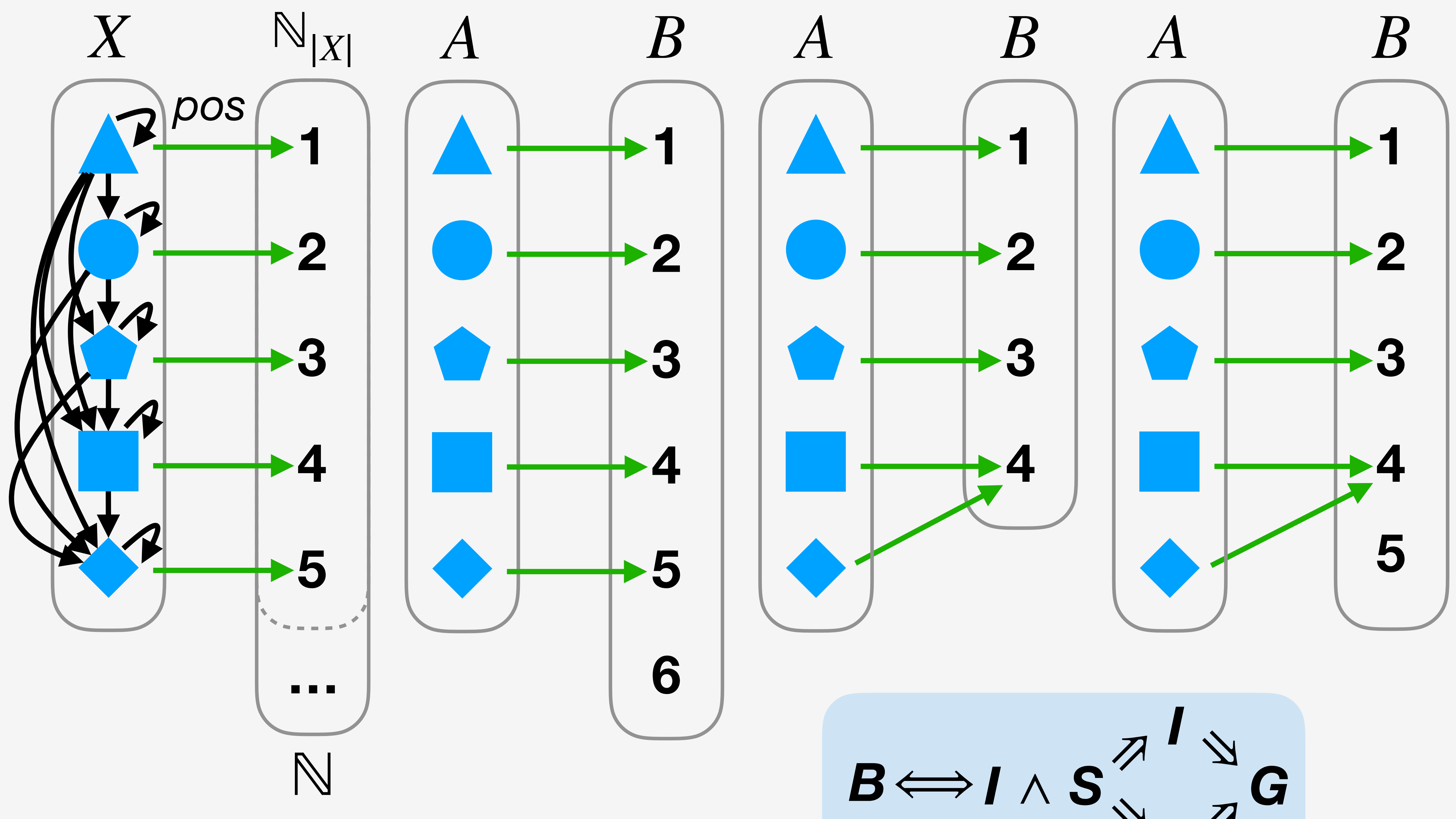

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PROOF *pos* is ***injective***:

Assume $x, y \in X : pos(x) = pos(y) \implies x = y$

By ***totality***, $xRy \lor yRx$

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PROOF *pos* is ***surjective (existence)***:

By ***injectivity*** and $\mathbb{N}_{|X|} = |X|$, *pos* must also be ***surjective***

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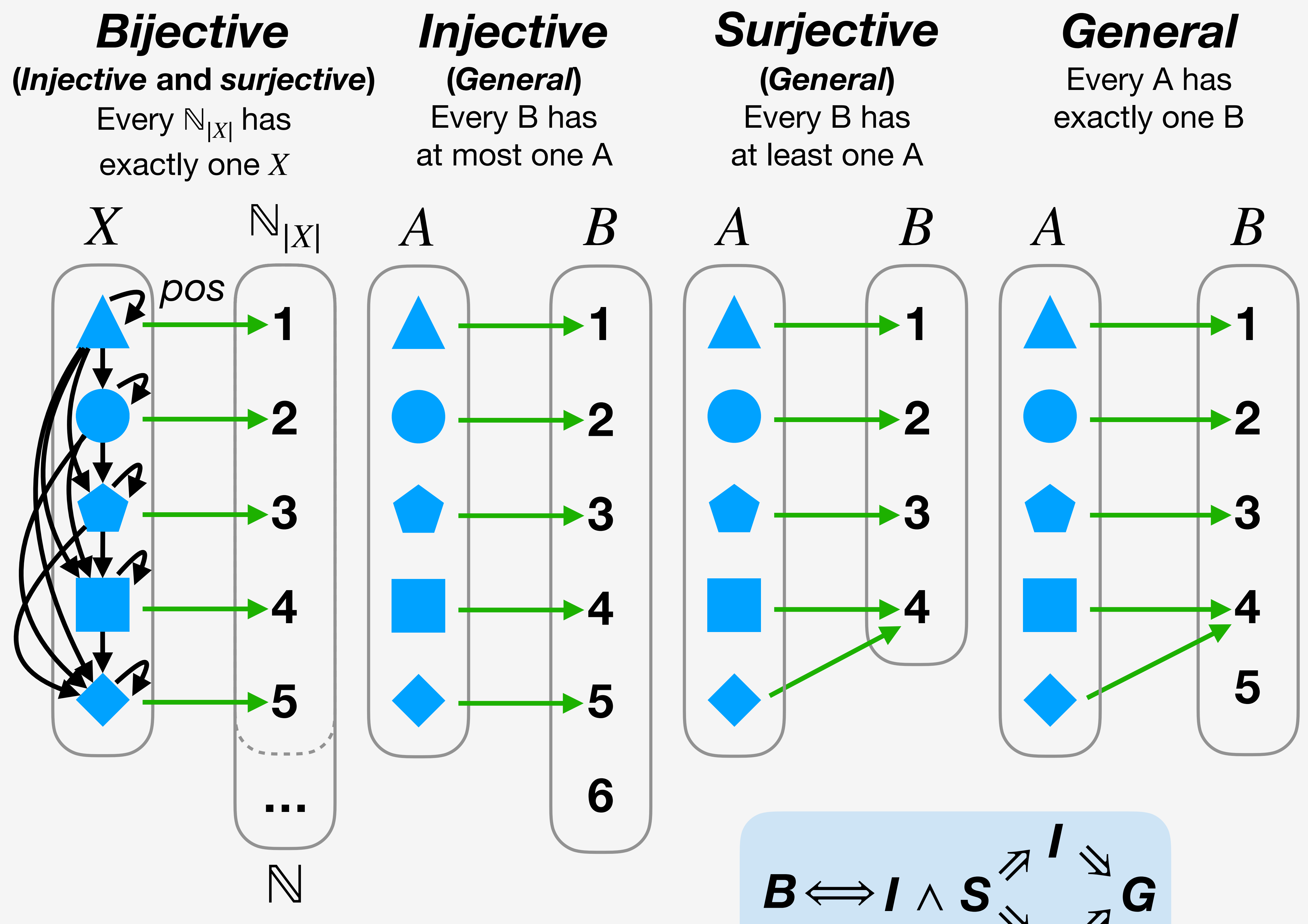

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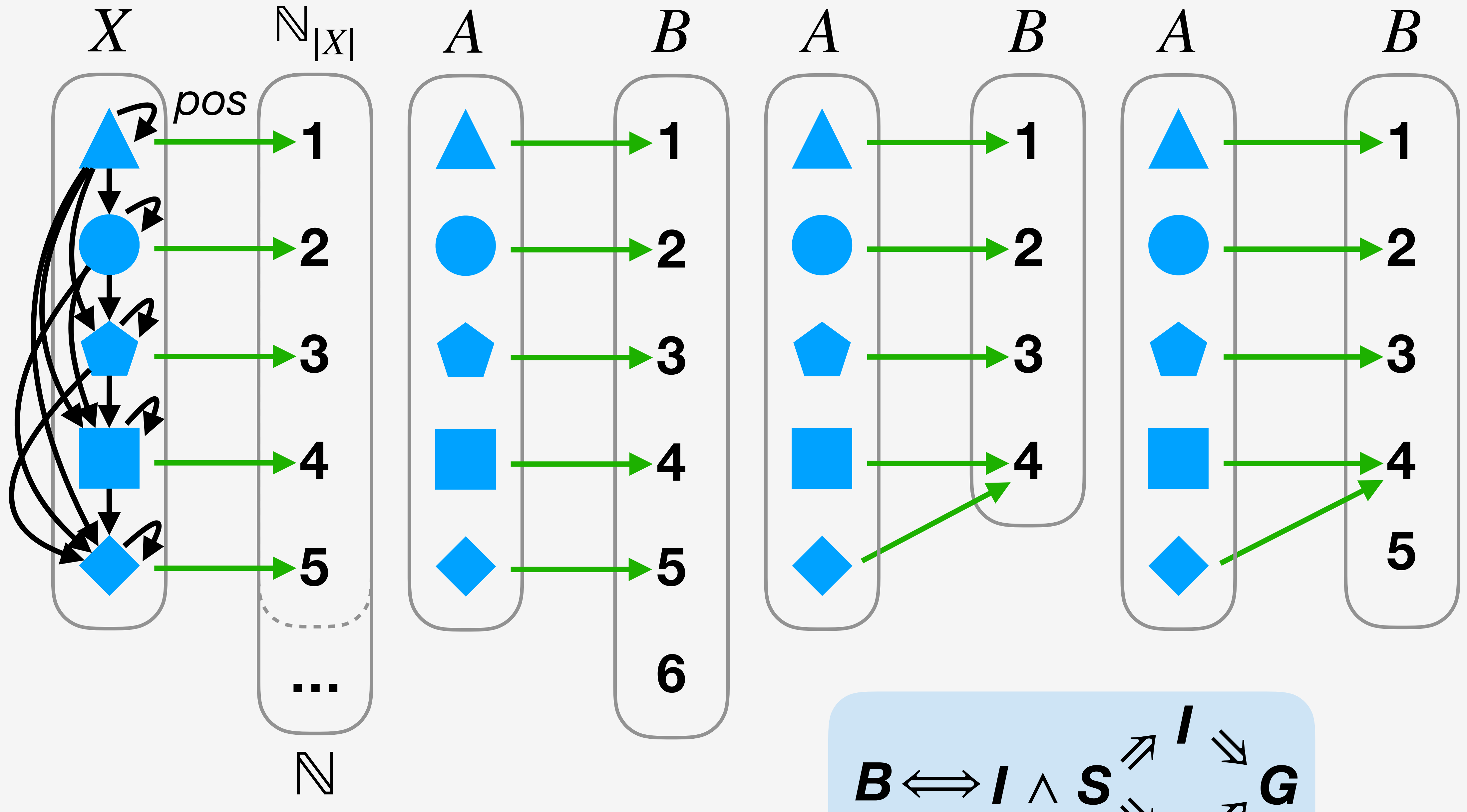

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